ELECTRICAL TECHNOLOGY

A TEXTBOOK FOR THE FOLLOWING EXAMINATIONS:
NATIONAL CERTIFICATE
CITY AND GUILDS
I.E.E.
B.Sc. ENGINEERING

BY

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FIFTH EDITION

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PREFACE

TO THE FIFTH EDITION

THIS new edition is substantially the same as the previous edition, but the opportunity has been taken to make certain small additions which, it is believed, will be of help to the student. In the chapter on Systems of Conductors the star-delta transformation has been applied to direct-current circuits, the unbalanced Wheatstone's Bridge network being taken as an illustrative example.

The section dealing with the action of the direct-current motor has been extended in order to illustrate the effect of the housing of the armature conductors in slots. The explanation of the torque production of the polyphase induction motor has also been amplified in order to illustrate the effect of the variable power factor of the rotor circuit with changing conditions of load. The M.M.F. of a three-phase winding has also been considered in greater detail, since it is not immediately obvious that the method of treatment given in the chapter on Rotating Magnetic Fields is applicable to the winding of an actual machine.

Other additions are of a minor nature, and there have been a few small corrections.

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PREFACE

TO THE FIRST EDITION

THIS textbook has been written with two main objects in view; first, to cover the ground of the more important examination syllabuses in Electrical Technology; and second, to be sold at a price which is within the means of all students. These two aims are diametrically opposite, but it is hoped that both have been fulfilled through the elimination of the more elementary portions of the subject, which should be thoroughly understood by students reaching this stage, and by the non-inclusion of half-tone blocks, which, although undoubtedly attractive, are of no real educational value.

Although the book is written primarily for examination purposes, the practical side has been kept in view, and it is hoped that it will appeal to practical engineers as well as to students.

The fundamental principles of Electrical Technology, both direct and alternate working, are covered, as are also certain principles of design, but no detailed designs are worked out, as these are beyond the scope of the book. The mathematical knowledge required of the reader is, on the whole, elementary, except in the chapter on "Electrical Oscillations," in which differential equations are used of necessity. There is a large number of worked examples in the text, and most of the chapters have questions to be answered; these are mainly drawn from examination papers set by the London University and by the City and Guilds of London Institute.

The great majority of the diagrams have been specially drawn, but a few have been kindly lent by Messrs. Pitman, and others by various manufacturing firms. These are acknowledged in the text.

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INTRODUCTION

1. Electrical Units.

Unit Magnetic Pole is that which, when placed at a distance of 1 centimetre from a similar pole in air, is repelled with a force of 1 dyne.

Unit Magnetic Field Strength. A magnetic field has unit strength when a unit magnetic pole situated in the field is acted on by a force of 1 dyne.

Unit Current is that current which, when flowing in a conductor of unit length bent in the form of an arc of a circle of unit radius, produces a magnetic field of unit strength at the centre. The current so defined is the *absolute* unit. At one time this was considered too large, and one-tenth of this was taken as the *practical* unit. The practical unit is the *ampere.*

The *International Ampere* is the unvarying current which, when passed through a specified solution of silver nitrate in water, deposits silver at the rate of 0·001118 of a gramme per second. The international ampere is equal to 0·99997 amperes and $0 \cdot 99997 \times 10^{-1}$ absolute units.

Unit Quantity of electricity passes across the cross section of a conductor when unit current flows for unit time. The practical unit is the *coulomb,* this being the quantity which passes when 1 ampere flows for 1 second.

Unit Potential Difference. The unit of potential difference (P.D.) exists between two points if unit work is done in transferring unit quantity from one to the other. If 1 erg of work is done when 1 absolute unit of current flows for 1 second, the P.D. between the two points is 1 absolute unit. This unit is too small for practical purposes, the practical unit, the *volt,* being equal to 10^8 absolute units. The volt is also the P.D. required at the ends of a practical unit of resistance (*see* below) to produce a current of 1 ampere. The *international volt* is the P.D. which, applied to a conductor of 1 international unit of resistance (*see* below), produces a current of 1 international ampere. The international volt is equal to 1·00049 volts or $1 \cdot 00049 \times 10^8$ absolute units.

Unit Resistance. A conductor has unit resistance (absolute) if an amount of energy equal to 1 erg per second is expended in sending 1 absolute unit of current through it. This unit is very small, and the practical unit, the *ohm,* is equal to 10^9 absolute units. The *international ohm* is the resistance offered to an unvarying electric current by a column of mercury at the temperature of melting ice, 14·4521 grammes in mass, of constant cross section, and of length 106·300 centimetres. The international ohm is equal to 1·00052 ohms or $1 \cdot 00052 \times 10^9$ absolute units.

2. Work and Power.

Power is the rate of doing work. If a quantity Q of electricity is passed through a circuit against a P.D. of E, then the work done is equal to EQ. The power is therefore EQ/t. But Q/t is the current. Hence, power is equal to EI. The absolute unit of power is the *erg per second,* and the practical unit is the *watt.*

$$1 \text{ watt} = 10^8 \times 10^{-1} = 10^7 \text{ absolute units.}$$
$$= 10^7 \text{ ergs per sec.}$$
$$= 1 \text{ joule per sec.}$$

Also $\quad 1 \text{ watt} = \dfrac{1}{746} \text{ of a h.p.}$

$\quad 1 \text{ kilowatt (kW)} = 1{,}000 \text{ watts.}$

From the relationship $I = E/R$, we have

$$\text{Power} \qquad W = E\,I$$
$$= E^2/R$$
$$= I^2 R$$

The unit of energy or work done has been fixed by the Board of Trade as the energy supplied in 1 hour when the power in the circuit is 1 kW.

$$\text{Hence,} \qquad \text{energy} = \text{kW} \times \text{hours}$$
$$= 1{\cdot}34 \times \text{h.p. hours.}$$

It is to be remembered that when the temperature of a body is raised, the necessary heat entails the expenditure of an equivalent amount of energy. If the heat H is measured in gramme calories, J is the mechanical equivalent of heat, and the energy is measured in watt-seconds or joules; then

$$J\,H = \text{watt-seconds}$$
$$= (\text{watt-seconds} \times 10^7) \text{ ergs.}$$

But J is numerically equal to $4{\cdot}2 \times 10^7$

$$\therefore\ 4{\cdot}2 \times 10^7\, H = \text{watt-seconds} \times 10^7$$

$$\text{or} \quad H = \frac{\text{watt-seconds}}{4{\cdot}2}$$

3. Capacitance.

The capacitance, C of a condenser is defined as the quantity of electricity necessary to produce a P.D. of 1 unit across its plates.

$$C = \frac{Q}{V}$$

The practical unit is the *farad*. Now the coulomb is equal to 10^{-1} absolute units, while the volt is equal to 10^8 absolute units. Hence,

$$\text{the farad} = \frac{10^{-1}}{10^8} \quad 10^{-9} \text{ absolute units.}$$

When the capacitance of a condenser is calculated from its dimensions the result is given in *electrostatic* units, the dimension of the electrostatic unit of capacitance being the cm. The absolute unit is an *electromagnetic* unit, and therefore, to convert from cm. unit to farads it is necessary to make use of the relationship between the electrostatic and electromagnetic units. It can be shown that

Electromagnetic unit of capacitance	=	Electrostatic unit of capacitance	$\times$ (velocity of light)^2
	=	Electrostatic unit	$\times (3 \times 10^{10})^2$
Farads	=	Electromagnetic units	$\times\ 10^{-9}$
Microfarads (μF.	=	Electromagnetic units	$\times\ 10^{-9} \times 10^{-6}$
	=	Electrostatic units	$\times\ 10^{-15} \times (9 \times 10^{20}$
	=	Electrostatic units.	$\times$ 900,000

EXAMPLE. Deduce an expression for the capacitance per mile of a concentric cable having an inner conductor of diameter d, an outer conductor of inside diameter D, and a dielectric of S.I.C., κ.

Such a cable constitutes a cylindrical condenser, for which we have the well-known formula

$$C = \frac{\kappa}{2\log_\varepsilon \frac{D}{d}} \text{ electrostatic units per cm. length}$$

$$= \frac{2 \cdot 54 \times 12 \times 5280}{2} \times \frac{\kappa}{\log_\varepsilon \frac{D}{d}} \text{ electrostatic units per mil}$$

$$= \frac{2 \cdot 54 \times 12 \times 5280}{2 \times 2 \cdot 3} \times \frac{\kappa}{\log_{10} \frac{D}{d}} \text{ electrostatic units per mile}$$

$$= \frac{2 \cdot 54 \times 12 \times 5280}{2 \times 2 \cdot 3 \times 900{,}000} \times \frac{\kappa}{\log_{10} \frac{D}{d}} \mu\text{F. per mile}$$

$$= \frac{\cdot 039\kappa}{\log_{10} \frac{D}{d}} \mu\text{F. per mile.}$$

4. Resistance of Conductors.

The resistance of a conductor of length l and uniform cross section a is given by the expression

$$R = \frac{\rho l}{a}$$

where ρ is a constant for any material at a constant temperature. This constant ρ is called the *specific resistance* or *resistivity*. The numerical value of ρ for a given conductor depends upon the units of l and a. If cm. units are used then ρ is in *ohms per centimetre cube,* whereas if inch units are used ρ is in *ohms per inch cube.* The numerical value of either unit is so very small that it is usual to express the specific resistance in microhms per cm. cube or per inch cube. The following values are useful—

Material.	Specific Resistance at 60°F. or 15·6°C.		Temperature Coefficient.	
	Microhms per cm. cube	Microhms per in. cube.	Per °C. at 15·6°C.	Per °F. at 60°F.
Copper (annealed)	1·696	·668	$\cdot 401 \times 10^{-2}$	$\cdot 223 \times 10^{-2}$
Copper (hand drawn)	1·731	·681	·401 „	·223 „
Silver (annealed)	1·560	·614	·376 „	·209 „
Aluminium	2·91	1·146	·38 „	·21 „
German Silver (50%Cu, 30%Ni, 20%Zn)	30	11·8	·04 „	·02 „
Manganin (84%Cu, 12%Mn, 4% Ni)	44	17·3	·00 „	·00 „
Eureka (60%Cu, 40%Ni)	49	19·3	·00 „	·00 „
Graphite	3,000	1,200	−·05 „	−·03 „

The specific resistance of most conductors increases with increase in temperature, exceptions being one or two alloys which have a negligible temperature coefficient, and carbon which has a negative temperature coefficient, its specific resistance therefore decreasing with an increase in temperature. For the range of temperature found in electrical machinery the law connecting resistance and temperature, t, can be regarded as linear.

If $\quad R_1 =$ resistance at $t_1°$

$\quad\quad R_2 =$ resistance at $t_2°$

Then $\quad R_2 = R_1 \{1 + a(t_2 - t_1)\}$

where a is a constant called the *temperature coefficient*. Using this expression, the mean rise of temperature of a winding can be determined very accurately from resistance measurements made at the beginning and end of a run. If R_1 and R_2 are the initial and final resistances, then

$$\text{Rise of temperature } (t_2 - t_1) = \frac{R_2 - R_1}{R_1 a}$$

5. Insulators.

An insulator is a substance whose specific resistance is so very many times greater than that of a conductor, such as copper, that it can almost be regarded as a non-conductor. The most important property of an insulator is its *disruptive* or *dielectric strength*. This is a measure of its ability to resist breakdown under the application of a high voltage, and is usually expressed in kilovolts per mm. or per mil thickness. The numerical value of the dielectric strength is of little value unless the following additional data are specified—(a) shape of the electrodes; (b) duration of the application; (c) frequency and wave form of the pressure if made with alternating voltage. It is necessary to specify the shape of the electrodes, since this governs the distribution of the electrostatic field in the material, and the breakdown is obviously determined by the *maximum* and not the average value of the potential gradient. It is also desirable to specify the thickness of the test piece, because the breakdown voltage is not proportional to the thickness. The relationship between breakdown voltage E and thickness can be expressed with fair accuracy by either

$$E = A \times t^{\frac{2}{3}} \quad . \quad . \quad . \text{Baur's Law}$$

$$E = B + Ct$$

where A, B, and C are constants for a given material.

The specific resistance of practically all insulators decreases rapidly with increase of temperature. The ratio of the specific resistances at 20°C. to the values at 30°C. for various insulators ranging from 1·0 to 2·0 for mica to 16·0 for yellow beeswax.

If the temperature becomes so high that softening or oxidizing takes place, insulators usually become unfit for further use, for which reason it is necessary to limit the temperature rise of all insulated conductors.

The properties of the more important insulators are given in the table on page xiii. They can be roughly classified as hygroscopic or non-hygroscopic.

ELECTRICAL TECHNOLOGY

PART ONE—DIRECT CURRENT

CHAPTER I

ELECTRO-MAGNETISM

1. Production of Magnetic Fields by Electric Current. Whenever a current-carrying conductor is situated in a magnetic field and is arranged at right angles to the lines of force, a mechanical force acts on the conductor, the direction of this force being given by the well-known "left-hand" rule. The magnitude of the force is given by

$$F = BIl \text{ dynes}$$

where B = flux density in lines per sq. cm.

I = current in electromagnetic c.g.s. units or ab-amperes.

l = length of conductor in cm.

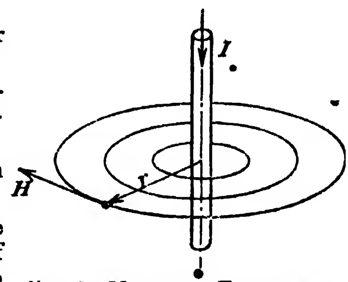


FIG. 1. MAGNETIC FIELD OF A STRAIGHT CONDUCTOR

If this conductor is moved in the field so as to cut a total flux of Φ lines of force, then work has to be done by (or against) this force F, the work done being equal to

$$w = \Phi I \text{ ergs.}$$

Now consider a long straight conductor carrying current and influenced by no field other than that produced by the current. The lines of force of this field are circles having their plane perpendicular to the conductor, and their centre at the conductor centre. Let H be the field strength at a distance r cm. from the centre, Then, if a unit north pole is placed at this distance, the force acting on it will be H dynes tangential to a line of force, Fig. 1. Hence. if this unit north pole is moved once round the circle of radius r against this force H, the work done will be

$$w = H \times 2\pi r \text{ ergs.}$$

But 4π lines of force emanate radially from a unit pole, and, in one tour of a circular line of force of the conductor's field, each of these 4π lines will have cut the conductor once.

$$\therefore \quad w = \Phi I = 4\pi I \text{ ergs.}$$

$$\therefore \quad 2\pi r H = 4\pi I$$

$$H = \frac{2I}{r} \text{ dynes}$$

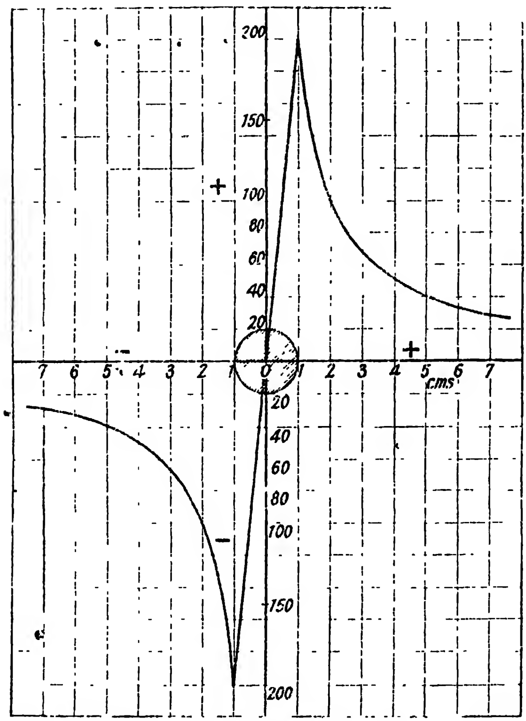


FIG. 2. MAGNETIC FIELD EXTERNAL TO A STRAIGHT CONDUCTOR

If the wire is surrounded by a medium of magnetic permeability unity then $B = H$, and

$$B = \frac{2I}{r}$$

If the current is expressed in amperes, then since this practical unit is one-tenth of the c.g.s. unit we have

$$B = \frac{\cdot 2I}{r}$$

This refers to the magnetic field external to the conductor. Let the conductor radius be a cm., then, if the current density is uniform over the cross section, the current enclosed by an internal line of force of radius $r(< a)$ will be

$$I_r = I \times \frac{r^2}{a^2}$$

and the field strength at a distance r from the axis will be

$$H = \frac{2I_r}{r} = \frac{2I}{r} \times \frac{r^2}{a^2}$$

$$= \frac{2Ir}{a^2}$$

Also $\qquad B \quad H = \frac{2Ir}{a}$

assuming the conductor has unit permeability.

The flux density inside the conductor is thus a linear function of the distance r from the axis. Just at the surface where

$$r = a,$$

we have $\qquad B = H = \dfrac{2I}{a}$

The direction of the lines of force is given by the well-known corkscrew rule.

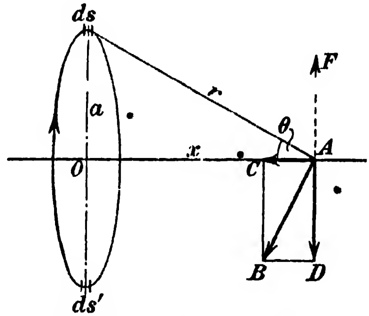


FIG. 3. CIRCULAR CONDUCTOR.

Fig. 2 shows the variation of flux density with distance for a conductor of radius 1 cm. carrying a current of 1,000 amperes.

2. **Field due to a Circular Current.** Fig. 3 shows a circular conductor carrying current. Considering an element ds of conductor the field strength at a distance r is given by Laplace's law, viz.

$$dH = \frac{I.\,ds.}{r^2}$$

Applying this to a point A distant x from the centre, the direction AO being perpendicular to the plane of the coil, we have for the field strength at A due to the element dS

$$\frac{I\,.\,ds}{r^2} \text{ along the direction } AB.$$

Component of this along $AO \quad \frac{I \,.\, ds}{} \sin \theta$

Component perpendicular to $AO = \frac{I \,.\, ds}{r^2} \cos \theta$

and acts along AD. Now if we take the diametrically opposite elements ds', the perpendicular component will be equal in magnitude to that due to ds but will act along AF, these two components thus neutralizing one another. Hence, if we consider the whole circle as made up of pairs of diametrically opposite elements, we see that the total perpendicular component will be zero. The total field strength at A is thus the sum of all the axial components.

$$\therefore\ H = \frac{I \sin \theta}{r^2} \times \Sigma ds$$

$$= \frac{I \sin \theta}{r^2} \times 2\pi a$$

$$= \frac{2\pi I a^2}{r^3} \text{ or } \frac{2\pi I a^2}{(a^2 + x^2)^{\frac{3}{2}}}$$

At the centre of the circle

$$x = 0,$$

and $$r = a.$$

$$\therefore\ H = \frac{2\pi I.}{a}$$

If the current is expressed in amperes then

$$H = \frac{2\pi I a^2}{(a^2 + x^2)^{\frac{3}{2}}}$$

3. Field due to a Solenoid. Let the solenoid be of length l and radius a and let it be uniformly wound with N turns. Then there will be N/l turns per unit length, and in an element of length dx, Fig. 4, there will be Ndx/l turns. Now an element of length can be regarded as a coil such as that in § 2, and therefore the field strength at the centre due to the element dx is

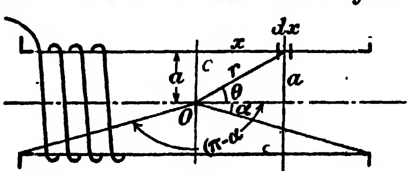


FIG. 4. SELF-INDUCTION OF A SOLENOID

$$dH = 2\pi I \times \frac{Ndx}{l} \times \frac{a}{r^2} \sin \theta$$

$$= \frac{2\pi I N \sin^3 \theta dx}{al}$$

Now $$x = a \cot \theta$$

$$\therefore\ dx = -\frac{a}{\sin^2 \theta}\ .\ d\theta.$$

$$\therefore\ dH = -\frac{2\pi IN \sin \theta\ .\ d\theta}{l}$$

$$\therefore\ H = -\frac{2\pi IN}{l}\int_{\pi-\alpha}^{\alpha} \sin \theta\ .\ d\theta$$

$$= \frac{2\pi IN}{l}\Big[\cos\ \theta\Big]_{\pi-\alpha}^{\alpha}$$

$$= \frac{4\pi IN\ \cos\ \alpha}{l}$$

If the solenoid is very long, the angle α is very small and cos α approximates to unity. The expression then becomes

$$H = \frac{4\pi IN}{l}$$

This expression is of very great practical importance. If I is in amperes then

$$H = \frac{\cdot 4\pi IN}{l} \simeq 1{\cdot}26\ \frac{IN}{l}$$

4. **Force Between Two Parallel Current-carrying Conductors.** Two parallel conductors are shown in Fig. 5, and it will be clear that either conductor can be considered as being situated in the field of the other. Thus conductor A is in a field of strength.

$$H = \frac{2I}{r} \text{ due to conductor } B.$$

∴ Force acting on A

$$F = BIl = HIl \text{ in air}$$

$$= \frac{2I^2l}{r} \text{ dynes, when } I \text{ is in c.g.s. units.}$$

An equal force acts on conductor B. If the currents are in the same direction the left-hand rule shows that the two

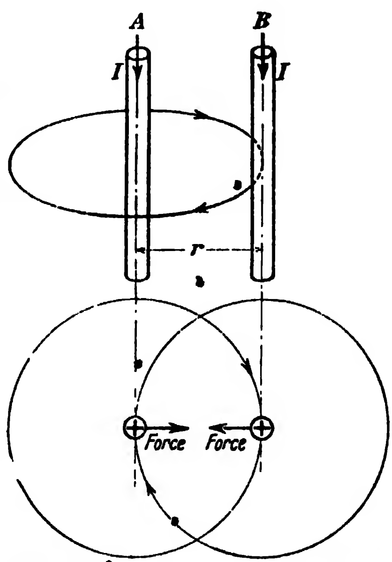


FIG. 5. MECHANICAL FORCE BETWEEN PARALLEL CONDUCTORS

forces produce attraction between the conductors, while if the currents are in opposite directions then the conductors repel one another.

Example. Two long parallel conductors, each carrying 500 amperes in the same direction, are spaced 6 in. apart. Calculate the force per foot run acting on each conductor.

$$I = 500 \text{ amperes} = 50 \text{ c.g.s. units.}$$

$$l = 12 \times 2{\cdot}54 \text{ cm.}$$

$$r = 6 \times 2{\cdot}54 \text{ cm.}$$

$$\therefore\ F = \frac{2 \times 50 \times 50 \times 12 \times 2{\cdot}54}{6 \times 2{\cdot}54} \text{ dynes.}$$

$$= 10{,}000 \text{ dynes per foot run}$$

$$= 10{\cdot}2 \text{ gm. weight, or } 0{\cdot}0225 \text{ lb. weight.}$$

5. **The Magnetic Circuit.** The path of the magnetic flux is spoken of as the magnetic circuit. Just as a flow of current in the electric circuit necessitates the presence of an electro-motive force, E.M.F., so the production of a magnetic flux requires the presence of a "magneto-motive force," M.M.F. Consider a uniform solenoid of length l cm., cross section a sq. cm., and having N turns. Then, if it is carrying a current of I amperes, the field strength inside is given by

$$H = \frac{4\pi}{10} \times \frac{IN}{l}$$

Calling the product of the current and the number of turns the "ampere-turns," AT, we have

$$H = \frac{1{\cdot}26 \times AT}{l} \text{ lines per sq. cm.}$$

If the solenoid is wound on a magnetic substance of permeability μ, then the flux density or induction density in the core is given by

$$B = \mu H$$

$$= \frac{1{\cdot}26 \times AT}{l/\mu} \text{ lines per sq. cm.}$$

Hence the total magnetic flux through the core is given by

$$\Phi = Ba \text{ lines of force}$$

$$\therefore\ \Phi = \frac{1{\cdot}26 \times AT}{l/a\mu}$$

Now in the electric circuit the resistance of a conductor of length l, cross section a, and specific resistance ρ, is given by

$$R = \frac{\rho l}{a} \text{ ohms}$$

$$= \frac{l}{a\sigma} \text{ ohms}$$

where σ, the reciprocal of ρ, is the specific conductivity. Hence, from Ohm's Law, we have current

$$I = \frac{\text{E.M.F.}}{l/a\sigma}$$

Comparing these expressions for flux and current, we see that they are analogous, and we have

$$\text{M.M.F.} = 1{\cdot}26 \times AT$$

$$\text{Reluctance} = \frac{l}{a\mu}$$

The magnetic reluctance is analogous to resistance in the electric circuit, M.M.F. is analogous to E.M.F., and flux is analogous to current. It is useful to bear this analogy in mind when making magnetic calculations, but it is not complete, for the following reasons—

(*a*) The electric current is a true " flow " but there is no flow in a magnetic flux, the word " flux " being, therefore, misleading in this respect.

(*b*) For a given temperature, the quantity σ, or its reciprocal ρ in the expression for electrical resistance, is independent of the strength of the current, but the magnetic permeability μ is not independent of the total flux.

(*c*) In the electric circuit energy is expended so long as the current flows, but in the magnetic circuit energy is expended only in creating the flux, and not in maintaining it.

From the equation

$$H = \frac{1{\cdot}26\, AT}{l}$$

we have, ampere-turns per cm. length, *at*,

$$at = \frac{AT}{l}$$

$$= \frac{H}{1{\cdot}26}$$

$$= {\cdot}8H \text{ very approximately.}$$

Example. An iron ring, 100 cm. mean circumference, is made from round iron of cross section 10 sq. cm.; its permeability is 500. If it is wound with 200 turns, what current will be required to produce a flux of 100,000 lines?

$$B = \frac{\text{Flux}}{\text{cross section}} = \frac{100{,}000}{10} = 10{,}000 \text{ lines per sq. cm.}$$

$$H = B/\mu = 10{,}000/500 = 20$$

$\therefore$ Ampere-turns per cm. length

$$at = \cdot 8H = 16$$

The length of the magnetic path is the circumference of the ring, and therefore

$$AT = at \times l$$

$$= 16 \times 100 = 1{,}600$$

$$\therefore \text{Current required} = \frac{AT}{N} = \frac{1{,}600}{200} = 8 \text{ amperes.}$$

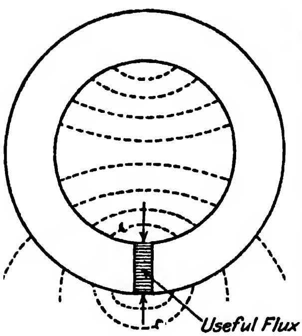


FIG. 6

MAGNETIC LEAKAGE

6. **Magnetic Leakage.** Fig. 6 shows a magnetized iron ring with a narrow air gap, and the flux which crosses the gap can be regarded as the "useful" flux. Some of the total flux produced does not cross the gap, but takes instead leakage paths, as shown by the dotted lines. For the purposes of calculation it is assumed that the iron carries the whole of the total flux throughout its entire length. The ratio

$$\frac{\text{Total flux}}{\text{Useful flux}}$$

is called the "leakage factor."

Example. The iron ring in the first example has a saw cut 2 mm. wide made in it. Find the current required to produce the flux of 100,000 lines across the air gap, given that the leakage factor is 1·3 and that the iron is such that when $B = 13{,}000$, $\mu = 300$.

The ampere-turns for the gap and for the iron are calculated separately.

Flux through gap $= 100{,}000$ lines

B in gap $= \frac{100{,}000}{10} = 10{,}000$ lines per sq. cm.

H in gap $= B/\mu = \frac{10{,}000}{1} = 10{,}000$, since $\mu = 1$ for air

$at = \cdot 8H$ $= 8{,}000$

$\therefore AT$, for gap $= 8{,}000 \times \cdot 2 = 1{,}600$

Flux through iron = useful flux × leakage factor

$= 100{,}000 \times 1{\cdot}3 = 130{,}000$ lines

$\therefore B$ in iron $= \frac{130{,}000}{10} = 13{,}000$ lines per sq. cm.

$\therefore H$ $= \frac{13{,}000}{300} = 43{\cdot}3$

$\therefore at$, for iron $= \cdot 8 \times 43{\cdot}3 = 34{\cdot}6$

$\therefore AT$, for iron $= 34{\cdot}6 \times 99{\cdot}8$

$= 3{,}450$

$\therefore$ Total ampere-turns $= 1{,}600 + 3{,}450 = 5{,}050$

$\therefore$ Current $= \frac{5{,}050}{200} = 25{\cdot}3$ amps.

7. **Magnetic Circuits in Parallel.** We see from the above example that when the different parts of a composite circuit are in series, that is, when they carry the same flux, the total M.M.F. required to produce a given flux is the sum of the M.M.F.s for the separate parts. If the different parts of a circuit are in parallel, then the necessary M.M.F. is that which will produce the required flux in one part of the circuit considered by itself. This is best illustrated by the calculation of the ironclad magnet shown in Fig. 7. The number of ampere-turns to produce a useful flux of 750,000 lines in the air gap is required, the leakage factor being taken as 1·4, and the permeability of the iron, 600. There are two magnetic paths in parallel, one being shown by the dotted lines, and it is therefore

necessary to consider one circuit only. The calculation is made in tabular form below.

Part of Circuit.	Area a.	Length l.	Flux Φ.	Flux density $B = \Phi/a$.	$H = B/\mu$.	$at = \cdot 8H$	$AT = at \times l$.
Poles . .	12 sq. in. = 77·5 sq. cm.	2 × 6 in. = 30·4 cm.	750,000 × 1.4 = 1,050,000	13,600 lines per sq. cm.	22·7	18·2	553
Air Gap .	77·5 sq. cm.	½ in. = 1·27 cm.	750,000	9,700 lines per sq. cm.	$H = B$ = 9,700	7,760	9,860
Yoke . .	6 sq. in. = 38·8 sq. cm.	Path $ABCDEF$ 20½ in. = 52 cm.	½ × flux per pole = 525,000	13,600	22·7	18·2	950

Total Ampere-turns = 11,360

Note that the magnetic calculations are never made beyond four significant figures.

The method of calculation for a generator field magnet is given in Chapter III.

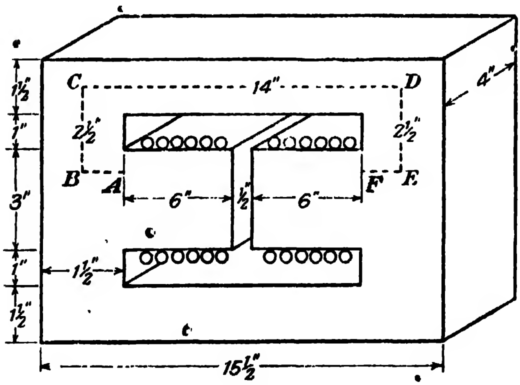


FIG. 7

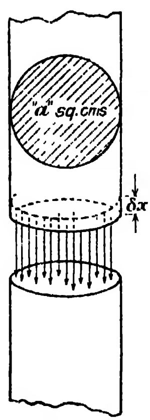


FIG. 8

8. **Lifting Power of an Electro-magnet.** It is shown in par. 10 that the energy stored in a magnetic field in air is $B^2/8\pi$ ergs per cc., where B is in lines per sq. cm. Consider two poles arranged in Fig. 8. Let each have an area a sq. cm. and let P be the force of attraction in dynes between them. Let one pole be moved a very small distance δx against the force P, then work done is obviously $P\delta x$ ergs.

But the volume of the magnetic field is increased by $a\delta x$ cc. and therefore the energy stored in the field is increased by $B^2/8\pi \times a\delta x$ ergs.

This is obviously equal to the work done in separating the poles so that

$$P\delta x = \frac{B^2}{8\pi} \times a\delta x$$

$$P = \frac{B^2a}{8\pi} \text{ dynes}$$

$$= \frac{B^2a}{8\pi \times 981} \text{ grammes wt.}$$

$$= \frac{B^2a}{8\pi \times 981 \times 454}$$

$$\text{or} \quad \frac{B^2a}{11{,}200{,}000} \text{ lb. wt.}$$

B still being in lines per sq. cm.

Owing to their enormous lifting powers, electro-magnets are used commercially for handling heavy magnetic materials such as steel rails and girders, castings, etc. The type used for lifting irregular scrap is illustrated in Fig. 9.

9. **Action of Iron and Steel under a Varying Magnetizing Force** Consider a straight solenoid having first of all a non-magnetic core. If current is passed through the solenoid a magnetic field will be set up, whose value inside the coil will be given by

$$H = \frac{1{\cdot}26 \times AT}{l} = \frac{1{\cdot}26NI}{l}$$

This field is a measure of the magnetizing force set up by the coil and we see that it is proportional to the current. The total number of lines of force is equal to the field strength H multiplied by the cross section of the solenoid. Now suppose that an iron core is substituted for the non-magnetic core, then there is immediately an enormous increase in the total number of lines of force due to the lines set up by the induced magnetism in the core. If $\mathscr{I}$ is the intensity of magnetization of the core, that is, the pole strength per unit area, then the number of lines per sq. cm. in the core due to the induced magnetism is $4\pi\mathscr{I}$. The flux density, B,

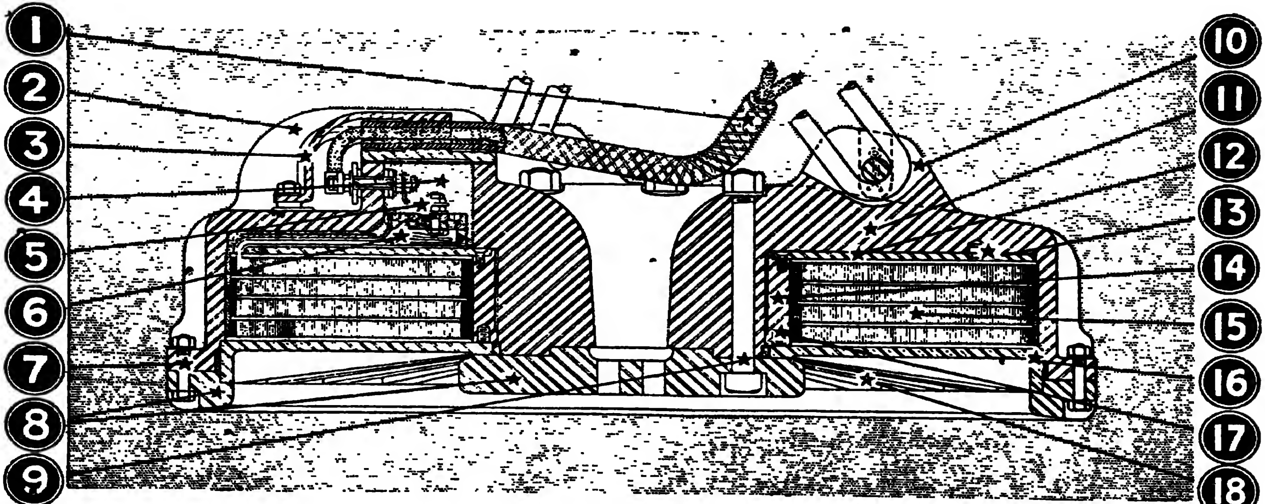


FIG. 9 (*By courtesy of the Igranic Electric Co., Ltd.*)

1. *Leads.*—Protected by armoured hose.
2. *Protecting Flange for Terminal Box.*
3. *Terminal Hood.*—Made of heavy cast steel.
4. *Terminal Cavity.*—Filled with a moisture-repelling, insulating compound.
5. *Coil Terminus Studs.*
6. *Leads from Coil to Terminal Stud.*—Extra length folded in a pocket under the stud making it possible to repair a broken stud without removing the coil from the case.
7. *Through Bolts.*—To secure outer pole shoe to magnet case. The nuts on these bolts are protected by ribs on the magnet case.
8. *Pole Shoes.*—Both outer and inner pole shoes are easily renewable.
9. *Through Bolt.*—One of five for holding the inner pole shoe in place.
10. *Suspension Lugs.*—Cast integral with the magnet body and heavy enough to withstand rough usage.
11. *Magnet Case.*—Made of special magnet steel.
12. *Top Plate of Coil Spool.*—Removable to facilitate inspection of the upper portion of the coil.
13. *Dowel.*—Prevents rotation of the spool in the magnet case.
14. *Core of Spool.*—Magnet steel, forms part of the inner pole of the magnet.
15. *Magnetizing Coil.*—Wound with strap copper and insulated with mica and asbestos.
16. *Coil Shield.*—Made of manganese steel, sealed at inner and outer edges with waterproof packing.
17. *Tapped Eye-Bolt Hole.*—One of four holes in the core of the magnet spool for inserting eye-bolts used in handling the spool.
18. *Heavy Protective Ribs.*—On the under side of the coil shield.

in the core is the sum of the lines per sq. cm. due to the original field and the induced magnetism, so that

$$B = H + 4\pi \mathcal{I}$$

and, as stated before, the permeability is given by

$$\mu = B/H$$

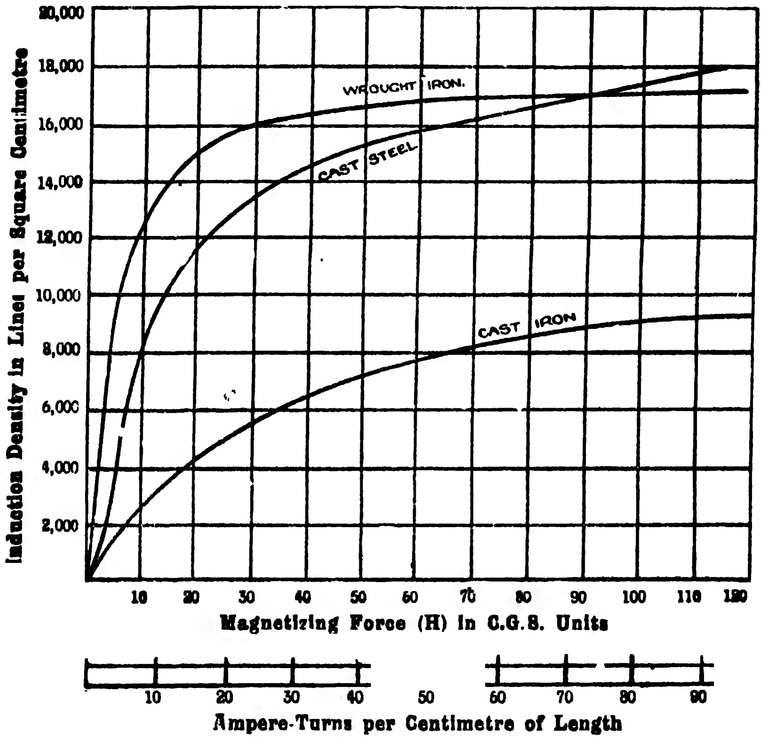


Fig. 10

Magnetization Curves for Iron and Steel

If H is increased from zero to a high value, and B plotted against H, the shapes of the resultihg curves for different materials are shown in Fig. 10. It will be noticed that the curves are first of all approximately straight lines through the origin, showing that B is roughly proportional to H. Then the curves begin to turn over, forming a "knee," and finally they become almost horizontal and exhibit very little increase in B for a large increase in H. In this final state the iron is said to be "saturated," and the phenomenon of saturation is explained by the molecular theory of magnetism, by the assumption that all the molecular magnets are pointing in

the same direction. It is useful to memorize the saturation values and the "knee" positions, since from these the curves can be reproduced roughly.

10. **Cycles of Magnetism.** If the magnetizing force applied to a specimen of iron is increased from zero to some maximum value, and the force is gradually reduced again to zero, it will be found that the new *B-H* curve for decreasing values of *H* lies somewhat

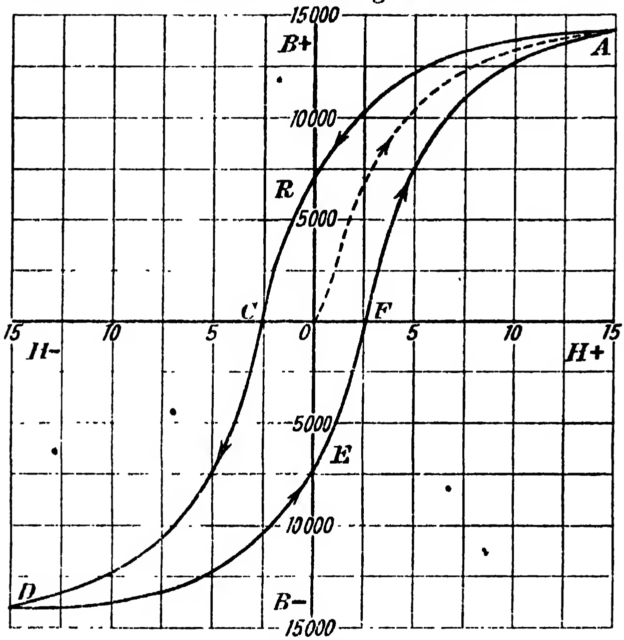


Fig. 11

Hysteresis Loop for Wrought Iron

above the original curve, and that when *H* is zero again *B* is finite. This effect is called *hysteresis*, since the values of *B* apparently lag behind those of *H*. The finite value of *B* when *H* is zero, *OR* in Fig. 11, is a measure of the residual magnetism. In order that the iron may be demagnetized it is necessary to apply a negative magnetizing force represented by *OC*. This negative force is called the *coercive force*. If now *H* is increased in this negative direction to its previous maximum value, the curve will reach a point *D*, at which the induction is equal to the previous maximum value, B_{max}. If, finally, *H* is gradually reduced to zero, reversed, and increased to its maximum value in the original direction, the curve *DEFA* will be traced out, the complete curve forming a closed loop, usually called a hysteresis loop. It will be seen that to take the iron

through the various states represented by this loop, an alternating magnetizing force has been applied. If these alternations are maintained, the value of B_{max} being the same during each cycle, then the iron will continue to go through the same series of changes.

The hysteresis loop can be regarded as a magnetic indicator diagram, and from this analogy we should expect that during each cycle an amount of energy represented by the area enclosed by the loop is expended. This is the case, as can be proved as follows. Suppose that the specimen is a ring of mean circumference l cm. and cross section a sq. cm., and let it have N turns. If the value of the magnetizing current at any instant is i* the magnetizing force will be

$$H = \frac{4\pi}{10} \times \frac{AT}{l} = \frac{4\pi Ni}{10l} \text{ if } i \text{ is in amperes}$$

and $4\pi Ni/l$ if i is in c.g.s. units.

If the value of the induction at the instant considered is B, then flux through the ring $= Ba$. Since the flux threads, or links with, the coil, an E.M.F. will be induced in the coil of magnitude.

$e =$ No. of turns $\times$ rate of change of flux . . . c.g.s. units

$$= N \times \frac{d}{dt} \text{ of } Ba$$

$$= Na \times \frac{dB}{dt}$$

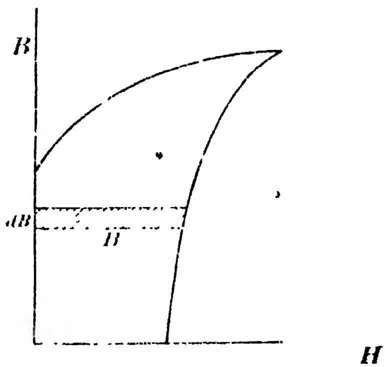


FIG. 12

Now the current i flowing at that instant will be opposed by this induced E.M.F., and therefore power will have to be expended in order to maintain the increase in i. We have

$$\text{Power at any instant} = ei = \frac{la}{4\pi} \times H \frac{dB}{dt}$$

Hence, work done during a small interval of time dt

$$= \frac{la}{4\pi} \times H \frac{dB}{dt} . dt$$

∴ Total work done

$$= \frac{la}{4\pi} . \int HdB.$$

* Small letters are used to denote instantaneous values of the current and voltage.

We see from Fig. 12 that HdB is the area of an elementary strip of the B-H curve, and therefore $\int HdB$ for a whole cycle is the area enclosed by the loop.

Again, la is the volume of the ring.

$$\text{Work done per cc.} = \frac{\text{Area of loop}}{4\pi}$$

The area of the loop is obviously measured in units of H and B. Thus, if one square on the diagram represents 1 unit of H and 100 units of B, its area will be 100. If H and B are in c.g.s. units, i.e. lines per sq. cm., then the above expression will give the work done in ergs per cc. per cycle. This work done in taking a specimen of iron through a cycle of magnetism is waste energy, since it becomes converted into heat, and it is therefore spoken of as the hysteresis loss.

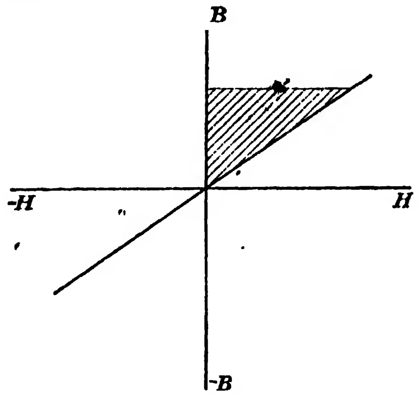


FIG. 13

Suppose the ring were non-magnetic (air, for example), then, since $B = H$ for such materials, the B-H curve is a straight line through the origin (Fig. 13). Therefore, work done in increasing the flux density from zero to some value B.

$$\frac{la}{4\pi}\int HdB = \frac{la}{4\pi}\int BdB$$

$\therefore$ Work done per cc. = energy per cc. of the magnetic field

$$= \frac{1}{4\pi}\int BdB$$

$$= \frac{1}{8\pi}B^2 \text{ or } \frac{1}{8\pi}H^2 \text{ ergs.}$$

In all classes of electrical machinery some part of the magnetic circuit is taken through rapid reversals of magnetism. Hysteresis loss therefore takes place, and in order that the efficiency and temperature rise of the machinery may be calculable it is necessary to know this loss. If the shape of the loop is known, then the loss can be calculated by determining the area of the loop, but this method, although suitable for testing specimens of iron, is not so suitable for calculations on machines. It is, therefore, preferable to use the method discovered by Steinmetz. He found that the hysteresis loss per cc. per cycle is very closely represented by an expression of the form of $\eta B_{max}^{1 \cdot 6}$ ergs, where η is a constant

called the hysteresis coefficient, and whose value depends upon the material. Hence, if the material is taken through f cycles per second,

$$\text{Hysteresis loss per cc.} = \eta B_{max}^{1\cdot6} f. \text{ ergs per sec.}$$
$$= \eta B_{max}^{1\cdot6} f \times 10^{-7} \text{ watts}$$

And for a volume of v ccs.,

$$\text{Hysteresis loss} = \eta v f B_{max}^{1\cdot6} \times 10^{-7} \text{ watts.}$$

Values of the hysteresis coefficient for different materials are given in the following table. These figures are taken from *Specification and Design*, by Miles Walker.

Material.	Hysteresis Coefficient.
Hard cast steel	0·028
Cast steel	0·003 to 0·012
Cast iron	0·011 to 0·016
Very soft iron	0·002
Good dynamo sheet steel .	0·002
Silicon steel (·2% Si) . .	0·0021
Silicon steel (4·8% Si). .	0·00076

Example. A cylinder of iron of volume 10,000 ccs. revolves for 30 minutes at a speed of 3,000 revolutions per minute (r.p.m.) in a two-pole field of strength 8,000 lines per sq. cm. If the hysteresis coefficient of the iron is 0·003, the specific heat of iron is 0·11, the loss due to eddy currents is equal to that due to hysteresis, and 30% of the heat produced is lost by radiation, find the temperature rise of the iron.

Taking the density of iron as 7·8, we have for the mass of the iron 78,000 grammes. Hence, if t°C. is the temperature rise, heat retained by the iron

$$= 78{,}000 \times 0{\cdot}11 \times t = 8{,}580\, t \text{ calories}$$

$$\therefore \text{ Total heat produced} = \frac{8{,}580\, t \times 100}{70} = 12{,}250\, t \text{ calories}$$
$$= 12{,}250 \times 4{\cdot}2 \times 10^7 \times t \text{ ergs.}$$

Again, hysteresis loss $= \eta B_{max}^{1\cdot6}$ ergs per cc. per cycle

$$\therefore \text{ Total hysteresis loss} = \cdot003 \times 8{,}000^{1\cdot6} \times 10{,}000 \times (30 \times 60 \times f) \text{ ergs.}$$

Now in a two-pole field there is one magnetic cycle per revolution.

$\therefore f = 3{,}000$ cycles per min. $= 50$ cycles per sec.

$\therefore$ Total hysteresis loss $= 475 \times 10^{10}$ ergs

$\therefore$ Total loss $= 950 \times 10^{10}$ ergs

$\therefore 12{,}250 \times 4{\cdot}2 \times 10^7 \times t = 950 \times 10^{10}$

$\therefore t = 18$°C.

If the hysteresis loops for different kinds of iron and steel are examined it will be found that they can be separated broadly into

three groups, of which the typical loops are as shown in Fig. 14. Loop 1 is for hard steel, the large value of the coercive force indicating that this material is suitable for permanent magnets. The area of the loop is also large, thus showing that hard steel is not suitable for rapid reversals of magnetism. Loop 2 rises very steeply, showing that the permeability is high, and its large intercept on the *B* axis indicates a good retentivity. This loop, which is typical of wrought iron and cast steel, shows that these materials are suitable for the cores of electro-magnets. The characteristic feature of loop 3 is its very small area; at the same time it rises steeply, indicating a high permeability. The material, alloyed sheet steel, which has such a loop, is therefore suitable for all purposes where the iron is subjected to rapid reversals of magnetism, as in armature and transformer cores.

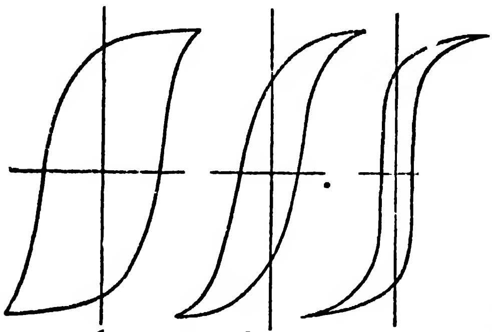


Fig. 14

Shapes of Hysteresis Loops for Different Materials

11. **Determination of B-H Curves.** Probably the most important method of determining a *B-H* curve is the Ballistic method, so called because a ballistic galvanometer is used. The classical form of specimen is that of a ring and the radial thickness is preferably kept small compared with the diameter, so that *H* will not vary appreciably over the cross section. The ring has two windings, one, the magnetizing winding, distributed over the whole circumference as evenly as possible; the other a secondary winding having 30 or 40 turns. The magnetizing coil is connected to a direct current

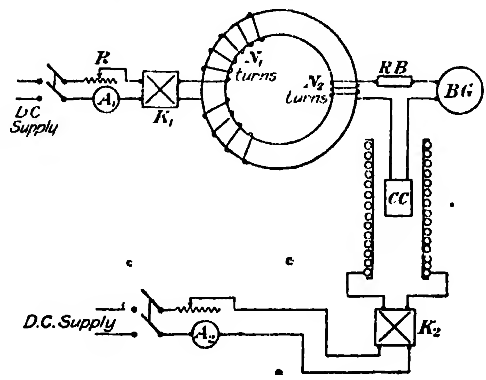


Fig. 15

Ballistic Method of Determining Magnetization Curves

supply through an adjustable resistance R (Fig. 15). An ammeter, A_1, measures the current, and a reversing switch, K, is included, so that the magnetizing current can be reversed. The secondary is connected to the ballistic galvanometer, and a resistance box, RB, and a calibrating coil, CC, are included in this circuit. The coil CC is placed in the middle of a large solenoid whose equipment is similar to that of the magnetizing coil for the specimen.

Let N_1 = No. of turns on magnetizing coil
l = mean circumference of specimen in cm.
a = cross section in sq. cm.

Then for any magnetizing current I, the magnetizing force

$$H = \frac{1 \cdot 26\, N_1 I}{l}$$

If, for any value of I, the switch K is reversed, the flux density B inside the ring is reversed, and it therefore changes from $+B$ to $-B$, a change of $2B$. Hence, the change of flux in the ring is $2Ba$, and if this change takes place in a small interval of time δt, then

$$\text{Rate of change of flux} = 2Ba/\delta t$$

Hence, if the secondary coil has N_2 turns,

$$\text{E.M.F. induced in the secondary, } e = N_2 \times \frac{2Ba}{\delta t}.$$

If the total resistance of the galvanometer circuit is R, the average current through the galvanometer will therefore be

$$\frac{2BaN_2}{R\,.\,\delta t}$$

Hence, quantity of electricity

$$= \text{Average current} \times \delta t$$
$$= \frac{2BaN_2}{R}$$

Now the "throw" of a Ballistic galvanometer, when corrected for logarithmic decrement, is proportional to the quantity of electricity which passes through it, provided that this quantity has passed through in a small interval of time, a condition which holds in this case. Hence, if θ is the corrected throw and K the galvanometer constant,

$$\text{Quantity of electricity} = K \times \theta$$

$$\therefore \frac{2BaN_2}{R} = K\theta$$

$$B = \frac{RK}{2aN_2} \times \theta$$

In order to be able to calculate B it is necessary to know the constant K. This constant *must be determined under the actual conditions of the experiment*, since it may vary if these conditions vary. It is determined by means of the calibrating coil, CC. A current, I_2 say, is passed through the large solenoid, in which CC hangs, and it produces through CC a field of strength $1{\cdot}26N_3I_2/l_2$, where N_3 and l_2 refer to the solenoid. If CC has a cross section of a_2 the flux through it will be $1{\cdot}26N_3a_2I_2/l_2$, and on reversing the switch K_2 there will be a change of flux through CC of

$$\frac{2{\cdot}52N_3a_2I_2}{l_2}$$

Hence, if CC has N_4 turns, the quantity of electricity which will pass through the galvanometer on reversing K_2 will, by the same reasoning as before, be

$$\frac{2{\cdot}52N_3N_4a_2I_2}{l_2R}$$

If the corrected galvanometer throw when K_2 is reversed is φ, then

$$\frac{2{\cdot}52N_3N_4a_2I_2}{l_2R} = K\varphi$$

$$K = \frac{2{\cdot}52N_3N_4a_2I_2}{l_2R\varphi}$$

Substituting this in the expression for B, we have

$$B = \left\{\frac{1{\cdot}26N_3N_4a_2I_2}{N_2l_2a\varphi}\right\} \times \theta$$

All the terms in the expression in the brackets are known, and therefore B can be calculated from observed values of θ. If it is not required to determine a hysteresis loop, the magnetizing current I is given a series of gradually increasing values, and the throw observed on reversing K_1. It will, of course, be necessary to make a preliminary test in order to be sure that the throw is of suitable magnitude when K_1 is reversed with I at its highest value. Before taking observations it will then be necessary to demagnetize the iron, this being most easily accomplished by applying a gradually decreasing alternating potential to the magnetizing coil.

If it is required to take the iron through a complete loop, then the procedure is somewhat different from the above. The current is suddenly increased in small steps and the galvanometer throw noted at each increase. Obviously for any given current the throw from which the corresponding flux density is calculated is the *sum* of all the previous throws, since the throw is only proportional to the change in flux and not to the total flux. When

the maximum current is reached, the current is reduced in small steps and at last brought to zero. Then K_1 is reversed and the current increased in steps again until the same maximum value has been attained. This procedure is followed until the complete loop has been traced out. Reversed galvanometer throws will obviously be experienced when B is being reduced. These are reckoned negative, and the total throw from which B is calculated at any step is the algebraic sum of all the throws up to that point.

12. **Magnetic Potentiometer Method.** The great disadvantage of a ring specimen from the practical point of view is that the specimen

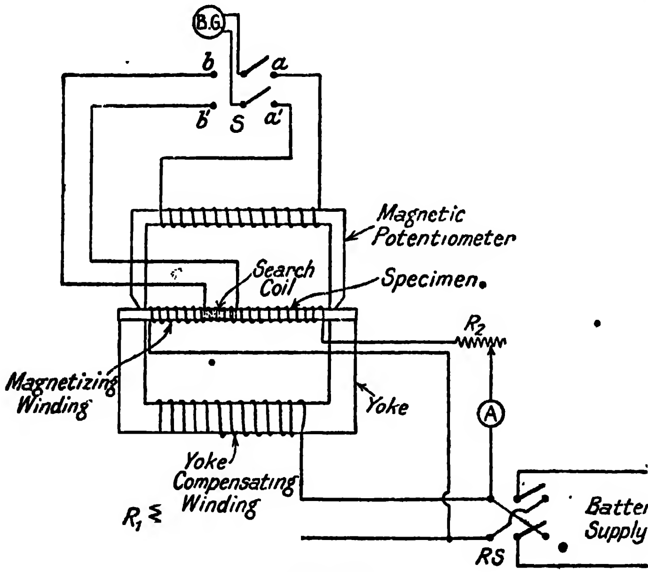


FIG. 16

itself is expensive, whether solid or whether made up of laminations, and, in addition, a separate magnetizing winding, put on by hand, is required for each one.

Practical methods involving samples, as distinct from materials in bulk, therefore require a more convenient shape, as, for example, a short rectangular bar.* In the Illiovici Permeameter the specimen is used to close the magnetic circuit of a massive yoke, as shown in Fig. 16. Both specimen and yoke carry separate windings, the function of the yoke winding being to provide the ampere-turns

* For complete information, see *Properties and Testing of Magnetic Materials*, by Spooner; *Applied Magnetism*, by Wall, or *Electrical Measurement and Measuring Instruments*, by Golding.

necessary for the yoke and to two gaps. The specimen is, of course, clamped down to the yoke but even then the gap ampere-turns may not be negligible, and furthermore will be different for every test. Compensation being thus provided for yoke and gaps, it is known that ampere-turns of the winding on the specimen will be required by the specimen only. The interest of the method lies in the method of insuring that the yoke winding carries just sufficient current to compensate for the yoke and gaps. The appliance used is the magnetic potentiometer, which is simply a thin strip or rod of flexible insulating material wound uniformly with a helix of thin wire. This coil is connected to a ballistic galvanometer. An E.M.F. will be induced in this coil either by a change in the magnetizing current producing the magnetic field in which the coil is situated, or by a rapid movement of one end of the helix from one point in the field to another. It can be shown that the galvanometer throw is proportional to the change in magnetic potential, this potential being given by $\int Hdl$*.

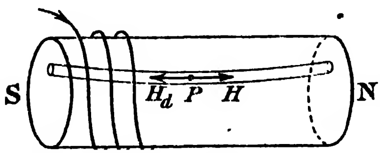


FIG. 17. DEMAGNETIZING FORCE

In the above test the magnetic potentiometer is applied to the ends of the specimen, as shown in the figure, and is kept in that position. In making the test the current in the ammeter A is adjusted to give the desired value of H and the throw-over switch S placed on contacts aa'. The current in the yoke compensating coil is now adjusted by means of R, until, on reversing both currents by means of RS, no galvanometer deflection is noted. When this adjustment is made, the magnetic potential between the two ends of the search coil will be zero, and therefore the M.M.F. of the coil on the specimen will be required to overcome the reluctance of the specimen only. The value of H in the specimen is now known to be

$$H = \frac{4\pi NI}{10l}$$

The switch S is now changed over to contacts bb', thus connecting the search coil to the galvanometer, and RS reversed. The galvanometer throw is noted, and the value of B calculated as for the previously described ballistic test.

13. Note on Permanent Magnets. Imagine a magnetized bar NS, Fig. 17, with a very small cavity in the direction of the lines of induction. Then, since no lines will emerge from the iron into the

* The theory of the magnetic potentiometer is given in Golding, *loc. cit.*, p. 331.

cavity, the walls of this cavity will not exhibit any form of polarity. The flux through the cavity will therefore be that corresponding to the magnetizing force H set up by the current, and not to the total flux density B, and, consequently, if a unit north pole P is placed within the cavity, it will be acted on by a force of H dynes. But it is clear that the unit pole will be repelled by the N pole of the magnetized bar and attracted by the S pole, these two together setting up a force H_d which is in direct opposition to H. In other words, the fact of the bar having polarized ends is responsible for the setting up of a self-demagnetizing effect. The magnitude of the demagnetizing force for round bars is given by the expression

$$H_d = \frac{K}{4\pi} . B$$

where B is the flux density and K is a constant depending on the ratio of length to diameter. When this ratio is large, as for long thin rods, K is small; for example, if the ratio is 200, the value of K is about 0·001. On the other hand, when the ratio is small as for short, thick bars, then K is much greater. Thus where the ratio is 10, K is equal to 0·2. This means that with short electro-magnets a very large proportion of the ampere-turns is required to overcome the demagnetizing effect.

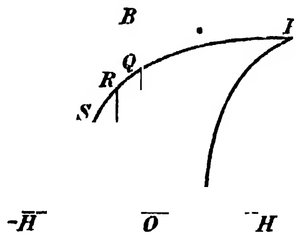


FIG. 18. DEMAGNETIZING FORCE

The significance of this in connection with permanent magnets is as follows: the demagnetizing effect in a closed ring is zero, but, in order that the magnetic flux may be available, all permanent magnets must have a gap, thus requiring two polarized ends, and so giving rise to self-demagnetization. Consider the portion of the hysteresis loop shown in Fig. 18, and let the magnetization be taken to the point P. On removal of the magnetizing force the flux density will decrease to OQ in the case of a closed ring, but, in a practical form of permanent magnet, the self-demagnetizing effect will cause a further reduction to, say, R. If the magnet is jarred, there will be a further reduction in flux density, and, without any special ageing process, most of the magnetization may be ultimately lost. To prevent this, it is usual to age permanent magnets by deliberately demagnetizing to some other point S, the corresponding negative value of H being now so large that there is little chance of further demagnetization.

Modern permanent magnets are made from an alloy of cobalt and iron, this material having a high coercive force, and retaining its magnetization over very long periods. The usual flux density is of

the order of 5,000, unless the gap is exceedingly small, when the small value of H_d will permit a higher degree of initial magnetization.

EXAMPLES ON CHAPTER I.

(1) A ring-shaped electro-magnet has an air gap 6 mm. long and 20 sq. cm. in area, the mean length of the core being 50 cm., and its cross section 10 sq. cm. Calculate the ampere-turns required to produce a field strength $H = 5{,}000$ in the gap. (Assume the permeability of the iron as 1,800.) (C. and G. II, 1908.)

Ans.—2,610.

(2) An iron ring of 10 in. mean diameter is made from ½ in. round iron, of which the particulars are

B	10,000	12,000	15,000
μ	2,500	1,800	600

If it has a saw cut $\frac{1}{16}$ in. wide and its leakage factor is 1·25, calculate how many ampere-turns are required to produce a flux density in the gap of 11,000 lines per sq. cm.

Ans.—2,195.

(3) A circular lifting magnet of the type shown in Fig. 4 has a diameter of 50 in., the inner circular pole is 16 in. in diameter, and the outer annular pole has the same area as the inner one. The mean length of the magnetic path through the magnet is 60 in., and the permeability of the steel can be taken as 1,000. If it is lifting steel plates of negligible reluctance, and the plates are separated ¼ in. from the magnet poles, find the ampere-turns necessary to produce a flux density in the poles of 10,000 lines per sq. cm.

Ans.—11,350.

(4) Calculate the force of attraction between the magnet and plates in the above example.

Ans.—23,200 lb.

(5) Find the force of attraction in pounds weight between the square-faced ends of two rods of iron placed in a solenoid. The area of the iron rods is 6 sq. cm., the permeability of the iron is 1,000, and the magnetic force produced in the iron by the solenoid is 14 c.g.s. units. (London Univ., 1907.)

Ans.—105.

(6) A smooth core armature, working in a four-pole field magnet, has a gap (from iron to iron) of 0·5 in. The area of surface of each pole is 1 sq. ft. The flux from each pole is 7 megalines. Find (*a*) the mechanical force with which the pole attracts the armature; (*b*) the amount of energy expressed in joules that is stored in the four gaps. (N.B.—746 joules = 550 ft.-lb. at London; 1 ft. = 30·48 cm.; 1 lb. = 453·6 gramme.) (C. and G. II, 1909.)

Ans.—(*a*) 4,710 lb. wt.; (*b*) 1,065 joules.

(7) The armature core of a dynamo is a cylinder of length 12 in. and internal and external diameters 12 and 7 in. respectively. It rotates at 1,000 r.p.m.

in a four-pole field, and from each pole a flux of $1{\cdot}55 \times 10^6$ emerges. If the Steinmetz coefficient for the sheet iron used in the core is 0·0025, and 80% of the core is iron, find the hysteresis loss in watts.

Ans.—101.

(8) The ascending and descending values of B and H for a half cycle are as follows—

Ascending—

H	1·9	2	3	4	4·5
B	0	2,000	5,800	7,000	7,300

Descending—

H	2·5	1	0	− 1	− 1·9
B	7,000	6,100	5,300	3,800	0

Plot the curve and calculate the energy dissipated in hysteresis in ergs per cc. per cycle. Calculate also the hysteresis coefficient.

Ans.—(*a*) 4,500 ergs; (*b*) 0·0029.

(9) Define the terms "Magneto-motive Force," "Magnetic Flux," and "Magnetic Reluctance," and prove the relation which holds between these quantities for a magnetic circuit.

Estimate the number of ampere-turns necessary to produce a flux of 100,000 lines round an iron ring of 6 sq. cm. cross section and 20 cm. mean diameter, having an air gap 2 mm. wide across it. The permeability of the iron may be taken to be 1,200. Neglect the leakage flux outside the 2 mm. air gap. (London Univ., 1922.)

Ans.—3370.

(10) Define hysteresis. Prove that when any material is subjected to cyclic changes of magnetism, a loss of energy is involved proportional to the area of the hysteresis loop. Calculate the loss of energy caused by hysteresis in 1 hr. in 50 kg. of iron when subjected to cyclic magnetic changes. The frequency is 25, the area of the hysteresis loop represents 2,400 ergs per c.cm., the density of the iron is 7·8. (London Univ., 1916.)

Ans.—138,000 joules.

CHAPTER II

ELECTROMOTIVE FORCE

1. **Production of an Electromotive Force.** An electromotive force can be produced in the following ways: (*a*) by chemical action as in a voltaic cell; (*b*) by the heating of a thermo-junction; (*c*) by electro-magnetic action. The third method is by far the most important, and it has two subdivisions. First, an E.M.F. can be produced by the motion of a conductor in a magnetic field; such an E.M.F. can be called a "dynamically induced" E.M.F. Second, an E.M.F. is produced when the flux which threads, or links with a coil, changes. In this case there is no motion of the coil relative to the magnetic field, and therefore the E.M.F. so produced is "statically induced."

2. **Dynamically Induced E.M.F.** In Fig. 19 three conductors, A, B, and C, are shown in cross section in a magnetic field, and the arrows attached to them indicate their directions of motion. Conductor A is moving in the direction of the lines of force and therefore, since it does not *cut* any of them, no E.M.F. is induced in it. Conductor B is moving in a direction perpendicular to its own length and perpendicular to the lines of force, and consequently, since it cuts the lines of force, it has an E.M.F. induced in it, of magnitude

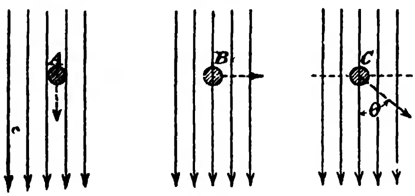


FIG. 19

DYNAMICALLY INDUCED E.M.F.

$$E = Hlv \text{ c.g.s. units}$$
$$= Hlv \times 10^{-8} \text{ volts}$$

where H is the field strength in c.g.s. units, l the length of conductor in cm., and v its velocity in cm. per second. The direction of this induced E.M.F. is given by Fleming's Right-hand Rule (Fig. 20). Hold the thumb and first finger of the right hand at right angles and bend the second finger so as to point at right angles to the plane of these two. Then if the first finger is pointed in the direction of the field, and the thumb in the direction of motion, the second finger will point in the direction of the induced E.M.F.

Conductor C in Fig. 19 is moving at an angle θ to the direction of the field. In this case the magnitude of the induced E.M.F. is proportional to the component of the velocity perpendicular to the

direction of the field. The modified form of the E.M.F. equation is, therefore,

$$E = Hlv \sin\theta \times 10^{-8} \text{ volts.}$$

Example. A conductor 12 in. long on the periphery of an armature of diameter 18 in. rotates at 1,000 r.p.m. If the field strength under the poles is 6,000 lines per sq. cm., find the E.M.F. induced in the conductor.

The lines of force in the air gap between poles and armature are radial, as shown in Fig. 21, and therefore the conductor cuts the

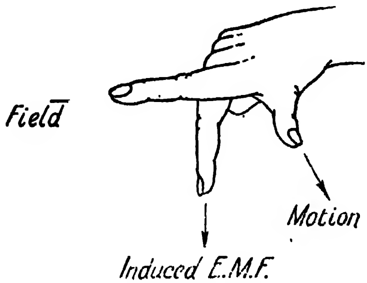


FIG. 20
FLEMING'S HAND RULE

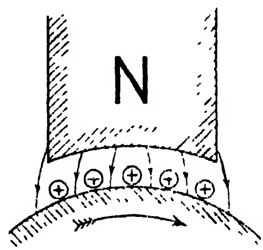


FIG. 21

lines of force at right angles.

$$H = 6{,}000 \text{ c.g.s. units}; \quad l = 12 \text{ in.} = 30{\cdot}5 \text{ cm.}$$

$$v = \pi \times \text{diameter} \times \text{rev. per sec.}$$

$$= \pi \times 18 \times 2{\cdot}54 \times \frac{1{,}000}{60} = 2{,}390 \text{ cm. per sec.}$$

$$\therefore E = 6{,}000 \times 30{\cdot}5 \times 2{,}390 \times 10^{-8} = 4{\cdot}4 \text{ volts.}$$

3. **Statically Induced E.M.F.** A statically induced E.M.F. may be (*a*) "mutually," (*b*) "self" induced. Consider first of all its production by the process of mutual induction. In Fig. 22 there are two coils, *A* and *B*, placed close together. *A* has a battery and switch connected to it, and *B* is connected to a galvanometer. If *A* carries either zero current or a finite steady current, there will be no deflection of the galvanometer, thus showing that there is no E.M.F. in the coil *B*. If, when the current in *A* is zero, the switch is suddenly closed, there will be a momentary deflection of the galvanometer, but not a permanent deflection so long as the switch is kept closed and the current in *A* is not varied. If now the switch is suddenly opened there will be another momentary deflection, this time in the reverse direction, and immediately afterwards the galvanometer needle will return to the zero position. These experiments show that an E.M.F. is induced in *B* whenever the

current in A is *changing*, but not when it is steady. Also the direction of the E.M.F. induced by an increase in current is opposite to that induced by a decrease in current. These phenomena are explained by the following laws—

1. Whenever the number of lines of force linking with a circuit changes, an E.M.F. is induced in the circuit proportional to the rate of change of flux.

This is Faraday's law of electromagnetic induction.

2. The direction of the induced E.M.F. is such that the current set up by it tends to stop the motion or change producing it.

This is known as Lenz's Law, and it follows from the fact that in order to set up an induced current some energy must be expended.

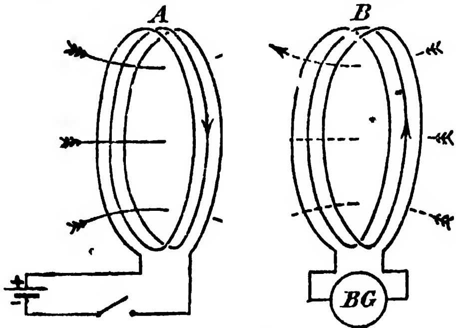


FIG. 22
MUTUALLY INDUCED E.M.F.

4. **Self Induced E.M.F.** In the case of a mutually induced E.M.F., the E.M.F. is set up by a change in flux through a coil when the flux is produced by a neighbouring current. Obviously an E.M.F. will be produced whenever the flux changes, no matter how the flux may be produced. Thus if a single coil carries a current it will produce a magnetic flux, and if the current changes the flux will change. The change in flux will induce an E.M.F. in the coil, which in this case is called a self induced E.M.F., because it is set up by a change in its own current instead of by a change in a neighbouring current. If the current increases, the self induced E.M.F. will oppose the current; whereas if the current decreases, it will act in the same direction as the current.

5. **Coefficients of Self and Mutual Induction.** Consider a solenoid of N turns, length l cm., and cross section a sq. cm. If it is carrying current, it will produce a flux; and if the current changes, the flux will change. We then have

$$\text{Self induced E.M.F.} = -(\text{Rate of change of flux}) \times N \text{ c.g.s. units}$$
$$= -(\text{Rate of change of flux}) \times N \times 10^{-8} \text{ volts}$$

$$\text{Now} \quad \text{Flux} = (\text{Flux set up by 1 ampere}) \times I$$

the quantity in the brackets being a constant (if the permeability of the core is constant), which can be determined from the data of the circuit. Therefore we can write

$$\begin{pmatrix}\text{Rate of change}\\ \text{of flux}\end{pmatrix} = \begin{pmatrix}\text{Flux set up}\\ \text{by 1 ampere}\end{pmatrix} \times \begin{matrix}\text{rate of change}\\ \text{of current}\end{matrix}$$

$$\therefore \begin{pmatrix}\text{Self induced}\\ \text{E.M.F.}\end{pmatrix} = -\left\{\begin{pmatrix}\text{Flux set up}\\ \text{by 1 ampere}\end{pmatrix} \times N \times 10^{-8}\right\} \times \begin{matrix}\text{rate of change}\\ \text{of current}\end{matrix}$$

The expression in the large brackets is a constant for any given circuit and it is called the "coefficient of self induction." It is given in practical units, henrys, since the factor 10^{-8} is included. It is represented by the symbol L. Hence we have

$$(\text{Self induced E.M.F.}) = -L \times \text{rate of change of current}$$

$$\text{or} \quad -L\frac{dI}{dt}$$

the minus sign being used because the E.M.F. is opposed to the change of current; in other words, it is the mathematical equivalent of Lenz's Law.

The law can be stated as follows. If an E.M.F. is induced in a circuit through a change of current in the circuit, its direction is such as to oppose the change of current. Thus, if the change is an increase in current, the E.M.F. will act in the opposite direction to the current; if the change is a decrease in current, the induced E.M.F. will act in the same direction as the current.

It is very easy to determine L for a simple circuit such as a solenoid. We have, for any current I,

$$\text{Flux} = \frac{1{\cdot}26\,NIa\mu}{l}$$

if the core is magnetic and of permeability μ. Hence, flux per ampere

$$= \frac{1{\cdot}26\,Na\mu}{l}$$

$$\therefore\ L = (\text{Flux per ampere}) \times N \times 10^{-8}$$

$$= \frac{1{\cdot}26\,N^2a\mu}{l} \times 10^{-8} \text{ henrys.}$$

The "coefficient of mutual induction" of a coil B relative to a coil A is obtained as follows—

Let A and B have N_1 and N_2 turns respectively,

$$\begin{pmatrix}\text{Mutually induced}\\ \text{E.M.F. in } B\end{pmatrix} = \left\{\begin{pmatrix}\text{Flux through } B\\ \text{due to 1 ampere in } A\end{pmatrix} \times N_2 \times 10^{-8}\right\}$$

$$\times\ (\text{rate of change of current in } A)$$

The expression in the large brackets is the coefficient of mutual

induction M, of B with respect to A. The flux produced by A, and therefore that portion of it which links with B, is proportional to N_1. We thus see that the coefficient of mutual induction of two coils is proportional to the product of their numbers of turns, whereas the coefficient of self induction of a single coil is proportional to the square of its turns.

6. **Rise and Decay of Current in an Inductive Circuit.** If a coil possessing no self-induction, that is, a coil which does not produce a magnetic field when currient is passed through it, has a P.D. of E volts applied to its terminals, the current produced reaches the Ohm's Law value of E/R instantaneously. If the coil possesses self-induction, the current theoretically takes infinite time to reach this value. With a non-inductive coil the applied P.D. has to overcome the Ohmic drop IR only, but in the case of an inductive coil it has, in addition, to overcome the back E.M.F. set up by self-induction, this back E.M.F. only becoming zero when the current is steady. Now the self induced E.M.F. is $-L\,di/dt$, where i is the instantaneous value of the current, and the applied P.D. therefore has to possess a component equal and opposite to this, as well as a component equal to the Ohmic drop iR. Hence, when the current is increasing we have

$$E = Ri + L\,di/dt$$

If we multiply both sides by idt we have an energy equation

$$Eidt = Ri^2dt + Lidt \times \frac{di}{dt}$$

$Eidt$ is the total energy supplied to the coil in time dt, Ri^2dt is the energy converted into heat due to the Ohmic resistance of the coil, and $Lidt \times di/dt$ is the energy required to establish the magnetic field whose presence is the cause of the coil's self-induction. The solution of the original equation is

$$i = \frac{E}{R}\left(1 - \varepsilon^{\frac{R}{L}\cdot t}\right)$$
$$= I\left(1 - \varepsilon^{\frac{R}{L}\cdot t}\right)$$

where I is the final value of the current.

Example. A constant P.D. of 1 volt is applied to a coil of resistance 1 ohm and inductance 1 henry; plot the curve of current against time.

The final value of the current is $I = E/R = 1$ amp., hence, for the current at any instant we have

$$i = I\left(1 - \varepsilon^{-\frac{R}{L}\cdot t}\right)$$
$$= 1 - \varepsilon^{-t}$$

when $t = 0{\cdot}1$ sec., $i = 1 - \varepsilon^{-{\cdot}1} = 1 - 2{\cdot}7183^{-{\cdot}1} = {\cdot}095$

when $t = 0{\cdot}2$ sec., $i = 1 - \varepsilon^{-{\cdot}2} = 1 - 2{\cdot}7183^{-{\cdot}2} = {\cdot}181$, and so on.

The curve is as shown in Fig. 23.

After an interval of time of L/R seconds the current reaches the value $i = I(1 - \varepsilon^{-1}) = 0{\cdot}6321I$.

This is a definite fraction of I, and the ratio L/R is therefore called the "time-constant" of the coil. The time-constant can thus be defined as the time required for the current to reach 0·6321 of its final value.

From the energy equation we saw that the energy imparted to the magnetic field in time dt was

$$Lidt \times di/dt$$

Hence, when the current has attained its final value I, the energy of the field will be

$$\int_0^I Lidi = \tfrac{1}{2}LI^2$$

Now L is the flux per ampere $\times N \times 10^{-8}$, and therefore the energy of the field is

$$\tfrac{1}{2} \times \text{current} \times \text{total flux} \times N \times 10^{-8}$$

Suppose there are two coils A and B, whose coefficients of self-induction are L_A and L_B respectively, and whose coefficient of mutual induction is M. If they are traversed by currents of I_A and I_B respectively, then

Flux through A

$$= \left(\frac{L_A I_A}{N} \times 10^8 + \frac{MI_B}{N} \times 10^8\right)$$

and therefore the energy of the field due to coil A is

$$\tfrac{1}{2}I_A \times \left(\frac{L_A I_A}{N} \times 10^8 + \frac{MI_B}{N} \times 10^8\right) \times N \times 10^{-8}$$

$$= \tfrac{1}{2}L_A I_A^2 + \tfrac{1}{2}MI_A I_B$$

Similarly, the energy of the field due to coil B is

$$\tfrac{1}{2}L_B I_B^2 + \tfrac{1}{2}MI_A I_B$$

The total energy is, therefore,

$$\tfrac{1}{2}L_A I_A^2 + MI_A I_B + \tfrac{1}{2}L_B I_B^2$$

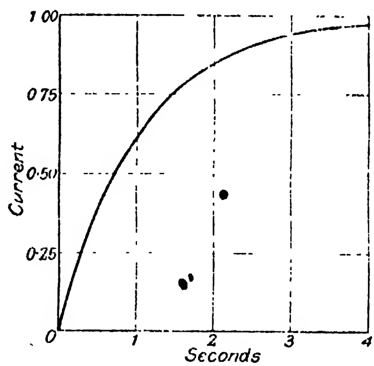


FIG. 23

We have now to consider the decay of the current when the applied P.D. is removed. Suppose, for instance, that the coil is suddenly short-circuited and the source of P.D. at the same time disconnected. Then, since the current now decreases, its rate of change di/dt is negative, and the E.M.F. equation becomes

$$0 = Ri + L\,di/dt$$

the solution of which is

$$i = \frac{E}{R} \times \varepsilon^{-\frac{R}{L}t} = I \cdot \varepsilon^{-\frac{R}{L}t}$$

This shows that the curve dies away exponentially, as shown in Fig. 24. The curves of rise and decay are complementary, for if they are drawn with the same origin as in Fig. 24 the sum of the ordinates at any instant is equal to the Ohm's Law value of the current I.

If both sides of the E.M.F. equation are multiplied by idt, another energy equation is obtained, namely,

$$O = Ri^2dt + Li\frac{di}{dt}.dt$$

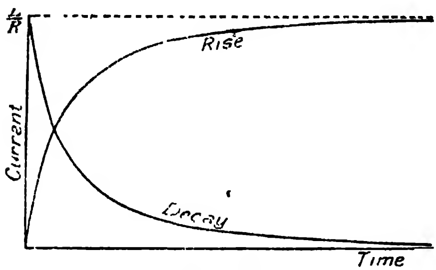


FIG. 24

This shows that, when the current is decaying, the whole of the energy for the production of heat in the winding is drawn from the stored energy of the magnetic field, and eventually, when the current is zero, the field also is of zero strength.

Example. A shunt dynamo gives an air gap flux which is practically proportional to the exciting current. When this current is 2 amperes, an air gap flux of 2,000,000 lines is produced. If there are 1,000 shunt turns of resistance 100 ohms, and they are carrying a current of 2 amperes, determine the equation of decay of the current if the shunt winding is suddenly short circuited.

$$L = \text{Flux per ampere} \times N \times 10^{-8} \text{ henrys}$$

$$= \frac{2{,}000{,}000}{2} \times 1{,}000 \times 10^{-8}$$

$$= 10 \text{ henrys}$$

$$R = 100$$

Initial current $I = 2$ amperes

$$i = I \times \varepsilon^{-\frac{R}{L}t}$$

$$= 2\varepsilon^{-10t}$$

7. **In the above discussion of the effect of self-induction we have assumed that the magnetization characteristic of the circuit, namely,**

the graph of flux against current, is a straight line, or, what amounts to the same thing, the permeability of the magnetic circuit is constant. If the path of the lines of flux is through iron, this is no longer the case, particularly if the iron is taken up to saturation point, the result being that the coefficient of self-induction is no longer a constant, but is a function of the current. To determine the self-induction it is, therefore, necessary to calculate the flux per ampere for a series of values of the current, the flux per ampere being given by the gradient to the curve of flux against current. A numerical example will make this clear.

A ring of iron whose magnetic characteristics are given by the following figures—

B	2,500	5,000	7,500	10,000
μ	1,250	1,200	1,000	800

has a mean circumference of 300 cm. and cross-section of 100 sq. cm. It has a magnetizing coil of 250 turns. Plot a curve of self-induction against magnetizing current.

The four values of H ($= B/\mu$) are equal to 2, 4·17, 7·5, and 12·5 respectively; hence, the ampere-turns per cm. length ($= \cdot 8H$) are equal to 1·6, 3·34, 6·0, and 10·0 respectively.

The total AT ($= at \times l$) are, therefore, 480, 1,002, 1,800, and 3,000 respectively. Dividing these values of AT by the number of turns we obtain the currents, whose values are, therefore, 1·92, 4·01, 7·2, and 12·0 respectively.

The total fluxes ($\Phi = BA$) are 250,000, 500,000, 750,000 and 1,000,000. The curve of Φ against I can now be plotted, and is given in Fig. 25. Tangents to the curve are drawn at the point corresponding to the above values of the current, and their gradients, in lines of force per ampere, are 121,800, 91,400, 63,500, and 42,600 respectively.

We now apply the general expression for the self-induction, and the corresponding values are—

$$
\begin{aligned}
&&&&\text{henry}\\
\text{when } I &= 1{\cdot}92; & L &= 1{\cdot}218 \times 10^5 \times 2{\cdot}5 \times 10^2 \times 10^{-8} &= 0{\cdot}305\\
\text{when } I &= 4{\cdot}01; & L &= 9{\cdot}14 \times 10^4 \times 2{\cdot}5 \times 10^2 \times 10^{-8} &= 0{\cdot}229\\
\text{when } I &= 7{\cdot}2; & L &= 6{\cdot}35 \times 10^4 \times 2{\cdot}5 \times 10^2 \times 10^{-8} &= 0{\cdot}159\\
\text{when } I &= 12{\cdot}0; & L &= 4{\cdot}26 \times 10^4 \times 2{\cdot}5 \times 10^2 \times 10^{-8} &= 0{\cdot}107
\end{aligned}
$$

If the current in such a circuit varies by a small amount, from, say, I to $(I + \delta I)$, then the self-induced E.M.F. will be calculated from the value of L corresponding to the mean current, namely, $(I + \frac{1}{2} . \delta I)$. On the other hand, if a current I in such a circuit is suddenly switched off, then the *mean* value of the self-induced E.M.F. will be given by the value of L corresponding to the current I.

The growth of the current in such a circuit is obviously not of the pure exponential form corresponding to a constant value for L.

The most convenient method of determining the curve is a step-by-step method based on the differential equation of the circuit. Whatever the shape of the magnetization curve, the E.M.F. equation is

$$E = Ri + N.\frac{d\Phi}{dt}.10^{-8}$$

If we use small differences, then the equation becomes

$$E = Ri + N.\frac{\delta\phi}{\delta t}.10^{-8}$$

$$\therefore\ \delta\phi = \frac{E - Ri}{N}.10^{8}.\delta t$$

Suppose, in the above example, that $R = 1{\cdot}0$ ohm, that $E = 20$,

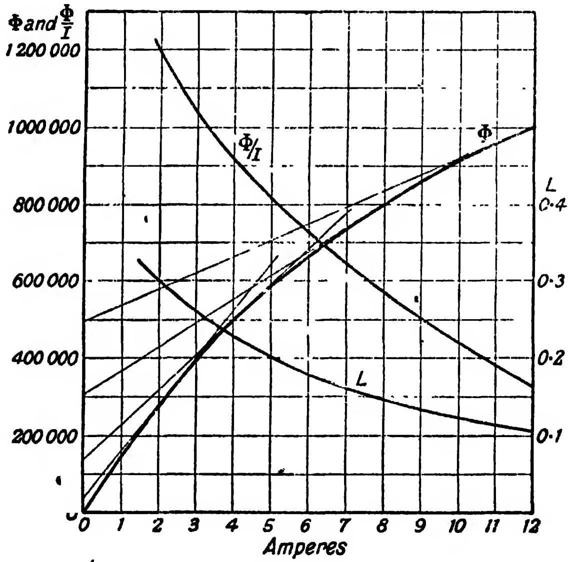


FIG. 25

and let δt be taken as $\cdot01$ of a second; then, since $N = 250$, we have

$$\delta\phi = \frac{20 - i}{2{\cdot}5 \times 10^2}.10^8.\delta t$$

$$= 4 \times 10^5\,(20 - i)\delta t$$

$$= 4 \times 10^3\,(20 - i)$$

when $t = 0$, $i = 0$, and $\phi = \Sigma\delta\phi = 0$;
when $t = \cdot01$, $\delta\phi = 8 \times 10^4$, $\phi = \Sigma\delta\phi = 8 \times 10^4$, and $i = \cdot58$,

as read off from the magnetization curve;
when $t = \cdot 02$, $\delta\phi = 7{\cdot}77 \times 10^4$, $\delta = (8 + 7{\cdot}77) \times 10^4 = 15{\cdot}77 \times 10^4$
and $i = 1{\cdot}1$;
when $t = \cdot 03$, $\delta\phi = 7{\cdot}56 \times 10^4$, $\phi = (15{\cdot}77 + 7{\cdot}56) \times 10^4$
$= 23{\cdot}33 \times 10^4$
and $i = 1{\cdot}7$, and so on.

In this way the curve of i against t can be determined as far as desired, and obviously the accuracy of the determination depends solely on the smallness of the intervals δt. The curve of Fig. 26 gives the curve of growth of current as so determined up to a time of 0·15 sec. after switching on. It will be seen that this curve is, up to $t = \cdot 13$ sec., concave upwards instead of concave downwards, as in the case of a circuit for which L is constant. The reason for this

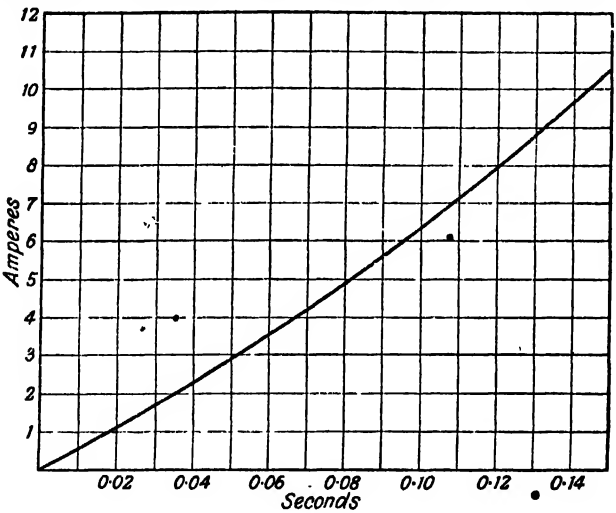


FIG. 26

is that the value of L is continually falling, as is indicated by the curve of L against I, and the growth of current curve will only assume the normal form when L has become sensibly constant.

8. **Inductance of a Pair of Parallel Conductors.** The lines of force due to either conductor are concentric circles round the conductor and also concentric circles inside the conductor (Fig. 27). Consider first of all the flux inside the conductor. The current inside a line of force of radius x is $I \times \frac{x^2}{r^2}$, and therefore the field strength at a distance x from the centre

$$= \frac{2 \times \text{current}}{\text{distance}} = 2I\frac{x^2}{r^2} \times \frac{1}{x} = \frac{2Ix}{r^2}, \ I \text{ being in c.g.s. units}$$

Hence, the flux through a cylindrical shell of radial thickness dx and axial length 1 cm. is

$$2\frac{Ix}{r^2} \times 1 \times dx = \frac{2Ixdx}{r^2}$$

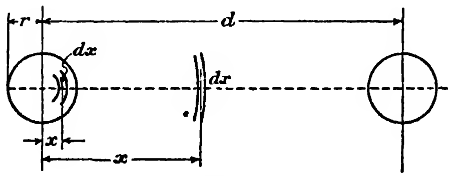


FIG. 27

But the flux links with only x^2/r^2 of the conductor, so that the linkage of the shell is

$$\frac{2I\,xdx}{r^2} \times \frac{x^2}{r^2} = \frac{2I\,x^3dx}{r^4}$$

$\therefore$ Total linkage inside the conductor

$$= \frac{2I}{r^4}\int_0^r x^3dx = \frac{I}{2}$$

Now consider the flux between the two conductors. At any distance x the field strength is again $2I/x$, and therefore the flux through a cylindrical shell of thickness dx and axial length 1 cm. is $2I\,dx/x$. Hence, linkage

$$= 2I\int_r^d \frac{dx}{x} = 2I\log_e\frac{d}{r}$$

Hence, the total linkage per conductor, reckoning the fluxes inside and outside the conductor, is

$$\left(\frac{I}{2} + 2I\log_e\frac{d}{r}\right)$$

The total linkage for both conductors is, therefore,

$$\left(I + 4I\log_e\frac{d}{r}\right)$$

This is expressed in c.g.s. units. If I is in amperes it must be divided by 10. Hence, the linkage per ampere is

$$\left(1 + 4\log_e\frac{d}{r}\right) \div 10.$$

This will be the E.M.F. in c.g.s. units induced in each cm. of the loop formed by the two conductors, if the current in the loop changes

at the rate of 1 amp. per sec. Hence, for the inductance per cm. of the loop in practical units, we have

$$L = \left(1 + 4 \log_\varepsilon \frac{d}{r}\right) \times 10^{-9} \text{ henrys per cm.}$$

$$= 4 \log_\varepsilon \frac{d}{r} \times 10^{-9} \text{ approximately.}$$

If the log is reduced to the base 10 and the inductance expressed in henrys per mile, we have, finally,

$$L = 14{\cdot}8 \times 10^{-4} \log_{10} \frac{d}{r}$$

EXAMPLES ON CHAPTER II.

(1) A conductor 12 in. long rotates about one end at 1,000 r.p.m. in a plane perpendicular to a magnetic field of strength 5,000 lines per sq. cm. Find the E.M.F. induced in it.

Ans.—2·42 volts.

(2) Define the coefficient of inductance of a magnetic circuit and show that it is proportional to the square of the magnetizing turns and inversely to the reluctance. Show how to find, by observation of the curve of rise of current when a steady voltage is applied to the inductance, the change of reluctance due to saturation of the iron. (London Univ., 1911.)

(3) A coil of resistance 10 ohms and inductance 1 henry has a current which increases uniformly at the rate of 10,000 amperes per second. Find the value of the necessary applied P.D. (*a*) when the current is 10 amperes, (*b*) when it is 50 amperes.

Ans.—(*a*) 10,100; (*b*) 10,500 volts.

(4) Plot a curve giving the value at each instant of an electric current which varies in the following way—At time *o* it is 4 amperes. It increases at the rate of 10 amperes per second for 2 seconds; it then remains constant for 2 seconds; it then decreases at the rate of 4 amperes per second for 6 seconds; it then follows the law $12 \sin \frac{2\pi}{20} . t$. Plot curves showing the voltage at the terminals of a resistance of 3 ohms and at the terminals of an inductance of 4 henrys respectively when this current is passed through them. (London Univ., 1921.)

(5) A certain choke coil has 1,500 turns. When 4 amperes are passed through it, the total magnetic flux threading the coil is 6,000,000 lines. The resistance of the coil is 20 ohms. Find an expression for the current in the coil immediately after it is switched on to a D.C. supply yielding 100 volts. Assume that the self induction of the coil is constant during the rise of current. (London Univ., 1921.)

Ans.—$i = 5\,(1 - \varepsilon^{-\cdot 89t})$.

(6) Define the unit of inductance. Obtain an expression by which the inductance of an anchor ring of *D* cm. mean diameter and *A* sq. cm. cross section, wound with *n* turns of wire may be calculated approximately, if the permeability (μ) of the iron is assumed constant. A field magnet coil wound with 1,500 turns of wire produces a flux of 2·8 megalines when carrying a a current of 2 amperes; estimate the inductance of the coil in henrys. (London Univ., 1923.)

Ans.—21 henrys.

(7) What methods may be used for limiting the voltage rise of an inductive coil when it is disconnected from a D.C. supply? Estimate the discharge resistance necessary to prevent the voltage rise exceeding 300 volts when a coil of large inductance having a resistance of 100 ohms is disconnected from a supply at 200 volts. (London Univ., 1922.)

Ans.—50 ohms.

(8) Prove, for any simple case, that the mechanical work done by, or on, a coil traversed by a steady current I due to a movement from a position in which the total magnetic flux through the coil is N_1, to another position, in which the total flux is N_2, is $I(N_1 - N_2)$. State precisely the units in which the different quantities are measured, and what determines whether the work done is positive or negative.

The coefficient of mutual induction M, in henrys, between two coils, one of which is fixed, and the other movable about an axis, is given by $M = a + b\theta$, where a and b are known constants, and θ is the deflection in degrees measured from some zero position of the moving coil. Calculate the torque between the two coils for any deflection θ, when the coils are traversed by two currents I_1 and I_2 amperes. State the unit of torque in which the result is expressed. (London Univ., 1915.)

(9) A transmission line consists of a pair of $\frac{1}{4}$ in. conductors spaced 8 ft. apart. Calculate the inductance of the loop formed by joining the two conductors at one end.

Ans.—4·25 millihenrys per mile.

CHAPTER III

SYSTEMS OF CONDUCTORS

1. **Series and Parallel Connections.** If a number of resistances R_1, R_2, R_3, etc., are connected in series, then the total resistance is given by

$$R = R_1 + R_2 + R_3 + \ldots \text{ ohms.}$$

If an E.M.F. E is applied to the whole circuit, the current is given by

$$I = E/R.$$

and this current flows through each of the resistances. The voltage divides itself between the various resistances in such a way that the "drop" along any one section is proportional to the resistance of that section. Thus denoting the drops by V_1, V_2, V_3, etc., we have

$$V_1 = \frac{R_1}{R} . E ; \quad V_2 = \frac{R_2}{R} . E ; \quad V_3 = \frac{R_3}{R} . E ; \text{ etc.}$$

If the various resistances are all connected in parallel, then for the total resistance we now have

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \ldots$$

or, denoting the reciprocal of resistance by the conductance G, the practical unit being the *mho,* we have

$$G = G_1 + G_2 + G_3 + \ldots \text{ mhos.}$$

The total current produced by an applied voltage E is now

$$I = \frac{E}{R} = EG$$

while the portions of this current carried by the various branches are given by

$$I_1 = \frac{G_1}{G} . I ; \quad I_2 = \frac{G_2}{G} . I ; \quad I_3 = \frac{G_3}{G} . I ; \quad \text{etc.}$$

2. **Kirchhoff's Laws.** Circuits which do not come within the category of simple series or parallel circuits, such as the above, can be solved by means of Kirchhoff's Laws. These are as follows—

First Law. The algebraic sum of the currents at any junction of conductors is zero. In other words, the sum of the currents flowing towards a junction is equal to the sum of the currents flowing away from the junction. Thus in Fig. 28 the currents I_1 and I_2 are flowing

towards the junction, while I_2, I_4, and I_5 are flowing away from the junction. From this first law we therefore have for this particular junction

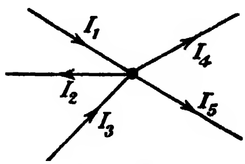


FIG. 28

$$I_1 + I_3 = I_2 + I_4 + I_5.$$

Second Law. In any closed circuit the algebraic sum of the potential drops in the various parts of the circuit is equal to the electro motive force acting round the circuit. To illustrate the law three possible cases are illustrated in Fig. 29.

(*a*) In Fig. 29 (*a*) the closed circuit does not contain a battery or other source of E.M.F., and therefore the algebraic sum of the

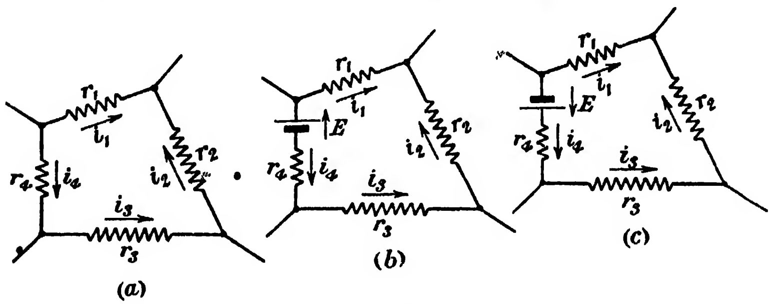


FIG. 29

various drops in volts is equal to zero. Taking a clockwise direction round the closed mesh as the positive direction, we therefore have

$$i_1r_1 - i_2r_2 - i_3r_3 - i_4r_4 = 0.$$

(*b*) In Fig. 29 (*b*) one arm contains a cell of E.M.F. E, and this E.M.F. tends to circulate current in the clockwise or positive direction

$$\therefore\; i_1r_1 - i_2r_2 - i_3r_3 - i_4r_4 = E$$

(*c*) In Fig. 29 (*c*) this cell is reversed, so that it tends to circulate current in the negative direction round the mesh. Its E.M.F. must therefore be reckoned negative, and we have

$$i_1r_1 - i_2r_2 - i_3r_3 - i_4r_4 = -E.$$

It is to be noted that in cases (*b*) and (*c*) the resistance r_4 must include the internal resistance of the cell.

As an example of the application of these two laws, take the case

of the well-known Wheatstone network, as shown in Fig. 30. The most important unknown quantity is the galvanometer current x. Denote the battery current by y and the current in r_1 by z, then we can use the first law to fix the other currents, as shown, in terms of three unknowns, x, y, and z. To obtain three equations we now apply the second law to three closed circuits, and we will choose $EBAC$, BAD, and DAC. We then have

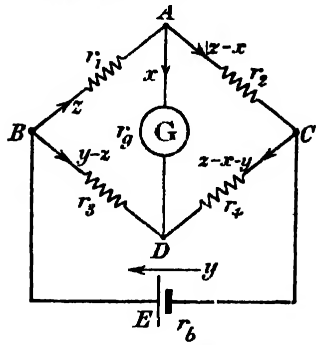


FIG. 30. WHEATSTONE'S BRIDGE NETWORK

$$r_1 z + r_2(z - x) + r_b y = E$$
$$r_1 z + r_g x - r_3(y - z) = 0$$
$$r_2(z - x) + r_4(z - x - y) - r_g x = 0$$

These reduce to

$$-r_2 x + r_b y + (r_1 + r_2)z = E$$
$$r_g x - r_3 y + (r_1 + r_3)z = 0$$
$$-(r_2 + r_4 + r_g)x - r_4 y + (r_2 + r_4)z = 0$$

Solving for x we have, in determinant form

$$x = \frac{\begin{vmatrix} E & r_b & (r_1 + r_2) \\ 0 & -r_3 & (r_1 + r_3) \\ 0 & -r_4 & (r_2 + r_4) \end{vmatrix}}{\begin{vmatrix} -r_2 & r_b & (r_1 + r_2) \\ r_g & -r_3 & (r_1 + r_3) \\ -(r_2 + r_4 + r_g) & -r_4 & (r_2 + r_4) \end{vmatrix}}$$

Evaluating the numerator we have

$$[E \times (-r_3) \times (r_2 + r_4) + r_b(r_1 + r_3) \times 0 + (r_1 + r_2) \times 0 \times (-r_4)]$$
$$- [0 \times (-r_3) \times (r_1 + r_2) + (-r_4)(r_1 + r_3)E + (r_2 + r_4) \times 0 \times r_b]$$
$$= E(r_1 r_4 - r_2 r_3).$$

Hence, denoting the denominator by Δ, we have

$$x = E\,\frac{r_1 r_4 - r_2 r_3}{\Delta}$$

$$\therefore\ x = 0 \text{ when } r_1 r_4 = r_2 r_3$$

or

$$r_2 \quad r_4$$

the well-known condition for a balance.

As a numerical example, let $E = 2$ volts, $r_1 = 10$, $r_2 = 20$, $r_3 = 30$, $r_4 = 40$, $r_g = 50$, and $r_b = 1$.

Then we have

$$x = \frac{\begin{vmatrix} 2 & 1 & 30 \\ 0 & -30 & 40 \\ 0 & -40 & 60 \end{vmatrix}}{\begin{vmatrix} -20 & 1 & 30 \\ 50 & -30 & 40 \\ -110 & -40 & 60 \end{vmatrix}} = \frac{\gamma}{\Delta} \text{ say}$$

Then

$$\gamma = [(2 \times -30 \times 60) + (1 \times 40 \times 0) + (30 \times 0 \times -40)] - [(0 \times -30 \times 30) + (-40 \times 40 \times 2) + (60 \times 0 \times 1)]$$

$$\Delta = [(-20 \times -30 \times 60) + (1 \times 40 \times -110) + (30 \times 50 \times -40)] - [(-110 \times -30 \times 30) + (-40 \times 40 \times -20) + (60 \times 50 \times 1)]$$

$$x = \frac{\gamma}{\Delta} \cdot \frac{-400}{-162{,}400}$$
$$= \cdot 00246 \text{ ampere.}$$

3. **Delta-star Transformation.** In some cases a network can be solved readily by means of what is called a delta-star transformation, and it is to be noted that this method is applicable to A.C.

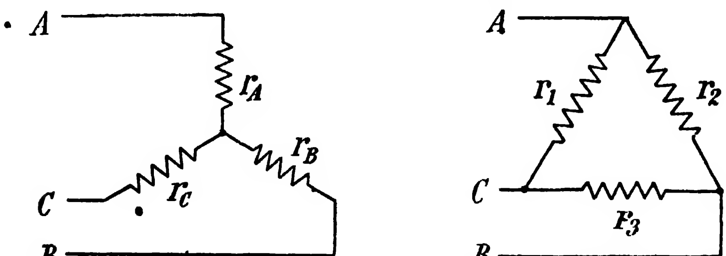


FIG. 31. STAR-DELTA CONVERSION

as well as to D.C. circuits.* Consider the two networks of Fig. 31. If these are identical then the resistance between any pair of lines will be the same when the third line is opened. Hence

$$\text{when } A \text{ is open } r_B + r_C = \frac{r_3 (r_1 + r_2)}{r_1 + r_2 + r_3}$$

$$\text{when } B \text{ is open } r_C + r_A = \frac{r_1 (r_2 + r_3)}{r_1 + r_2 + r_3}$$

$$\text{when } C \text{ is open } r_A + r_B = \frac{r_2 (r_3 + r_1)}{r_1 + r_2 + r_3}.$$

* See p. 256.

The solution of these equations is

$$r_A = \frac{r_1 r_2}{r_1 + r_2 + r_3}$$

$$r_B = \frac{r_2 r_3}{r_1 + r_2 + r_3}$$

$$r_C = \frac{r_3 r_1}{r_1 + r_2 + r_3}$$

As an example, the previous Wheatstone's Bridge network is given

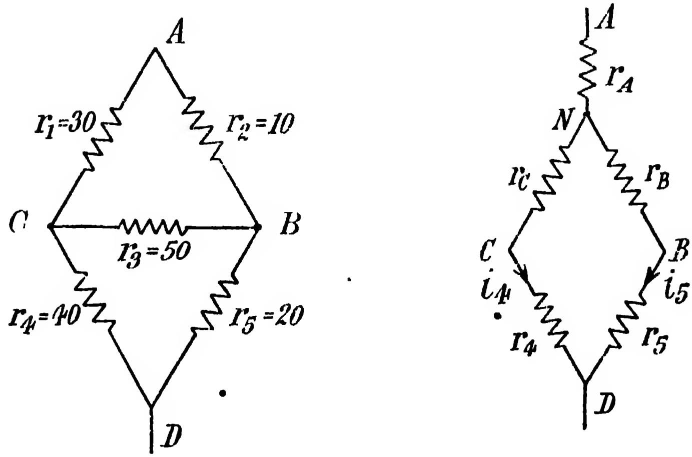


FIG. 32

in Fig. 32, the right-hand figure of which gives the equivalent network after the delta ABC has been converted to the equivalent star.

$$r_A = \frac{30 \times 10}{30 + 10 + 50} = 3 \cdot 33$$

$$r_B = \frac{10 \times 50}{30 + 10 + 50} = 5 \cdot 56$$

$$r_C = \frac{50 \times 30}{30 + 10 + 50} = 16 \cdot 67$$

The network thus further simplifies to that of Fig. 33. The resistance of section ND is

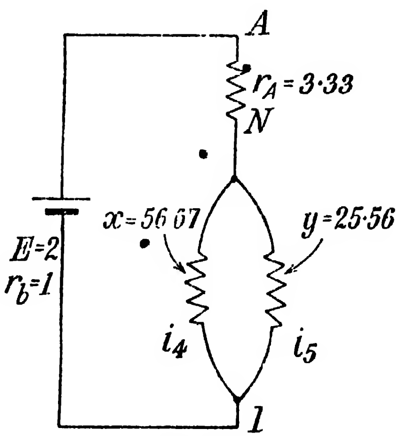


FIG. 33

$$\frac{xy}{x+y} = \frac{56\cdot67 \times 25\cdot56}{56\cdot67 + 25\cdot56} = 17\cdot61 \text{ ohms.}$$

The total circuit resistance is thus

$$1 + 3\cdot33 + 17\cdot61 = 21\cdot94 \text{ ohms}$$

and the current delivered by the cell is

$$\frac{2}{21\cdot94} = 0\cdot0912 \text{ amp.}$$

$$\text{Current } L_4 = \frac{25\cdot56}{82\cdot23} \times 0\cdot0912 = 0\cdot0283 \text{ amp.}$$

and $$L_5 = \frac{56\cdot67}{82\cdot23} \times 0\cdot0912 = 0\cdot0629 \text{ amp.}$$

The P.D. of point B with respect to $D = 0\cdot0629 \times 20 = 1\cdot258$

The P.D. of point C with respect to $D = 0\cdot0283 \times 40 \;\; 1\cdot132$

Thus point B is at a higher potential than C, the difference being

$$1\cdot258 - 1\cdot132 = 0\cdot126 \text{ volt.}$$

The galvanometer current is therefore

$$\frac{0\cdot126}{50} = 0\cdot00252 \text{ amp}$$

The difference between the two values represents slide-rule errors, and shows that when an accurate result is desired it is preferable to use log tables.

As a second example, take the case of a triangular pyramid, $ABCD$, built up of six wires whose resistances are as follows: AB 1 ohm, AC 1 ohm, CB 2 ohms, AD 2 ohms, DC 1 ohm, and DB 1 ohm. Calculate its resistance between the points A and B.

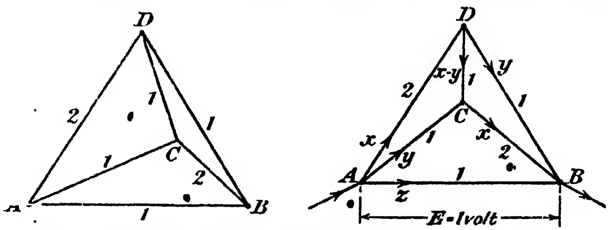


FIG. 34. NETWORK ILLUSTRATING KIRCHHOFF'S LAWS

The pyramid is shown in Fig. 34, along with the equivalent "flat" circuit. Imagine that an E.M.F. of 1 volt is applied between the points A and B, then, inserting symbols for the currents in the various resistances, we obtain the distribution shown, since it is clear that the currents in AD and CB will be equal, and also the

currents in AC and DB. The drop of volts is equal to E (i.e. 1 volt) along $(AD + DB)$, along $(AC + CB)$, and along AB. The first two of these really give the same condition, and therefore to obtain a third equation we will equate the total drop in the closed circuit ADC to zero.

$$\therefore\ 2x + y = 1$$

$$z = 1$$

$$2x + x - y - y = 0$$

or

$$3x - 2y = 0$$

The solution of this is

$$x = \frac{\;}{7}, \quad y = \frac{\;}{7}, \quad z = 1$$

The total current fed in at A is thus

$$\frac{2}{7} + \frac{3}{7} + 1 = 1\frac{5}{7}$$

Hence, as one volt is applied between A and B, the reciprocal of the current gives the total resistance, viz.

$$R = \frac{7}{12}\text{ ohm.}$$

As a third example, consider the following. Two cells are connected in parallel and supply a circuit of 1 ohm. Their E.M.F.'s are 2·05 and 2·15 volts, and their internal resistances 0·05 and 0·04 ohm respectively. Calculate the current in each cell.

The circuit is given in Fig. 35. Denoting the two battery currents by x and y, Kirchhoff's first law tells that the external current is $(x + y)$. Applying the second law to the circuits $ABCD$ and $AFCD$ in turn, we have

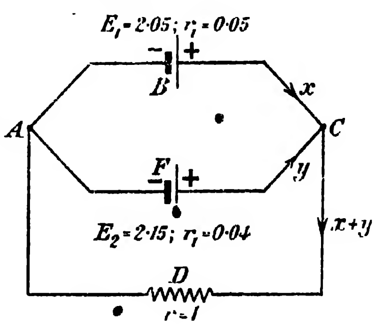


FIG. 35.

$$0{\cdot}05x + 1(x + y) = 2{\cdot}05$$

$$0{\cdot}04y + 1(x + y) = 2{\cdot}15$$

$$\text{Hence } 1{\cdot}05x + y = 2{\cdot}05$$

$$\text{and } x + 1{\cdot}04y = 2{\cdot}15$$

The solution is

$$x = -0{\cdot}19 \text{ amp}, \quad y = 2{\cdot}25 \text{ amp}.$$

CHAPTER IV

ELECTROSTATICS

1. General Principles. It is now known that the atom of any given element, instead of being indivisible, is made up of electrons and protons, both of which are entirely different in their nature from the atom of which they form a part. An electron is a particle of negative electricity which, when removed from an atom, exhibits none of the properties of ordinary matter. All electrons are exactly alike, even if the atoms from which they are derived are different.

The proton is the opposite of the electron, being a particle of positive electricity. The positive charge on the proton is exactly equal to the negative charge on the electron, but the mass of the proton is so very much greater than that of the electron that to all intents and purposes the mass of an atom can be said to be the sum of the masses of all the protons in it. The mass of a proton is $1{\cdot}63 \times 10^{-24}$ gramme.

The structure of an atom is, according to the Bohr theory, that of a solar system in miniature,* the central sun, or nucleus, being an aggregation of protons with their positive charges, with, in some cases, some electrons with their negative charges. The planets are a series of electrons rotating round the protons in a series of orbits. The simplest atom is that of hydrogen, which has a single proton for its nucleus and a single electron rotating round the nucleus. The radius of the orbit is of the order of 10,000 times the diameter of the electron. In more complex atoms the nucleus may have protons and electrons together, with the protons preponderating. Thus, in the case of helium, the nucleus has 4 protons and 2 electrons, while, as planets, there are 2 electrons. A more complicated case is that of the copper atom in which the nucleus has 64 protons and 35 electrons, while, as planets, there are 29 electrons.

In the case of all elements the atom is electrically neutral, since there are exactly as many electrons as protons. The atomic weight is (neglecting the masses of the electrons) equal to the number of protons in the atom. The atomic number of an element is the excess of protons over electrons in the nucleus, or, what amounts to the same thing, the number of planetary electrons. It is this number which determines the chemical properties of the atom. So far, the greatest known atomic number is 92, that for uranium, and therefore, if we assume that there must be an element to each atomic number, there must be at least 92 elements.

* This is not in accordance with the most recent views, but it is sufficiently accurate for the purpose of a mental picture.

2. Electrification. Since the aggregate of the positive electricity on the protons of any atom is exactly equal to the aggregate of negative electricity on the electrons, it follows that any normal atom is not electrified from the point of view of having any electrical influence external to itself. If some of the outer electrons are removed then the atom remains an atom, but its chemical nature may be altered, and, what is immediately obvious, the balance of positive to negative electricity will be upset. There will, in fact, be a surplus of positive electricity, and, if this removal of some of the electrons has been common to all the atoms in a body, this body will be charged positively, or will have acquired a positive charge.

If, on the other hand, an electron is added instead of removed there will be a preponderance of negative electricity, and if this addition has been made to all the atoms of a body this body will be charged negatively.

The process by which an originally neutral atom becomes positively charged through the removal of electrons is called ionization, and the atom itself is called an ion.

Summarizing, we can say that positive electrification is the result of a deficiency of electrons, while negative electrification is the result of an excess of electrons. The total deficiency, or excess, of electrons is called the charge.

3. Coulomb's Law. It is well known that bodies with like charges repel one another, while bodies with unlike charges attract. In the case of bodies which are so small that their charges can be regarded as point charges, the magnitude of this force in air (or more strictly *in vacuo*) is given by

$$F = \frac{q_1 q_2}{d^2} \text{ dynes}$$

where q_1 and q_2 are the magnitudes of the charges and d is their distance apart in cm. The force is positive when the charges repel and negative when they attract. From this expression we can define the unit charge as that which repels an equal similar charge, separated a distance of 1 cm. from it in air, with a force of one dyne. This is called the electrostatic c.g.s. unit of charge or quantity.

In a medium other than air, then, provided that the medium is an insulator, i.e. not a conductor of electricity, we have

$$F = \frac{q_1 q_2}{\kappa d^2} \text{ dynes}$$

where κ is a ratio or factor called the *dielectric constant* of the medium, or sometimes the *specific inductive capacitance*.

4. Electric Fields. The electric field is the space, in the neighbourhood of a charge, where forces of repulsion are exerted on like charges. The strength of a field at any point is defined as the force

in dynes which would act on a unit positive charge placed at that point, while the direction of the field is given by the direction of this force.

As with magnetic fields, it is useful to make use of the concept of lines, or tubes, of force. In the case of an electric field these lines emanate from charges, starting at a positive charge, and ending at a negative charge. If the charges are situated on isolated conducting bodies the lines of force will thus pass from one body to the other, in this respect differing from magnetic lines of force, all of which form closed circuits.

The above definitions of field strength is in terms of the force on a unit charge. Where the medium is air, field strength is also

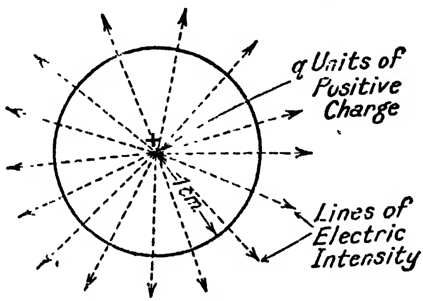


FIG.

equal to the number of lines of force per unit area of normal cross section. From this it follows that the number of lines emanating from a charge of q units is $4\pi q$. For imagine a point charge of q units at the centre of a sphere of radius 1 cm., Fig. 36, the medium being air. A unit positive charge placed at any point on the sphere will be repelled with a force of

$$\frac{q \times 1}{1^2} = q \text{ dynes.}$$

Hence the field strength anywhere on the sphere is q, and therefore there are q lines of force passing through each square centimetre. But the area of the sphere is 4π, and therefore the total number of lines emanating from the charge is $4\pi q$.

5. **Flux.** The total number of lines of force is called the flux. Thus in the case of an isolated charge of q units, the flux will be

$$\Psi = 4\pi q,$$

independently of the nature of the medium. If the charge is situated in a medium of dielectric constant κ, instead of in air, then, at any point on the sphere of radius 1 cm., the force on unit charge will be

$$F = \frac{q \times 1}{\kappa \times 1^2} = \frac{q}{\kappa}$$

If we denote the flux density by D, we have $F = \frac{D}{\kappa}$, an expression analogous to $H = \frac{B}{\mu}$ for the magnetic field.

Hence the number of lines per sq. cm. will be

$$D = F\kappa = \frac{q}{\kappa} \cdot \kappa = q$$

$$\therefore \Psi = 4\pi q$$

6. Electrical Intensity Inside a Charged Hollow Spherical Conductor. Fig. 37 shows a hollow sphere charged to a surface density σ, the surface density being the charge per unit area. Consider a cone with apex at any point P, and cutting off small areas ω_1 and ω_2. The charges on the two areas are then $\sigma\omega_1$ and $\sigma\omega_2$, and the field strength due to them at the point P is

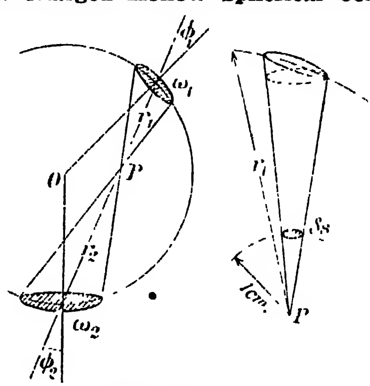


FIG. 37

$$\frac{\sigma\omega_1}{r_1^2} \quad \frac{\sigma\omega_2}{r_2^2}, \text{ assuming } \kappa = 1$$

The projection of area ω_1 perpendicular to the axis of the cone is $\omega_1 \cos \phi_1$ and therefore if the solid angle of the cone is δs, we have when $\kappa = 1$

$$\frac{\omega_1 \cos \phi_1}{\delta s} = \frac{r_1^2}{1}$$

$$\delta s = \frac{\omega_1 \cos \phi_1}{r_1^2}$$

Similarly

$$\delta s = \omega_2 \cos \phi_2$$

But

$$\phi_1 = \phi_2$$

$$\omega_1 = \omega_2$$

showing that the field strength at P, due to the areas ω_1 and ω_2, is zero.

Similarly we can divide the area of the sphere into a large number of pairs of small areas, such as ω_1 and ω_2, and, since each pair will produce zero field strength at P, it follows that the field strength is zero for the whole of the sphere. Now P is any point, proving that the field strength at any point within the sphere is zero.

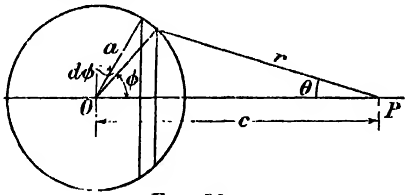


FIG. 38

7. The Field Strength at a Point External to a Charged Sphere. Considering an annular zone of angular width $d\phi$, Fig. 38, the charge on this ring is given by

$$2\pi a \sin\phi \times a d\phi \times \sigma = 2\pi a^2 \sigma \sin\phi \, . \, d\phi.$$

Hence the component along OP of the force on a unit charge at P due to the zone is

$$\frac{2\pi a^2 \sigma \sin\phi \, . \, d\phi}{r^2} \, . \cos\theta, \text{ again assuming } \kappa = 1.$$

Hence total force on a unit charge at P, i.e. the field strength at P

$$F = 2\pi a^2 \sigma \int \frac{\cos\theta \sin\phi \, . \, d\phi}{r^2}$$

Now $\qquad r^2 = a^2 + c^2 - 2ac\cos\phi$

and $\qquad a^2 = r^2 + c^2 - 2rc\cos\theta$

$$\therefore\ rdr = ac\sin\phi \, . \, d\phi \quad \text{or} \quad \sin\phi \, . \, d\phi = \frac{r\, . \, dr}{ac}$$

and $\qquad \cos\theta = \dfrac{r^2 + c^2 - a^2}{2cr}$

$$\therefore\ F = 2\pi a^2 \sigma \int_{c-a}^{c+a} \frac{1}{r^2} \times \frac{r^2 + c^2 - a^2}{2cr} \times \frac{r\, . \, dr}{ac}$$

$$= \frac{\pi a \sigma}{c^2} \int_{c-a}^{c+a} \left(1 + \frac{c^2 - a^2}{r^2}\right) dr$$

$$= \frac{\pi a \sigma}{c^2} \left[r - \frac{c^2 - a^2}{r} \right]_{c-a}^{c+a}$$

$$= \frac{4\pi a^2 \sigma}{c^2}$$

But $4\pi a^2\sigma$ is the total charge q while c is the distance of the point P from the centre O. Hence, to a point external to it, the sphere behaves as though its charge were concentrated at the centre.

8. **Gauss's Theorem.** Let a charge q be entirely enclosed by a surface of any shape whatsoever, and let the field strength at any point P on the surface be F. Then if P is situated within a very small area ω, Fig. 39, the number of lines of force crossing this element will be

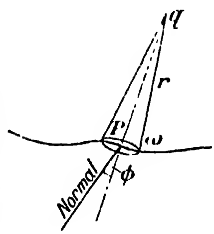


FIG. 39

$$\delta\Psi = F\kappa\omega \cos\phi$$

But $$F = \frac{q}{\kappa r^2}$$

$$\therefore\ \delta\Psi = \frac{q}{r^2}\omega\cos\phi$$

Hence total flux crossing the whole surface

$$\Psi = \Sigma\frac{q}{r^2}\omega\cos\phi$$

If the area ω subtends a solid angle δs at the charge, then

$$\frac{\omega\cos\phi}{\delta s} = r^2 \quad \text{(as before)}$$

$$\therefore\ \Psi = \Sigma q\delta s = q\Sigma\delta s$$
$$= 4\pi q.$$

Hence the total flux traversing a surface completely surrounding a charge of q units is $4\pi q$. This is Gauss's Theorem. *If there are a number of charges inside the surface, then the total flux is*

$$\Psi = 4\pi(q_1 + q_2 + q_3 + \ldots),$$

the summation being an algebraic sum, positive charges being reckoned positive, and vice versa.

9. **Coulomb's Theorem.** Consider again the small area ω in Fig. 36, and let the surface density be σ units per sq. cm. Thus total charge on the area ω is $\sigma\omega$. Hence, from Gauss's Theorem the total flux radiating from this charge is $4\pi\sigma\omega$. If the surface under consideration is a conducting surface, then we have seen that no flux can exist inside and consequently the whole of the flux $4\pi\sigma\omega$ passes outward normally. Hence flux density at the surface

$$= \frac{4\pi\sigma\omega}{\omega} = 4\pi\sigma$$

Hence field strength at a point close to the surface $= \frac{4\pi\sigma}{\kappa}$ dynes,

and is in a direction normal to the surface. This is Coulomb's Theorem.

10. **Potential.** If a mass is moved from one point to another in the gravitational field, work will be done against or by the force of gravity according as the vertical distance above the earth is increased or decreased. This work done is added to, or deducted from, the potential energy of the mass. Thus the total potential energy of the mass is a function of the position of the mass with respect to the earth. The case of the electrostatic field round a charged body is analogous to that of the gravitational field round the earth. If a small charge is introduced into the field, it will be repelled by, or attracted by, the body, and any change of position will necessitate that work shall be done against, or by, the force acting as the charge. We can thus regard potential in the electrostatic field in exactly the same way as we regard potential in the gravitational field.

In general, the potential at any point in an electric field is defined as the work done, in ergs, in moving a unit positive charge from an infinite distance to that point. The potential difference between two points in the field is the work done, in ergs, in moving a unit positive charge from the point of lower potential to that of higher potential.

If the two points are separated a very small distance ds, then if the field strength is F, the work done is obviously Fds. Let the difference of potential be dV, then

$$dV = -Fds$$

$$\therefore \ F = -\frac{dV}{ds}$$

showing that the field strength at any point is given by the potential gradient at that point. In this expression the positive direction of s is taken down the gradient.

11. **Equipotential Surfaces.** An equipotential surface is a surface such that all points on it are at the same potential. For such a surface, the potential gradient is obviously zero and therefore the field strength F can have no component along the surface. It therefore follows that the lines of force always cross an equipotential surface normally.

12. **Electrostatic Capacity, or Capacitance.** Consider a system composed of two insulated conductors disposed in such a way that, if a charge of electricity is imparted to one of them, all the lines of force emanating from this charge will end on the other conductor. Then if the charge is $+q$, there will be an induced charge of $-q$ on this other conductor. At any point in the field, the field strength will be given by

$$F = -\frac{dV}{ds}.$$

and therefore the potential difference between the two conductors will be given by

$$V = -\int F \cdot ds$$

But, from Coulomb's Law, we know that V is directly proportional to q, since V is proportional to F, in fact, for any particular system, the ratio of q to V is a constant, depending only on the shape and dimensions of the electrostatic field, and the medium the lines of force have to traverse. This ratio is called the capacity of the field, or the capacitance, and denoting it by C, we have

$$C = \frac{q}{V}$$

Thus the capacitance can be defined as the charge required to raise the potential difference to one unit.

In order to determine the dimensions of capacitance consider again the relationship

$$F = \frac{q_1 q_2}{\kappa d^2} = \frac{q^2}{\kappa d^2}$$

Now all charges are dimensionally the same irrespective of their magnitudes, so that we can put

$$q_1 = q_2 = q$$

$$\therefore\ q^2 = \kappa F d^2$$

$$\therefore\ [q^2] = [\kappa]\,[F]\,[L^2]$$

Now $$V = \frac{\text{Work done}}{\text{Charge}}$$

$$[F]\,[L]$$

and $\therefore$ since $$C = \frac{q}{V}$$

$$[C] = \frac{q^2}{[F]\,[L]}$$

$$= \frac{[\kappa]\,[F]\,[L^2]}{[F]\,[L]}$$

$$= [\kappa]\,[L]$$

Neglecting the dimensions of κ, since they cannot be expressed directly in terms of $[L]$, $[M]$ and $[T]$, we see that the dimensions of capacity is a length. Thus a sphere of one cm. radius in air has unit capacitance.

In c.g.s. units the unit of capacitance is thus the centimetre, but this is too small for practical purposes. More convenient units are

The farad $= 9 \times 10^{11}$ electrostatic c.g.s. units.

The microfarad $= 9 \times 10^{5}$ electrostatic c.g.s. units.

The derivation of the practical from the c.g.s. unit is given on p. xii.

The derivation of the practical unit of potential difference (P.D.), the volt, from the electrostatic c.g.s. unit of potential, is as follows—

$$1 \text{ electrostatic c.g.s. unit} = \frac{1 \text{ erg}}{1 \text{ E.S. unit of quantity}}$$

The volt is the P.D. between two points in an electric circuit in which the movement of one coulomb of electricity from one point to the other necessitates the expenditure of energy equal to one joule.

Now $\qquad 1 \text{ coulomb} = 3 \times 10^{9}$ electrostatic units of quantity

and $\qquad 1 \text{ joule} = 10^{7}$ ergs; or $1 \text{ erg} = 10^{-7}$ joule

$$\therefore \begin{array}{l}1 \text{ electrostatic c.g.s.}\\ \text{unit of P.D.}\end{array} = \frac{1 \text{ erg}}{1 \text{ E.S. unit of quantity}}$$

$$= \frac{10^{-7} \text{ joule}}{\frac{1}{3} \times 10^{-9} \text{ coulomb}}$$

$$= 300 \text{ joules per coulomb}$$

$$= 300 \text{ volts}$$

In practical units we have

$$C \text{ (farads)} = \frac{q \text{ (coulombs)}}{V \text{ (volts)}}$$

13. **Examples.** (i) ISOLATED SPHERE. Let the radius of the sphere be a cm., and let the charge be q. Then at any distance x from the centre the intensity is $q/\kappa x^2$. The work done in moving a unit positive charge a distance dx along a line of force is thus

$$\frac{q \cdot dx}{\kappa x^2}$$

$\therefore$ Work done in bringing a unit positive charge from infinity up to a distance r from the centre, viz. the potential at distance r is

$$V_r = \int^{r} \frac{-q \cdot dx}{\kappa x^2}$$

$$\frac{q}{\kappa r}$$

If the unit charge is brought from an infinite distance right up to the sphere, then putting $r = a$, we have for the potential of the sphere

$$V = \frac{q}{\kappa a}$$

But
$$C = \frac{q}{V}$$

$$= q \div \frac{q}{\kappa a}$$

$$= \kappa a$$

If $\kappa =$ unity then $C = a$, the radius of the sphere.

(ii) CONCENTRIC SPHERES. Let the radius of the inner sphere be a cm., and the inner radius of the outer sphere b cm.; let the inner sphere have a charge of $+q$, then there will be a charge of $-q$ induced on the inner surface of the outer sphere. The potential of the inner sphere due to its own charge is $+\frac{q}{\kappa a}$, and that throughout the space inside the outer sphere $-\frac{q}{\kappa b}$

$\therefore$ Resultant potential of inner sphere

$$V = \frac{q}{\kappa a} - \frac{q}{\kappa b} = \frac{q}{\kappa}\left(\frac{b-a}{ab}\right)$$

$$\therefore C = \frac{q}{V} = \kappa \cdot \frac{ab}{b-a}.$$

(iii) PAIR OF PARALLEL PLATES. Let two parallel plates A and B, Fig. 40, be each of area A sq. cm. and separated a distance d cm.; let B be earthed. If A is charged, the intensity in the space between the plates will be uniform except for a little fringing of flux at the boundary. Neglecting this, we see that if the density on the inner surface of A is σ, that on the inner surface of B will be $-\sigma$. Hence potential difference between A and B

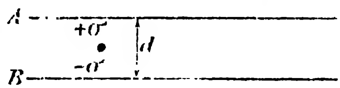


FIG. 40

$$V = \int_0^d \frac{4\pi\sigma}{\kappa} \cdot dx = \frac{4\pi\sigma d}{\kappa}$$

But $$\sigma = \frac{q}{A}$$

$$\therefore\ V = \frac{4\pi qd}{\kappa A}$$

$$\therefore\ C = \frac{q}{V} = \frac{\kappa A}{4\pi d}$$

(iv) COAXIAL CYLINDERS. In electrical engineering practice, conductors are almost invariably wires of circular section, and two cases of very great practical importance are thus a wire surrounded by a cylindrical metal sheath as in the case of a lead-covered cable, and the case of parallel wires as in an overhead transmission line. Take first the case of coaxial cylinders. Fig. 41 shows a core of radius r cm. surrounded by a lead sheath of inner radius R cm., the space between them being filled with dielectric of constant κ. If each centimetre of inner core has a charge of q units, we know from Gauss's Theorem that $4\pi q$ lines of force will emanate from each centimetre of core, and obviously from the symmetry of the system these lines will be radial. The surface area of a coaxial cylinder of radius x and of unit length is $2\pi x$, so that the intensity at a distance x from the centre is

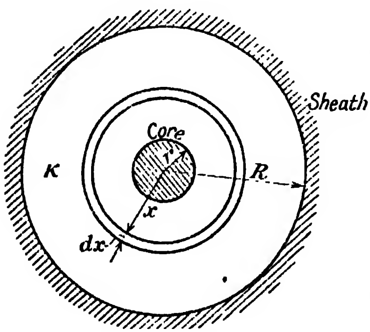


FIG. 41. TO ILLUSTRATE THE INSULATION RESISTANCE OF A CABLE

$$F_x = \frac{4\pi q}{\kappa \times 2\pi x} = \frac{2q}{\kappa x}$$

$\therefore$ P.D. between core and sheath

$$V = \int_r^R F_x \, . \, dx$$

$$= \int_r^R \frac{2q}{\kappa x} \, . \, dx$$

$$= \frac{2q}{\kappa} \log_e \frac{R}{r}$$

$$\therefore\ C = \frac{q}{V} = \frac{\kappa}{2 \log_e \frac{R}{r}} \text{ E.S. units per cm. length}$$

or $$C = \frac{\cdot 039\,\kappa}{\log_{10} \dfrac{R}{r}} \quad \mu\text{F per mile (see p. xiii)}$$

Now we have seen that the intensity is also equal to the voltage gradient, and denoting this by g we therefore have

$$g = \frac{2q}{\kappa x}$$

But $$q = \frac{\kappa V}{2 \log_e \dfrac{R}{r}}$$

$$\therefore\ g = \frac{V}{x \log_e \dfrac{R}{r}} = \frac{V}{2{\cdot}3\, x \log_{10} \dfrac{R}{r}} \text{ volts per cm.}$$

when V is expressed in volts.

At the surface of the inner conductor, where $x = r$, we have the maximum potential gradient of

$$g_{max} \doteq \frac{V}{2{\cdot}3 r \log_{10} \dfrac{R}{r}}$$

As a numerical example, take the case of a cable having a core of diameter ·927 in. with an impregnated paper insulation of thickness ·65 in., dielectric constant 3·5, working pressure 66,000 volts.

$$r = \cdot 464 \text{ in.}$$

$$R = \cdot 464 + \cdot 65 = 1{\cdot}114 \text{ in.}$$

$$\therefore\ \frac{R}{r} = \frac{1{\cdot}114}{\cdot 464} = 2{\cdot}41,\ \therefore\ \log_{10} \frac{R}{r} = \cdot 382$$

$$\therefore\ C = \frac{\cdot 039 \times 3{\cdot}5}{\cdot 382}$$

$$= 0{\cdot}357\ \mu\text{F per mile}$$

$$g_{max} = \frac{\sqrt{2} \times 66{,}000}{2{\cdot}3 \times (\cdot 464 \times 2{\cdot}54) \times \cdot 382}$$

$$= 86{,}500 \text{ volts per cm.}$$

The factor $\sqrt{2}$ is introduced into the numerator because the voltage will be alternating, and the maximum value is then $\sqrt{2}$ times the stated voltage (see p. 208).

(v) Parallel Conductors. Fig. 42 shows two parallel conductors, each of radius r, separated a distance d, and it is assumed that d is large compared with r. If conductor A has a charge of $+q$ per unit length, this will induce on B a charge of $-q$ per unit length. Hence

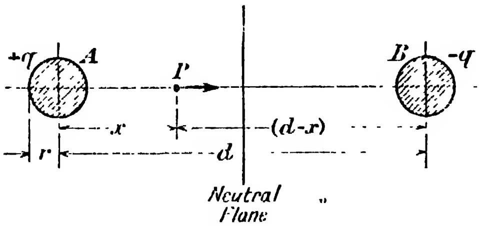


Fig. 42

Intensity at any point P due to A

$$= \frac{2q}{x} \text{ towards } B.$$

Intensity at P due to $B = \frac{2q}{d-x}$ in the same direction.

Hence total intensity at P

$$F_x = \frac{2q}{x} + \frac{2q}{d-x}$$

$$= 2q\left(\frac{1}{x} + \frac{1}{d-x}\right)$$

For air, the dielectric constant is unity so that κ does not appear in the equations. For the P.D. between the wires we have

$$V = \int_r^{d-r} F_x dx$$

$$= \int_r^{d-r} 2q\left(\frac{1}{x} + \frac{1}{d-x}\right)dx$$

$$= 2q\Big[\log_e x - \log_e (d-x)\Big]_r^{d-r}$$

$$= 4q \log_e \frac{d-r}{r}$$

But $$C = \frac{q}{V}$$

$$\therefore\ C = \frac{1}{4\ \log_e \frac{d-r}{r}} \text{ E.S. units per cm. length}$$

$$= \frac{\quad}{4\ \log_e \frac{d-r}{r}} \text{ since } r \text{ is small.}$$

Converting to practical units we have

$$C \simeq \frac{\cdot 0194}{\log_{10}\frac{d}{r}}\ \mu\text{F per mile}$$

Thus if $d = 12$ ft. $= 144$ in., and $2r = 0{\cdot}77$ in. as on the 132 kV British Grid lines

$$\frac{d}{r} = \frac{144}{\cdot 385} = 374$$

$$\therefore\ \log_{10}\frac{d}{r} = 2{\cdot}5729$$

$$\therefore\ C = \frac{\cdot 0194}{2{\cdot}5729}$$

$$= \cdot 755 \times 10^{-2}\ \mu\text{F per mile.}$$

To determine the voltage gradient it is convenient to regard the centre plane as a neutral plane so that if we denote the potential of one conductor with respect to this plane by V', we have

$$V' = \frac{V}{2} = 2q\log_e\frac{d-r}{r}$$

At point P the potential with respect to the neutral plane is

$$V'_x = \int_x^{\frac{d}{2}} 2q\left(\frac{1}{x} + \frac{1}{d-x}\right)dx$$

$$= 2q\log_e\frac{d-x}{x}$$

Hence for the voltage gradient we have

$$g = \frac{dV'_x}{dx} = -2q\frac{d}{x(d-x)}$$

But $$2q = \frac{V'}{\log_e\frac{d-r}{r}}$$

$$\therefore\ g = V' \times \frac{d}{x(d-x)} \times \frac{1}{\log_e\frac{d-r}{r}}$$

The gradient is a maximum at the conductor surface, where $x = r$, and since $(d - r) \simeq d$, we have

$$g = \frac{V'}{x} \times \frac{1}{\log_e \frac{d-r}{r}} = \frac{V'}{2\cdot3x \log_{10} \frac{d-r}{r}}$$

$$g_{max} = \frac{V'}{r} \times \frac{1}{\log_e \frac{d-r}{r}}$$

$$= \frac{V'}{r \log_e \frac{d-r}{r}} = \frac{V'}{2\cdot3r \log_{10} \frac{d-r}{r}}$$

Fig. 43 shows the variation of g between two conductors each of 1 in. diameter, spaced 10 ft. with a steady P.D. of 200,000 volts maintained between them. It will be seen that as x increases, g decreases exceedingly rapidly at first.

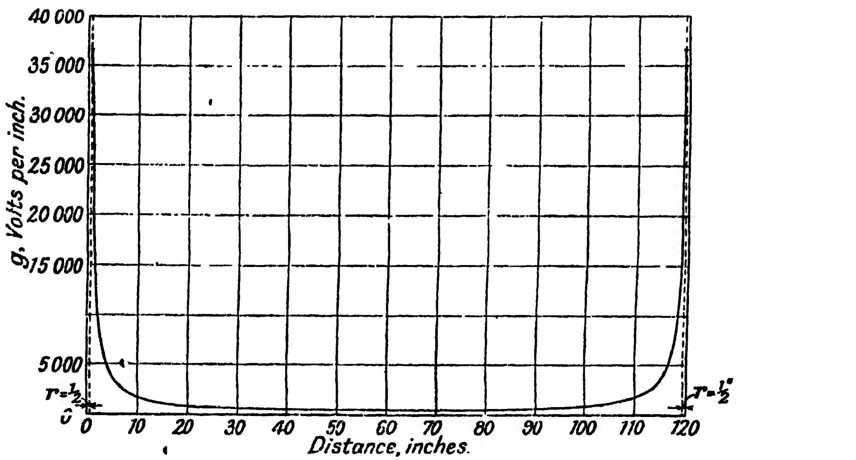


Fig. 43. Potential Gradient between Parallel Conductors

$$V' \simeq \frac{200,000}{2} = 100,000 \text{ volts}$$

$$d = 120 \text{ in.}$$

$$r = \cdot5 \text{ in.}$$

$$\therefore \log_{10} \frac{d-r}{r} = \log_{10} 239 = 2\cdot3784$$

$$\therefore g = \frac{100,000}{2\cdot3 \times 2\cdot3784x} = \frac{18,300}{x} \text{ volts per cm.}$$

In the case of the two parallel conductors, it is of interest to note that the lines of force are circles. Due to any one conductor alone, the lines of force are straight lines radiating from the conductor, and the intensity at any point is then inversely proportional to the distance from the wire. Thus at point P, Fig. 44,

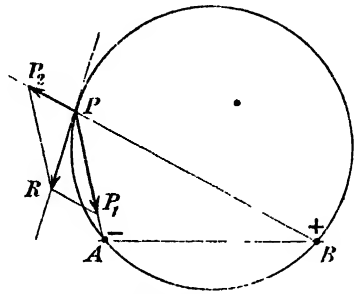


FIG. 44

Intensity due to $A \propto \frac{1}{PA}$ directed along PA.

Intensity due to $B \propto \frac{1}{PB}$ directed along BP.

Represent these by PP_1 and PP_2 respectively, then the total intensity at P will be represented by the resultant PR, and this means that PR must be a tangent to the line of force passing through P.

Now $$\frac{PP_1}{PP_2} = \frac{1}{PA} \div \frac{1}{PB} = \frac{PB}{PA}$$

$$\therefore \frac{P_1R}{PP_1} = \frac{PA}{PB}$$

also $$\widehat{PP_1R} = APB$$

So that the triangle PP_1R and BPA are similar. Hence the circle through the points AP and B must have PR as a tangent line.

14. Systems of Condensers. If a number of condensers of individual capacitances C_1, C_2, C_3 are connected in series, then the total capacitance is given by

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \cdot$$

The charge on each condenser is the same, so that if V_1, V_2, V_3, etc., are the P.D.s on the individual condensers, and V the total applied P.D., we have

$$q = CV = C_1V_1 = C_2V_2 = C_3V_3, \text{ etc.}$$

$$\therefore \quad V_1 = \frac{q_1}{C_1}$$

and
$$V = \frac{q}{C}$$

$$\therefore \quad \frac{V_1}{V} = \frac{C}{C_1}$$

Similarly
$$\frac{V_2}{V} = \frac{C}{C_2}; \quad \frac{V_3}{V} = \frac{C}{C_3}; \text{ etc.}$$

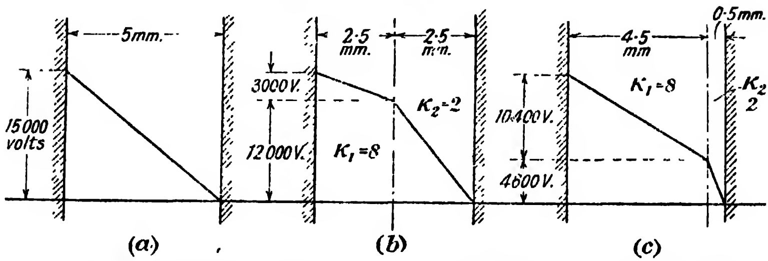


FIG. 45. POTENTIAL GRADIENT IN HOMOGENEOUS AND COMPOSITE DIELECTRICS

This is of great importance in connection with the compound dielectrics used in the manufacture of electrical machinery and other apparatus. Take, for example, the slot of a high voltage alternator. The armature conductors are at a high potential with respect to the core, and conductors and core are separated by dielectric, with the result that the system constitutes a condenser. If the dielectric is homogeneous, then the potential gradient will be uniform, but if the dielectric consists of layers of different materials with different dielectric constants, then the gradient will no longer be uniform but will be different for each material. Fig. 45 shows three such cases, the total thickness of insulation being 5 mm. in each case, and the total P.D. 15,000 volts. In case (*a*) the material is homogeneous so that g is uniform and equal to

$$g = \frac{15{,}000}{5} = 3{,}000 \text{ volts per mm.}$$

In case (*b*) there are two materials of dielectric constants 8 and 2, but of equal thicknesses. We can regard this system as the equivalent of two condensers in series, the capacitances being

$$C_1 = \frac{\kappa_1 A}{4\pi d} = \frac{8A}{4\pi \times \cdot 25} \quad 8k \text{ say}$$

$$C_2 = \frac{\kappa_2 A}{4\pi d} = \frac{2A}{4\pi \times \cdot 25} = 2k \text{ say}$$

Hence total capacitance

$$C = \frac{C_1 C_2}{C_1 + C_2} = \frac{16}{10} k$$

$$\therefore\ V_1 = \frac{C}{C_1} V = \frac{16}{10} \times \frac{1}{8} \times 15{,}000$$

$$= 3{,}000 \text{ volts}$$

$$\therefore\ g_1 = \frac{3{,}000}{2 \cdot 5} = 1{,}200 \text{ volts per mm.}$$

$$V_2 = \frac{C}{C_2} V = \frac{16}{10} \times \frac{1}{2} \times 15{,}000$$

$$= 12{,}000 \text{ volts}$$

$$\therefore\ g_2 = \frac{12{,}000}{2 \cdot 5} = 4{,}800 \text{ volts per mm.}$$

This shows that the gradient across the material of low dielectric constant is greater than that across the material with the high constant.

Now take the more likely case of Fig. (*c*). Then

$$C_1 = \frac{\kappa_1 A}{4\pi d_1} = \frac{8A}{4\pi \times \cdot 45} = 17 \cdot 8k,$$

where $k = \dfrac{A}{4\pi}$

$$C_2 = \frac{\kappa_2 A}{4\pi d_2} = \frac{2A}{4\pi \times \cdot 05} = 40k.$$

$$\therefore\ C = \frac{C_1 C_2}{C_1 + C_2} = \frac{17 \cdot 8 \times 40k}{57 \cdot 8} = 12 \cdot 3k.$$

$$\therefore\ V_1 = \frac{C}{C_1} V = \frac{12 \cdot 3}{17 \cdot 8} \times 15{,}000 = 10{,}400 \text{ volts}$$

and $$V_2 = \frac{C}{C_2} V = \frac{12 \cdot 3}{40} \times 15{,}000 = 4{,}600 \text{ volts.}$$

$$\therefore\ g_1 = \frac{10{,}400}{4 \cdot 5} = 2{,}320 \text{ volts per mm}$$

and $$g_2 = \frac{4{,}600}{\cdot 5} = 9{,}200 \text{ volts per mm.}$$

The expression for the capacitance of a parallel plate condenser with compound dielectric follows immediately from the above. Let

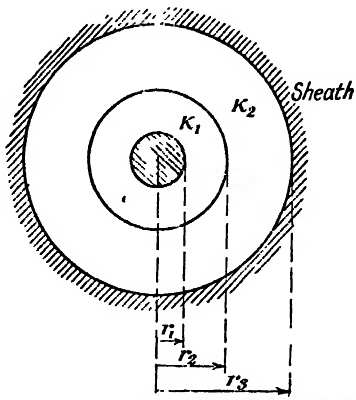


FIG. 46. CABLE WITH COMPOSITE DIELECTRIC

the distance between the plates be d cm., and let there be t cm. of dielectrics of constant κ_1 and $(d-t)$ cm. of dielectric of constant κ_2. Regarding each of these as forming separate condensers we have

$$C_1 = \frac{\kappa_1 A}{4\pi t}$$

$$C_2 = \frac{\kappa_2 A}{4\pi(d-t)}$$

Hence, since C_1 and C_2 are in series, we have

$$C = \frac{C_1 C_2}{C_1 + C_2}$$

$$= \frac{A}{4\pi}\left\{\frac{\frac{\kappa_1}{t}\cdot\frac{\kappa_2}{d-t}}{\frac{\kappa_1}{t}+\frac{\kappa_2}{d-t}}\right\}$$

$$= \frac{\kappa_1\kappa_2 A}{4\pi\{\kappa_1(d-t)+\kappa_2 t}$$

$$= \frac{A}{4\pi\left\{\frac{d-t}{\kappa_2}+\frac{t}{\kappa_1}\right\}}$$

Another practical example involving two condensers in series is an insulated cable with compound dielectric.

Let r_1 = radius of core

r_2 = radius of inner dielectric of constant κ_1

r_3 = radius of outer dielectric of constant κ_2.

Then regarding each dielectric as a separate condenser we have, Fig. 46,

$$C_1 = \frac{\cdot 039\ \kappa_1}{\log_{10} \frac{r_2}{r_1}}\ \mu\text{F per mile}$$

$$C_2 = \frac{\cdot 039\ \kappa_2}{\log_{10} \frac{r_3}{r_2}}\ \mu\text{F per mile}$$

$$\therefore \frac{V_2}{V_1} = \frac{C_1}{C_2} = \frac{\kappa_1 \log_{10} \frac{r_3}{r_2}}{\kappa_2 \log_{10} \frac{r_2}{r_1}}$$

As a numerical example, let $r_1 = 3$ mm., $r_2 = 6$ mm., $r_3 = 10$ mm., $\kappa_1 = 7$, $\kappa_2 = 5$.

$$\therefore \kappa_1 \log_{10} \frac{r_3}{r_2} = 7 \log_{10} \frac{10}{6} = 1{\cdot}553$$

$$\kappa_2 \log_{10} \frac{r_2}{r_1} = 5 \log_{10} \frac{6}{3} = 1{\cdot}505$$

$$\therefore \frac{V_2}{V_1} = \frac{1{\cdot}553}{1{\cdot}505} = 1{\cdot}03$$

$$\therefore \frac{V}{V_1} = \frac{2{\cdot}03}{1}$$

$$V_1 = \frac{V}{2{\cdot}03}$$

Thus suppose V is 10,000 volts, then

$$V_1 = \frac{10{,}000}{203} = 4{,}870 \text{ volts}$$

and $$V_2 = 10{,}000 - 4{,}870 = 5{,}130 \text{ volts.}$$

15. Condensers in Parallel. If a number of condensers are connected in parallel, then for the total capacitance we have

$$C = C_1 + C_2 + C_3 + \ldots$$

Each condenser is subjected to the whole of the applied P.D.

16. Energy Stored in an Electric Field. Since the lines of force of an electric field start on one conductor and end on another, it will be realized that any electrostatic field is associated with a condenser. Now the shapes of the lines of force depend on the shape, size, and relative positions of the conductors and therefore on the capacitance of the condenser formed by them. Consider for simplicity a parallel plate condenser. The two plates attract one another with a certain force F, and if their separation is increased by a small amount dx, work equal to Fdx will be expended. As there is no possibility of heat production, the whole of this work will be stored in the extended electric field in the form of potential energy. Thus the electric field is the seat of potential energy, just as a magnetic field is a seat of potential energy.

Let the P.D. between the plates be v, then v is the work done in taking a unit positive charge along one of the lines of force against the potential gradient. Hence, if the charge on the condenser is increased by a small amount dq, the work done must be vdq. Now the P.D. v is associated with a charge q according to the law

$$q = Cv$$
$$\therefore\ dq = Cdv$$

$\therefore$ Work done by increasing the charge by dq

$$vdq = Cv\,.\,dv$$

Hence work done in increasing the P.D. from zero to V

$$= \int_0^V Cvdv$$
$$= \tfrac{1}{2}CV^2.$$

If C and V are expressed in electrostatic units, then the work done will be in ergs. If C and V are in practical units—viz. C in farads and V in volts—then the work done will be in joules. Thus the stored energy in the electric field is $\frac{1}{2}CV^2$ joules. Compare this with the expression $\frac{1}{2}LI^2$ for the stored energy in a magnetic field.

Example. A condenser of capacitance 3 microfarads is charged to a P.D. of 10,000 volts. Calculate the stored energy in ergs, ft.-lb., joules, and watt-hours.

$$V = 10{,}000 \text{ volts} = \frac{10{,}000}{300}$$
$$= 33\cdot3 \text{ E.S. c.g.s. units}$$
$$C = 3 \times 10^{-6} \text{ farads}$$
$$= 3 \times 10^{-6} \times 9 \times 10^{11} \text{ E.S. c.g.s. units}$$
$$= 27 \times 10^5 \text{ E.S. c.g.s units.}$$

(i) $$\therefore\ \text{Ergs} = \tfrac{1}{2} \times (27 \times 10^5) \times (33\cdot3)^2$$
$$= 1\cdot5 \times 10^9$$

(ii)
$$\text{Joules} = \tfrac{1}{2} \times (3 \times 10^{-6}) \times (10^4)^2$$
$$= 150.$$

Alternatively, since one joule $= 10^7$ ergs, we have

$$\text{Joules} = 1{\cdot}5 \times 10^9 \times 10^{-7}$$
$$= 150 \text{ as before}$$

$$\text{One joule} = \text{one watt-second}$$

(iii) $\therefore$ One watt-hour $= \dfrac{\text{one joule}}{(60)^2}$

$$\therefore \text{ Watt-hours} = \frac{150}{3600} = {\cdot}042.$$

(iv)
$$\text{One erg} = \text{one dyne-cm.}$$
$$= \frac{1}{981 \times 454} \times \frac{1}{2{\cdot}54 \times 12} \text{ ft.-lb.}$$

$$\therefore \text{ Ft.-lb.} = \frac{1{\cdot}5 \times 10^9}{981 \times 454 \times 2{\cdot}54 \times 12}$$
$$= 110.$$

17. **Charge and Discharge of a Condenser.** Let a circuit be made up of a battery of E.M.F. E, a resistance R ohms, a condenser of capacitance C farads, and a switch S, as shown in Fig. 47. At the instant S is closed the P.D. across the condenser will be zero, but this P.D. will gradually increase until eventually it is equal to E. While the P.D. is increasing, the charge q on the condenser plates will be increasing, and the whole of this quantity will be derived from the battery. This means that while the P.D. is changing, the battery will be delivering current. This current, however, will not flow across the dielectric, but the quantity of electricity delivered by it will be stored by the condenser. Such a current is called a displacement current, to distinguish it from a current in the ordinary sense which flows right round a circuit.

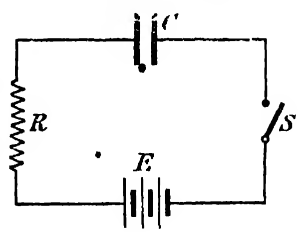


Fig. 47

Let $\qquad v =$ P.D. across condenser terminals at any instant

The displacement current at any instant is given by

$$i = \frac{dq}{dt} = \frac{d}{dt}(Cv)$$

$$= \frac{Cdv}{dt}$$

But we also have

$$i = \frac{\text{drop in volt along } R}{R}$$

$$= \frac{E - v}{R}$$

$$\therefore \frac{E - v}{R} = C \cdot \frac{dv}{dt}$$

or
$$v = E - CR\frac{dv}{dt}$$

The solution of this is

$$v = E(1 - \varepsilon^{-\frac{t}{CR}})$$

$$\therefore q = EC(1 - \varepsilon^{-\frac{t}{CR}})$$

the condenser thus acquiring a P.D. equal to E in infinite time.

Now let the terminals of the charged condenser be joined by a wire of resistance R, then the initial conditions are now $v = E$ and $i = 0$. The current i will now decrease instead of increasing, so that we have

$$i = -\frac{dq}{dt} = -C\frac{dv}{dt}$$

Also the drop in volts along the resistance R is now equal to the condenser P.D., so that

$$v = iR \text{ or } i = \frac{v}{R}$$

$$\therefore \frac{v}{R} = -C\frac{dv}{dt}$$

or
$$v = -CR\frac{dv}{dt}$$

The solution of this is

$$v = E\varepsilon^{-\frac{t}{CR}}$$

since the value of v when $t = 0$ is E.

$$\therefore\ q = CE\varepsilon^{-\frac{t}{CR}}$$

Suppose we wish to find the time taken for the condenser P.D. to drop from the initial value E to some lower value V, then at this P.D. we have

$$q' = CV = CE\varepsilon^{-\frac{t'}{CR}}$$

$$\therefore\ \frac{V}{E} = \varepsilon^{-\frac{t'}{CR}}$$

$$\therefore\ \frac{E}{V} = \varepsilon^{\frac{t'}{CR}}$$

$$\therefore\ \log_\varepsilon\left(\frac{E}{V}\right) = \frac{t'}{CR}$$

$$\therefore\ t' = CR\log_\varepsilon\frac{E}{V}$$

18. **Oscillatory Charge and Discharge of a Condenser.** If a circuit contains resistance and capacitance in series, then rise of current during charging of the condenser, and the decay of current during discharging, are both exponential in character. There are no current oscillations of any kind. Now consider a circuit having resistance, inductance, and capacitance in series. At any instant the quantity of electricity stored in the condenser is given by

$$q = \int i\,.\,dt.$$

The P.D. across the condenser terminals is equal to

$$\frac{q}{C} = \frac{1}{C}\int i\,.\,dt$$

Hence, if at any instant the E.M.F. impressed on the circuit is e, we have

$$Ri + L\frac{di}{dt} + \frac{1}{C}\int i\,.\,dt = e$$

Put $e = E_{max}\sin\omega t$, and differentiate with respect to t. Then

$$R\frac{di}{dt} + L\frac{d^2i}{dt^2} + \frac{i}{C} = \omega E_{max}\cos\omega t$$

The solution of this is

$$i = A\ \sin(\omega t - \alpha) + B\varepsilon^{-\frac{R}{2L}\cdot t}\sin\,(\omega' t - \beta)$$

where $A = \dfrac{E_m}{\sqrt{R^2 + \left(L\omega - \dfrac{1}{C\omega}\right)^2}}$

$$\omega' = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

and B, a, and β, are constants.

The first term of the expression for i represents a forced oscillation which obeys the laws which will be considered in detail in Chapter XIV. The second term represents a free oscillation, sometimes called a "transient," of frequency

$$f' = \frac{\omega'}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

This oscillation has a damping term $e^{-\frac{R}{2L}t}$ showing that its amplitude dies away exponentially as shown in Fig 48. The term $R/2L$ is called the "attenuation constant." It will be seen that the resistance R has some influence on the frequency of the free oscillation, but if $R^2/4L^2$ is small compared with $1/LC$ as, for example, in the oscillatory circuits of wireless apparatus, then the frequency of the free oscillation becomes

$$f' \simeq \frac{1}{2\pi}\sqrt{\frac{1}{LC}}$$

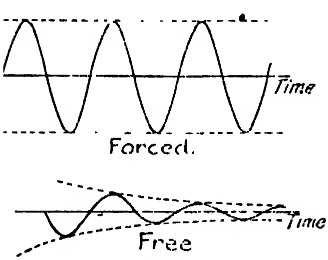


FIG. 48

ILLUSTRATING FREE AND FORCED OSCILLATIONS

Now suppose that the circuit, instead of having an external E.M.F. impressed on it, is allowed to discharge itself after the condenser has previously been charged up. The E.M.F. equation then becomes

$$R\frac{di}{dt} + L\frac{d^2i}{dt^2} + \frac{i}{C} = 0$$

$$\text{or}\ \frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0$$

To solve this, put $i = \varepsilon^{\lambda t} y$

$$\therefore\ \frac{d^2i}{dt^2} = \left(\frac{d^2y}{dt^2} + 2\lambda\frac{dy}{dt} + \lambda^2 y\right)\varepsilon^{\lambda}$$

$$\frac{R}{L}\frac{di}{dt} = \frac{R}{L}\left(\frac{dy}{dt} + \lambda y\right)\varepsilon^{\lambda t}$$

$$\frac{i}{LC} = \frac{1}{LC} \cdot y\varepsilon^{\lambda t}$$

$$\therefore \quad \frac{d^2y}{dt^2} + \left(2\lambda + \frac{R}{L}\right)\frac{dy}{dt} + \left(\lambda^2 + \frac{R}{L} \cdot \lambda + \frac{1}{LC}\right)y = 0$$

Now put $\lambda = -\frac{R}{2L}$ so that $i = y\varepsilon^{-\frac{R}{2L} \cdot t}$

$$\therefore \quad \frac{d^2y}{dt^2} + \left(\frac{1}{LC} - \frac{R^2}{4L^2}\right)y = 0$$

There are three cases—

(a) Let $\frac{1}{LC} > \frac{R^2}{4L^2}$. Put $\left(\frac{1}{LC} - \frac{R^2}{4L^2}\right) = \omega^2$

$$\therefore \quad \frac{d^2y}{dt^2} + \omega^2 y = 0$$

$$\therefore \quad y = A \cos \omega t + B \sin \omega t$$

$$\therefore \quad i = (A \cos \omega t + B \sin \omega t)\varepsilon^{-\frac{R}{2L} \cdot t}$$

$$= \alpha\varepsilon^{-\frac{R}{2L} \cdot t}(\cos \omega t + \varphi)$$

where α and φ are constants. This represents a simple harmonic oscillation whose amplitude $\alpha\varepsilon^{-\frac{R}{2L} \cdot t}$ continually diminishes as t increases, as shown in Fig. 49. The period

$$\frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\frac{1}{LC}}\left\{1 - \frac{R^2C}{4L}\right\}^{\frac{1}{2}}}$$

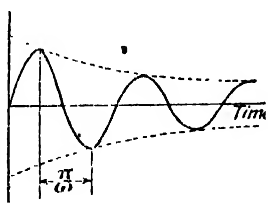


FIG. 49

DISCHARGE CURRENT OF A CONDENSER

If the resistance were zero, the period would be $2\pi\sqrt{LC}$, showing that the presence of resistance lengthens the period in the ratio

$$\frac{1}{\left(1 - \frac{R^2C}{4L}\right)^{\frac{1}{2}}}$$

If R is small, then $R^2C/4L$ can be neglected as a small quantity of the second order, and the change in period is then negligible.

To find the points of maximum amplitude, we have

$$\frac{di}{dt} = 0$$

This equation has the form

$$C \cos \omega t + D \sin \omega t = 0$$

$$\therefore \tan \omega t = -\frac{C}{D}$$

The solution of this is an arithmetic progression whose common difference is π. The corresponding increments of time are therefore π/ω, and the ratio of consecutive amplitudes in positive and negative directions is

$$\varepsilon^{\frac{R}{2L}\cdot\frac{\pi}{\omega}}$$

The log of this, namely,

$$\frac{\pi R}{2L\omega} \log_{10}\varepsilon,$$

is called the *logarithmic decrement.*

(*b*) Let $\dfrac{1}{LC} < \dfrac{R^2}{4L^2}$

$$\text{Put } \frac{R^2}{4L^2} - \frac{1}{LC} = m^2$$

$$\therefore \quad \frac{d^2y}{dt^2} - m^2 y = 0$$

$$\therefore \quad y = A\varepsilon^{mt} + B\varepsilon^{-mt}$$

$$\text{and } i = A\varepsilon^{-\left(\frac{R}{2L} - m\right)t} + B\varepsilon^{-\left(\frac{R}{2L} + m\right)t}$$

Since $R/2L > m$, both terms diminish asymptotically to zero, and the current is aperiodic instead of oscillatory.

(*c*) Let $\qquad \dfrac{1}{LC} = \dfrac{R^2}{4L^2}$

Then $\qquad \dfrac{d^2y}{dt^2} = 0 \; ; \; \therefore y = At + B$

$$\therefore i = (At + B)\varepsilon^{-\frac{R}{2L}t}$$

The current in this case approaches zero asymptotically, and there is at the most one passage through the zero position.

19. **Dielectrics.** An insulator is a material which has a very high specific resistance in comparison with a good conductor of electricity such as copper or aluminium. For this reason conductors which have to be maintained at a high potential with respect to earth are

either entirely sheathed in insulating material or are supported by insulators. Even so, there is always a certain amount of leakage to earth, although the leakage current which passes through the insulating material is exceedingly small in comparison with the useful current carried by the conductor. Thus the specific resistance of india-rubber is of the order of 2×10^{14} to 10^{15} ohms per cm. cube, as compared with about 1·7 microhms per cm. cube for copper.

The insulation resistance is the resistance to true leakage current (i.e. not including surface leakage). In the case of insulated cables, the useful current flows axially along the core whereas the leakage current flows radially from the core to the sheath. It therefore follows that the insulation resistance of a cable is inversely proportional to its length, whereas the conductor resistance is directly proportional to the length.

Consider unit length of single core cable of core radius r and insulation radius R, as in Fig. 50. An annular ring of radius x will have a surface area of $2\pi x \times 1 = 2\pi x$ sq. cm., and since the flow of leakage current is radial as shown, such a ring of thickness dx will offer to the leakage current a path of length dx and area $2\pi x$.

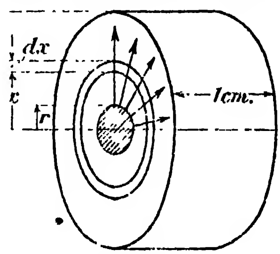


FIG 50

Hence, for the resistance of the ring, we have

$$dR = \frac{\rho dx}{2\pi x}$$

Hence insulation resistance per cm. length of cable

$$R = \frac{\rho}{2\pi}\int_r^R \frac{dx}{x}$$

$$= \frac{\rho}{2\pi}\log_e \frac{R}{r}$$

$$= \frac{2{\cdot}3\rho}{2\pi}\log_{10}\frac{R}{r}$$

Hence, for a length l cm. of cable, we have for the insulation resistance

$$R = \frac{2{\cdot}3\rho}{2\pi l}\log_{10}\frac{R}{r}$$

Example. $r = 1{\cdot}3$ cm., $R = 4$ cm., $\rho = 8 \times 10^{14}$ ohms per cm. cube, $= 8 \times 10^{8}$ megohms per cm. cube. Find the insulation resistance per mile of cable.

$$\log_{10}\frac{R}{r} = \log\frac{4}{1{\cdot}3} = {\cdot}488$$

$$l = 5280 \times 12 \times 2{\cdot}54$$
$$= 16{\cdot}2 \times 10^4 \text{ cm.}$$

$$\therefore\ R = \frac{2{\cdot}3 \times 8 \times 10^8 \times {\cdot}488}{2\pi \times 16{\cdot}2 \times 10^4}$$

= 883 megohms per mile, or, more correctly, 883 megohm miles.

The following are the important electrical properties of insulating materials—

(i) Specific resistance: this has already been considered above.

(ii) Dielectric strength or breakdown strength.

(iii) Dielectric loss when subjected to alternating electric stresses.

(iv) Dielectric constant, or specific inductive capacity.

Certain mechanical properties are also essential depending on the use to which the material is put. Thus for the insulating materials used in the manufacture of electrical machinery, the following are essential—

(v) Sufficient mechanical strength to withstand vibration, and the bending and abrasion experienced during the manufacture of the machine.

(vi) Good heat conductivity.

(vii) Ability to withstand the maximum temperature attained, without change in physical properties or in chemical composition.*

The breakdown strength naturally depends upon the thickness of the material but it is not, as might be expected, proportional to the thickness. Thus if the breakdown strength of a certain sample is given as x kilovolts per mm, this value x refers to a definite thickness of sample. The breakdown strength is also dependent on the shape of the electrodes to which the voltage is applied, the frequency and wave form of the voltage (when alternating), the time of application, and on the time taken to bring up the voltage from zero to the breakdown value, also on temperature and moisture content of test piece. It will therefore be seen that figures of breakdown value mean very little unless the test conditions are very completely specified.†

Provided that the test conditions are identical and only the

* Insulating materials are discussed very fully in *Electrical Insulating Materials*, by Monkhouse.

† The standard test conditions can be found in the appendix to Monkhouse's book, and in *Electrical Engineers' Data Book*, Vol. ii.

thickness of the material is varied, the relationship between breakdown voltage V and thickness t is given approximately by Baur's Law, viz.

$$V = at^{\frac{2}{3}}$$

where a is a constant depending on the material and also on the thickness t.

The reason for the decrease in the electric strength with increase in thickness is probably due to the fact that a thick material cannot get rid of internally produced heat as well as a thin material, a smaller applied voltage per unit thickness therefore being required to produce a given maximum temperature. This is in agreement with the thermal theory of breakdown given by Professor Miles Walker.* According to this theory the action of a solid dielectric is governed by the fact that all such dielectrics have a very marked temperature coefficient of electric conductivity, or, what amounts to the same thing, a decrease in resistance with increase in temperature. When a dielectric is stressed by the application of an alternating potential some losses are produced, as explained later, and these losses result in the production of heat. Consequently the resistance decreases, causing an increase in current and therefore an increase in heat production, and this in turn produces a still further reduction in resistance. The process is thus cumulative and either (*a*) a stable condition will be reached if the applied voltage is not sufficient to break down the material, or (*b*) the temperature will increase until the material changes its chemical composition, and then breaks down.

Dielectric loss when the material is subjected to alternating voltage is of little importance so long as the voltage is low, but it becomes of very considerable importance in high tension work. For example, insulated cables are now made for working voltages of 100,000 and more, and the dielectric loss is a limiting factor in their design. Imagine a condenser charged by the application of a steady voltage, then there is an initial rush of current which follows the exponential law previously given, but, after this, instead of falling to zero the current persists at a low value for an appreciable time. The current has been described as "soaking in" the dielectric. Similarly, when the condenser is discharged, there is an initial rush of current, and the P.D. very quickly falls to zero. If, however, the discharging connection is removed and the condenser left for some time, it will be found that a charge has been acquired, and this charge in turn can be removed by the process of discharging. With some dielectrics this can be repeated several times. The first of these phenomena is called *absorption*, and all dielectrics are absorptive to a certain degree. The charge recovered

* *Journ. I.E.E.*, Vol. xlix, p. 71.

Material	Dielectric Strength in kilovolts per mm. at 50 cycles	Specific Resistance when Dry at 25° C., in megohms per cm. cube	Dielectric Constant	Affected by Moisture	Safe Temperature in ° C.	Uses
Asbestos . . .	3–4·5	$1{\cdot}6 \times 10^6$	—	Absorbent	500 or more	Covering of wires in very highly rated machines.
Bakelite . . .	20–25	—	5–6	No	200	Bakelized paper made up in form of boards. Many uses.
Bitumen (vulcanized)	14	—	4·5	No	About 60	Low-voltage mining cables. Cable-box filling compound.
Cotton . . .	3–4	1,000 upwards according to dryness	—	Absorbent	90	Covering for wires.
Ebonite . . .	10–40	$2 - 100 \times 10^6$	2–3	Slightly	40	Covers for resistance boxes, etc.
Empire cloth . .	10–20	As cotton	2	Absorbent if varnish layer is cracked	90	Wrapping for groups of wires, e.g. armature coils.
Fibre . . .	5	As cotton	4–6	Ditto	90	In sheet form, slot linings.
Glass . . .	5–12	5×10^6 to infinity	3–8	No, except on surface	Room temperatures only or may crack	Not used in electrical practice except to a small extent for transmission line insulators.
Guttapercha . .	10–20	5 to 25×10^6	3–5	No	40	Covering for submarine cables.
India-rubber . .	10–25	2 to 10×10^6	2–3	Slightly	40	Cable insulation.
Marble . . .	6	400	8	Somewhat absorbent	Room temperatures	Formerly used for switchboards.
Mica . . .	40–150	5 to 100×10^6	3–8	No	500 or more	Not generally used in the pure form.
Micanite . . .	30	10 to $6{,}000 \times 10^6$	6–8	No	130 when under pressure	Commutator segments. Slot linings for high voltage machines. Bushes.
Paper . . .	4–10	As cotton	2	Absorbent	90	Cable insulation when oil impregnated. Covering for transformer conductors.
Paraffin wax . .	8	3×10^{10}	2	No	Under 50	Cable insulation when oil impregnated. Covering for transformer conductors.
Porcelain . . .	9–20	1 to $1{,}000 \times 10^6$	4–7	Not when vitreous and glazed	Room temperatures	Insulators for overhead lines.
Shellac . . .	5–20	9×10^9	2·5–3·5	No	Under 60	Insulating varnishes. Cement for manufacture of micanite.
Slate . . .	3	40	7	Rather absorbent	Room temperatures	For face-plates or starters, etc., not now used for switchboards.

or retained after an initial discharge is called the ***residual charge.*** These phenomena are explained by the assumption that there is a viscous movement of the molecules of a dielectric when the material is stressed electrically, and that internal forces have to be overcome before the molecules can align themselves correctly. This is analogous to the Weber theory of magnetization. In charging the condenser, there is molecular movement which is rapid at first and corresponds to the initial charging current, the movement thereafter being much slower. It is clear that with such a molecular mechanism, work will have to be done and therefore power expended when the process of charging and discharging is repeated in rapid cycles. This power is the dielectric loss, and the energy conveyed is converted into heat, which raises the temperature of the dielectric. Compare this with the hysteresis loss in magnetic materials subjected to rapid cycles of magnetization.

It is not possible to give a full discussion of the properties of the many dielectrics used in electrical engineering practice, but the table (on p. 76) giving particulars of the more usual materials will be of value.*

For more complete information on the properties of dielectrics and their applications to electrical plant and apparatus, see *Insulation of Electrical Apparatus*, by D. F. Miner.

* This is based largely on a table given by Miles Walker in ***Specification and Design of Electrical Machinery.***

CHAPTER V

CONSTRUCTION OF DIRECT CURRENT MACHINES

1. **Essential Features.** In order to set up a dynamically induced E.M.F. three things are required, namely, a magnetic field, a conductor, and motion of the conductor in the field. In dynamo-electric machinery the magnetic field is produced by the "field-magnet," the conductors in which the E.M.F. is induced are placed on the "armature," and the necessary motion is obtained by the rotation of the armature within the field magnet. We shall see that the E.M.F. induced is an alternating E.M.F., that is, alternately positive and negative. To obtain a direct or continuous E.M.F. this alternating E.M.F. is rectified by means of a "commutator" which rotates with the armature.

2. **The Field Magnet.** Practically all modern direct current machines have a field magnet consisting of a circular yoke with inwardly projecting poles, the polarity of which is alternately N. and S. A typical four-pole frame is shown in Fig. 51. The dotted lines indicate the paths of the main magnetic fluxes and the full lines the leakage paths. It will be seen that the flux per pole divides at the yoke, so that the cross section of the yoke carries only half the flux per pole. Similarly the cross section of the armature core carries half the flux which enters the armature from the air gap.

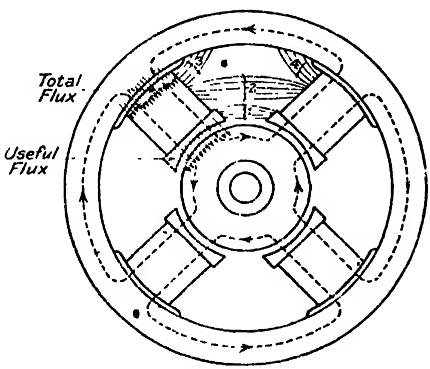


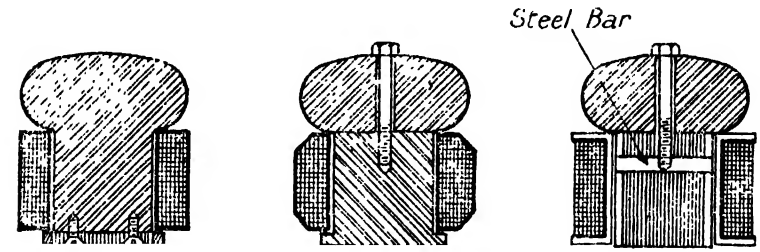
FIG. 51
FLUXES IN A TYPICAL FIELD MAGNET

The yoke of a large machine is almost invariably made of cast steel, because this material has roughly twice the permeability of cast iron. Only half the weight of cast steel need therefore be used, an important consideration in large machines. The poles may be cast steel, cast integral with the yoke, wrought iron, or built up of sheet, i.e. laminated. Typical methods of fastening the poles to the yoke are shown in Fig. 52.

In the case of small machines, cast iron is often used for both yoke and poles, because here the main consideration is cheapness, both from the point of view of the materials used and also the

amount of machining required by individual parts. Cast iron poles obviously require much less machining than, say, wrought iron poles. The extra weight due to the use of cast iron for small machines is not so important as with large machines.

The calculation of the ampere-turns per pole for a given magnetic circuit is made as follows. The mean path of the flux (*see* Fig. 50)



Magnet Core and Yoke cast solid. Separate pole shoe. *Magnet Core bolted to Yoke.* *Magnet Core laminated and bolted to Yoke.*

FIG. 52

is divided up between the various parts of the circuit. The ampere-turns per pole have to overcome the reluctance of half the mean circuit, i.e. the path A to F, Fig. 53

Let l_y = mean magnetic length of yoke $= AB$
l_p = length of pole $= BC$
l_g = length of air gap $= CD$
l_t = length of a tooth $= DE$
l_c = mean length in core $= EF$

Let corresponding suffixes refer to the areas a_y, a_p, etc.

In the case of the air gap an approximation can be made by taking for a_g the mean of the pole face area and the area of the tops of all the flux-carrying teeth per pole. If the teeth are not saturated it is sufficient to take the mean area at the middle, but if they are saturated, the areas at the top, middle, and root are calculated. Let these be $_1a_t$, $_2a_t$, and $_3a_t$. The required useful flux per pole is known; let this be Φ, then the total flux per pole is $\lambda\Phi$ where λ is the leakage

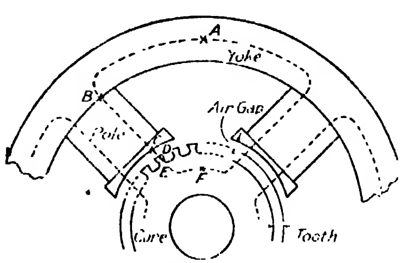


FIG. 53

factor. λ can be estimated with sufficient accuracy from previous designs, its average value for an orthodox design being 1·2.

The complete calculation can be set out in tabular form as below—

Part.	Flux.	Area.	Flux Density.	Ampere-turns per cm. Length.	Total Ampere-turns.
Yoke . . .	$\lambda\Phi/2$	a_y	$\lambda\Phi/2a_y$	at_y	$at_y \times l_y$
Pole . . .	$\lambda\Phi$	a_p	$\lambda\Phi/a_p$	at_p	$at_p \times l_p$
Air Gap . .	Φ	a_g	$\Phi/a_g = B_g$	$at_g = 0{\cdot}8B_g$	$at_g \times l_g$
Teeth: Top .	Φ	$_1a_t$	$\Phi/_1a_t$	$_1at_t$	
Teeth: Middle .		$_2a_t$	$\Phi/_2a_t$	$_2at_t$	
Teeth: Root .		$_3a_t$	$\Phi/_3a_t$	$_3at_t$	
Teeth: Mean .				$at_m = \dfrac{_1at_t + 4\,_2at_t + _3at_t}{6}$	$at_m \times l_t$
Core . . .	$\Phi/2$	a_c	$\Phi/2a_c$	at_c	$at_c \times l_c$

The ampere-turns per cm. length in column 5 are of course read off from magnetization curves.

3. **The Armature.** The armature consists of the core and the winding. The core obviously has to be magnetic in order to provide a path of low reluctance to the lines of force, and since iron is a good conductor of electricity, the rotation of a solid core in a strong magnetic field would result in very heavy eddy currents (or Foucault currents) being set up. These eddy currents would necessitate such a heavy expenditure of energy, and would cause such a large amount of heat to be produced, that it is imperative to eliminate them as far as possible. Since iron is a conductor it is impossible entirely to eliminate the production of these currents, but they can be kept down to reasonable proportions by building up the core of thin sheets lightly insulated from one another by varnish, thin paper, or even the oxide coating they acquire when they are "pickled." These sheets or discs are about $\frac{1}{40}$ in. thick. Up to about 3 ft. diameter they can be cut out in complete rings. but for large armatures it is necessary to cut them out in segments. There are thus two typical constructions for a direct current armature according as the discs are in one piece or in segments. Fig. 54 shows the construction of a fairly small armature, about 100 kW capacity. The core discs are mounted on a cast iron hub A, which is keyed to the shaft, and they are compressed between two end flanges F_1 and F_2 by means of long bolts. These end flanges also serve as coil supports. Because of the various losses which take place in the armature, heat is produced, which must be got rid of if the armature is to keep cool. This necessitates ventilation,

which is effected by separating the core discs into "packets" by interposing a number of special spacing discs *H*, two in the figure. These form ventilating ducts, which communicate with the external air by means of holes *D* in the hub, and the fanning action of the rotating armature draws air through these ducts, thus producing

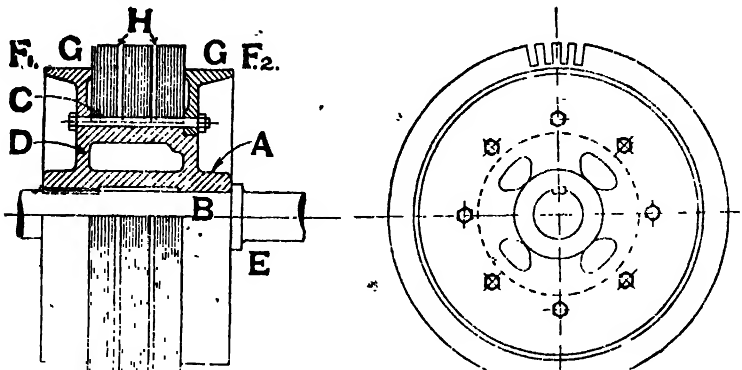


FIG. 54

ARMATURE CONSTRUCTION FOR SMALL MACHINE

efficient ventilation. In the case of large machines the hub takes the form of a "spider," that is, a flywheel without rim, and the segments of the core discs are dovetailed to this, as shown in Fig. 55.

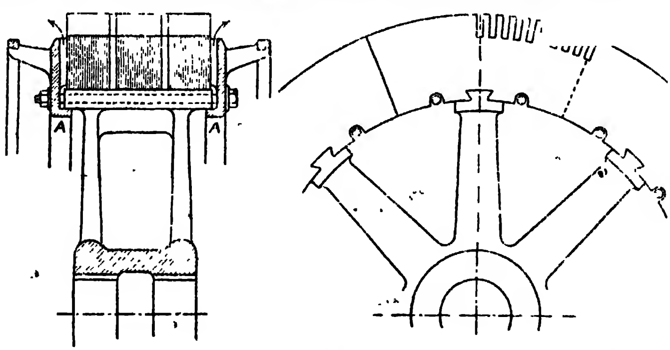


FIG. 55

ARMATURE CONSTRUCTION FOR LARGE MACHINE

The segments in any one layer butt one against another, but the joints in adjacent layers are staggered.

The core discs before being assembled have parallel sided slots notched out from their peripheries, and therefore in the completed core a series of axial slots is formed in which the armature winding is housed.

4. **The Commutator and Brush-Gear.** The commutator, the function of which is to rectify the alternating voltage induced in the armature conductors, is a cylindrical structure built up of segments of high-conductivity, hard-drawn copper, insulated from one another by mica. Fig. 56 shows the construction for a medium sized machine. The hub, *A*, of the commutator is mounted on an extension of the armature hub, a mechanically sound arrangement. The hub is provided at one end with a flange, *B*, and between this flange and a clamping ring, *C*, the commutator segments are mounted as shown. The segments have V-grooves so that they cannot fly out under the action of centrifugal force, and the insulation in these grooves is in the form of conical micanite rings.

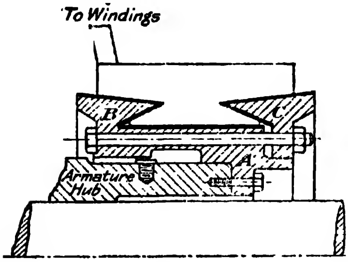


Fig. 56
Commutator Construction

The brushes, which are almost invariably of some form of carbon, are housed in brush holders. There are several forms of holder, the box type probably being the best. Fig. 57 shows a typical holder of this type. The holder, *A*, is mounted on a spindle, which passes through *B*, and the brush, *G*, can slide in a rectangular slot. It is pressed on the commutator by a spring, *C*, whose tension can be adjusted by placing the small lever, *E*, in one of the notches shown dotted. At the top of the brush a flexible copper "pig-tail" is mounted, and this conveys the current collected by the brush to the holder.

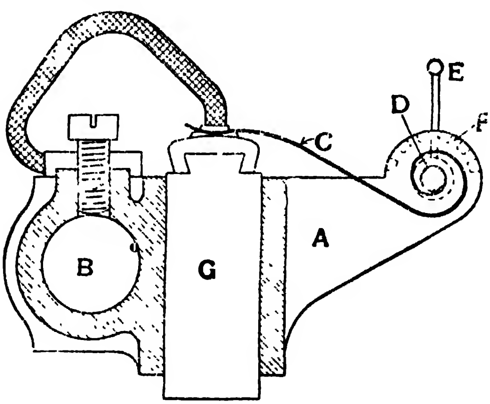


Fig. 57
Box Type Brush Holder

5. **Armature Windings.** All modern windings are what are called lap or wave winding, or modifications of these. The differences in the two types are illustrated in Fig. 58, and it will be seen that these differences consist merely in the arrangement of the

end connections at the front of the armature. The various "pitches" are also shown in this figure. The length of the end connection at the back, measured in terms of armature conductors, is called the back pitch y_B; the length of the front connection is the front pitch y_F. The two conductors which join two commutator bars, which are consecutive in the scheme of the winding, constitute a winding element, and the distance between the first and last conductors in an element is called the resultant pitch y. Hence,

$$y = y_B - y_F \text{ for a lap winding}$$
$$y = y_B + y_F \text{ for a wave winding}$$

The winding element may be regarded as the "repeat" of which the whole winding is composed, and it is obvious that all the conductors must be included. A reference to the figure shows that the end connections of consecutive conductors point alternately to right and left, and from this it follows that the partial pitches y_B and y_F must be *odd* numbers, and therefore y is an *even* number. In a simple lap winding y, reckoned in conductors, is 2. We also see from the directions of the arrows, which, in Fig. 55, represent the directions of the E.M.F.s in the various conductors, that since the poles are alternately N. and S., the partial pitches y_B and y_F must be approximately equal to the pole pitch measured in armature conductors.

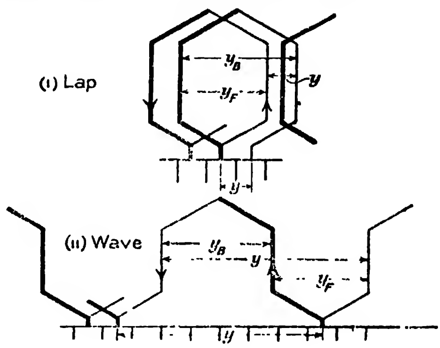


Fig. 58
Winding Pitches

6. Example of a Simple Lap Winding.

Let No. of poles, $2p$ $= 4$
No. of conductors, $N = 24$
$\therefore$ pole pitch $= 24/4 = 6$

Now the partial pitches must both be about the same as the pole pitch, but they must be odd numbers. Again, their difference must be equal to 2. Suitable values are, therefore,

$$y_B = 7; \quad y_F = 5.$$

The complete winding shown in the form of a developed view is given in Fig. 59. The connections to the commutator bars are

also shown, and it will be seen that there are half as many bars as conductors.

It is now necessary to find the brush positions, and this requires that the directions of the E.M.F.s should be inserted. As diagrams of armature windings are liable to be somewhat confusing, it is a good plan to draw the equivalent ring or spiral winding. The ring winding, now obsolete, consists simply of a solenoid wound on an annular ring, with equidistant tappings taken to the commutator bars. As there are no cross-overs of the end connections it is very easy to understand. The equivalent ring is shown in

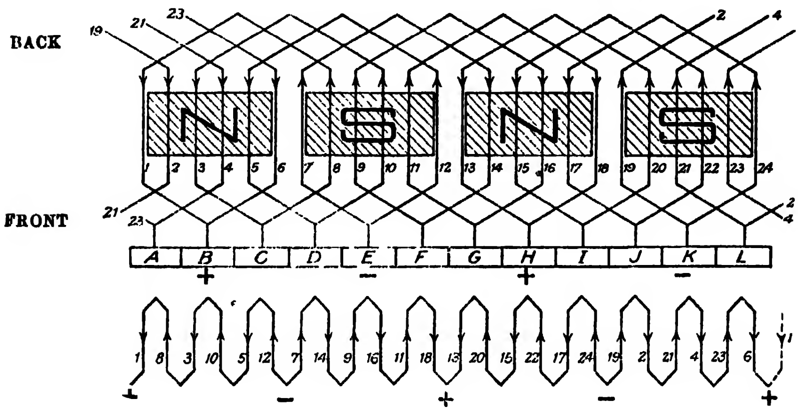


FIG. 59
SIMPLE LAP WINDING

the bottom half of figure, and the directions of the E.M.F.s are obtained by reference to the actual diagram. Thus, if the E.M.F.s in the conductors under the N. poles are assumed to be, say, downwards, then those in the conductors under the S. poles will be upwards. There will, of course, be some conductors in the inter-polar gaps, but if there is any doubt as to the direction of the E.M.F. in any particular conductor it need not be inserted. Tracing the general trend of the E.M.F.s through the equivalent ring, we find that there are two meeting points of two E.M.F.s and two separating points of two E.M.F.s. These are obviously the positions at which the + and − brushes respectively should be placed. It may happen that, according to the position of the poles relative to the winding, some of the brush positions may apparently come at the back of the armature. The brush, in such a case, must obviously be placed opposite to this position, but at the front.

The brushes of the same polarity are connected together, and therefore the armature winding is divided into four paths in parallel.

In general, the number of parallel paths through a lap winding is equal to the number of poles; the terminal E.M.F. is equal to the E.M.F. induced in one parallel path; the current delivered to the external circuit is equal to the current in each armature conductor multiplied by the number of parallel paths.

7. **Example of a Simple Wave Winding.** Sometimes called a series winding. In the simple lap winding all the conductors can be included in the winding if the number of conductors N is an even number. Any even number of conductors will not necessarily do for a wave winding, for when a number of winding elements equal to the number of pairs of poles has been passed through, the winding returns practically to the starting point, namely, two conductors in front of, or behind, the starting point. Hence, the product of the resultant pitch and the number of pairs of poles must be two greater or less than the number of conductors.

$$y \times p = N \pm 2$$

$$y = \frac{N \pm 2}{p}$$

$$\therefore \text{ Mean pitch, i.e. } \frac{y_B + y_F}{2} = \frac{y}{2} = \frac{N \pm 2}{2p}$$

As an example, take again $2p = 4$ and let N be in the neighbourhood of 24 as before. We see that $N = 24$ will not do, because the quotient of (24 ± 2) divided by 4 is not a whole number. $N = 26$ will do, and we then have $6\frac{1}{2}$ for the pole pitch. Suitable values for the partial pitches are therefore

$$\frac{y}{2} = \frac{26 \pm 2}{4} = 6 \text{ or } 7$$

Hence, $\quad y_B = y_F = 7$, if we take $y = 14$.

If we take $y = 12$, then one of the partial pitches must be equal to 7 and the other to 5, and in such a case it is usual to make y_B the smaller, because the value of y_B fixes the width of the armature coils, and therefore the amount of copper used in the coil ends when multi-turn coils are used. The values $y_B = 5$ and $y_F = 7$ are used in the winding shown in Fig. 60. It is assumed that the winding is in front of the poles and that it is moving from left to right, the induced E.M.F.s being therefore downwards in front of the N. poles and upwards in front of the S. poles. The equivalent ring winding is now drawn, and it will be noticed that the conductors in this winding are numbered, not consecutively, but in the order in which they are connected together in the scheme of the winding. On inserting the directions of the E.M.F.s as obtained from the actual winding, we see that the winding divides itself electrically into two halves, namely, the portion of the winding lying between points P and Q on the equivalent diagram, and that lying between

points R and P. In the first part, the E.M.F.s induced in the various conductors act in series with a general trend from left to right, while in the second part the general trend is from right to left. Hence, there can only be two parallel paths through the winding, and therefore only two brush sets are required, one positive, the other negative. The brush positions are found as follows. The point P is the separating point of the E.M.F.s in the two halves of the winding, and it therefore corresponds to the position of the negative brush. But it is at the back of the winding and not at the commutator end. We thus have two alternative positions for

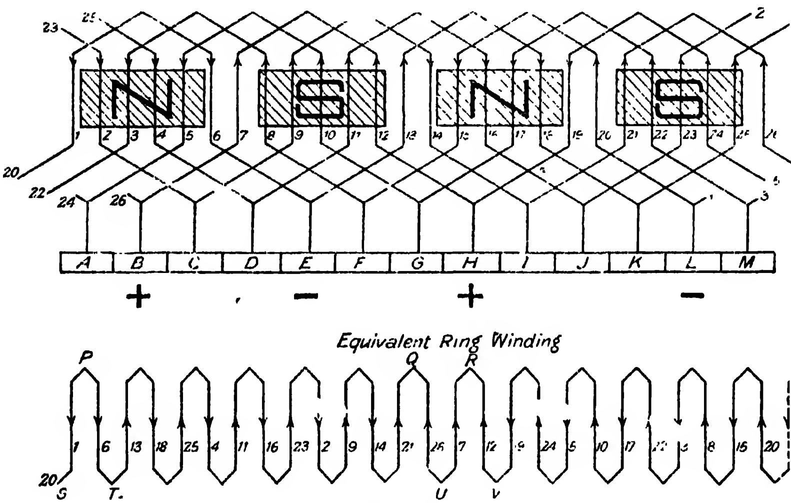


FIG. 60
SIMPLE WAVE WINDING

the negative brush, namely, points S and T on the equivalent winding, which correspond to the points on the actual winding from which tappings are taken to the commutator segments L and E respectively. For the placing of the positive brush, we see from the diagram that there are two meeting points of E.M.F.s, namely, Q and R, and both of these are at the back of the winding. They are separated by one loop only, that composed of conductors 26 and 7, and therefore the middle point U of the loop is the required position. This corresponds to the tapping brought out to the commutator segment B. It appears, therefore, that the E.M.F.s in these two conductors oppose the E.M.F.s acting in the parallel paths in which they lie, but on referring to the actual winding it will be seen that they are each situated almost in the middle of an interpolar gap, the E.M.F.s induced in them being therefore very small.

For one positive brush position, we thus see that there are two alternative positions for the negative brush. Now consider the two conductors 26 and 7 situated between the points Q and R. The winding is moving across the pole faces so that its position relative to the poles as represented by the figure is only instantaneous. We see that conductor 26 is just about to move from the influence of a S. to that of a N. pole, and the E.M.F. in it is therefore on the point of reversing. But conductor 7 is already past the point where its E.M.F. is reversed, so that its E.M.F. will not change in direction but will gradually increase in magnitude. This means that in a very short interval of time the point R will become the meeting point of two E.M.F.s, and since it is at the back of the winding, there will be two alternative positions for the positive brush, namely, point U, which has already been considered, and the point V, corresponding to commutator segment H. Similarly, when the armature has moved a little farther forward the segment I adjacent to H will become the second alternative position for the positive brush. Hence, if one positive brush is making contact with segment B and not either of the adjacent segments, then the second positive brush, if used, should be in contact with both of the segments H and I.

If brushes are placed in both alternative positions for both positive and negative, the effect is merely to short-circuit one loop of the winding between two brushes of the same polarity. Thus, whether only two or four brushes are used, the number of parallel paths still remains at two. Also these loops during the very short period of short circuit occupy a position almost symmetrical with respect to one of the poles, so that the actual E.M.F. acting round the loop and tending to produce circulating currents through the loops and connections to brushes of like polarity is very small.

The E.M.F. developed by a simple wave winding is therefore equal to that induced in one half of the total number of armature conductors all connected in series, whereas the current delivered to the external circuit is twice the current in an individual armature conductor.

8. **Multi-turn Windings.** The single-turn windings, described above, are used whenever possible. Such windings necessitate half as many commutator bars as armature conductors. If the voltage to be generated is high, particularly in a small machine, it will obviously be impossible to house only two conductors in a slot, one at the top and the other at the bottom, since this would necessitate too many slots. In such a case, six or eight conductors may be placed in one slot, those at the top being taped together, and also those at the bottom. The coils so formed are still single-turn coils, since they each consist of three or four single turns, insulated from one another, and mounted together for convenience.

In the case of small machines it is necessary to adopt multi-turn coils. The winding in such a case is identical with that for single-turn coils, except that each loop of the simpler winding is replaced by a coil of the required number of turns. This is illustrated clearly by Fig. 61. Comparing the multi-turn lap and wave windings, it will be seen that the arrangement of the coils in the two cases is identical. It is only the arrangement of the connections to the commutator which decides whether the winding will be lap or wave.

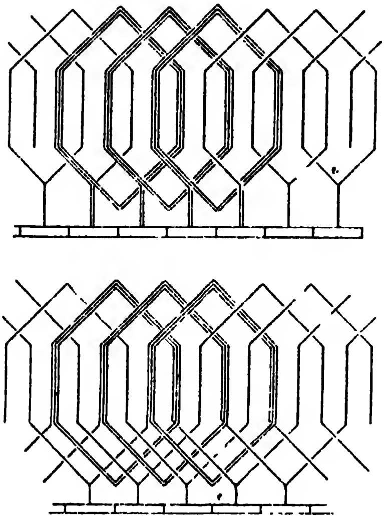


FIG. 61
WINDINGS WITH MULTI-TURN COILS

Since the construction of an actual winding is by no means obvious from an examination of the developed diagrams, such as those given in Figs. 59 and 60, Fig. 62 has been drawn. This figure shows the actual arrangement of a four-pole lap-wound armature, having as many elements as the winding in Fig. 59, namely, 24. Each coil consists of several turns, and there are two coil sides per slot, each coil side consisting of only one element. There are, therefore, half

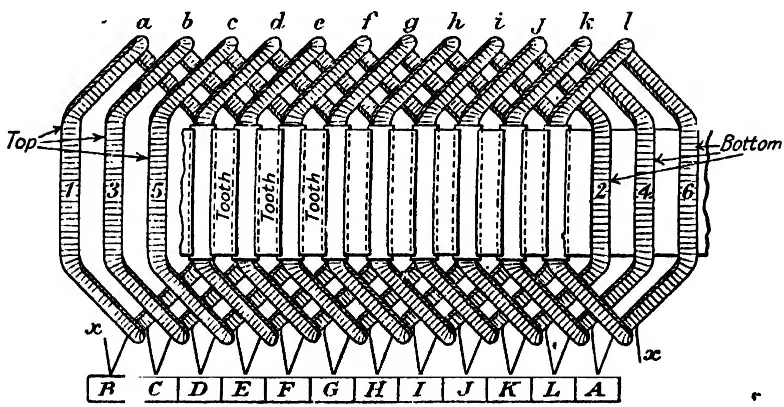


FIG. 62. FOUR-POLE LAP-WOUND ARMATURE

as many slots as elements, that is, 12 slots in the example. It would, of course, be possible to tape up coils a and b into a composite coil

with two elements per coil side, similarly with coils *c* and *d*, and so on. In such a case the number of slots would be only one-half the

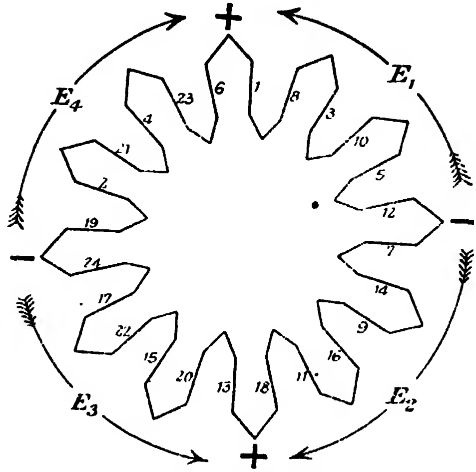


FIG. 63. BALANCE OF E.M.F.S IN A LAP WINDING

number required when there are only two elements per slot. Similarly, if coils *a*, *b*, and *c* were taped up into a composite coil, likewise

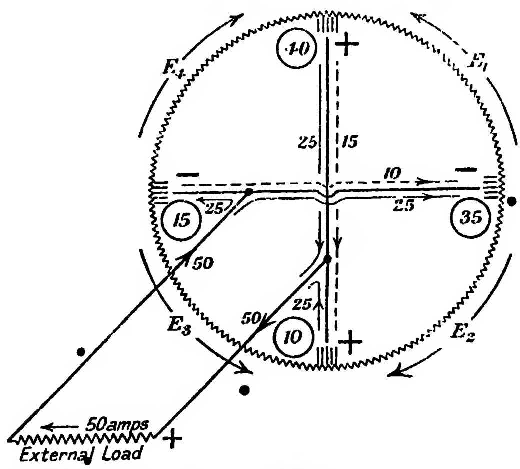


FIG. 64. EFFECT OF BRUSH CONNECTIONS ON DISTRIBUTION OF CURRENT IN A LAP WINDING

d, *e*, and *f*, and so on, the winding would still be essentially the same, although it would now only require one-third as many slots as when there are only two elements per slot. For a clear understanding of

the completed winding it is therefore necessary to realize that (*a*) the winding is arranged in two layers, and (*b*) that what appears to be a single coil on the armature may in reality consist of several quite independent coils all taped up together. This is indicated by a comparison of the number of commutator bars and of coil sides.

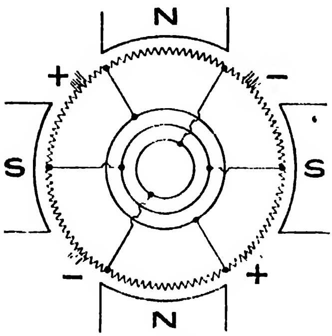


FIG. 65

EQUALIZER CONNECTIONS

9. **Equalizer Rings.** In a lap winding any path from brush to brush consists of conductors under the influence of two adjacent poles only and not all the poles. If the fluxes from all the poles are equal, the E.M.F.s induced in all the paths will be equal, and when there is no current in the external circuit there will be zero current in the armature. Actually it is impossible to obtain absolutely equal fluxes from each pole, the result being that the E.M.F.s in the various parallel paths are not equal. The resulting E.M.F. acting round the armature may give rise to large currents even with no external load. When the armature is delivering current the circulating current is superposed on the load current and produces unequal heating in the armature.

Referring, again, to the lap-winding of Fig. 59, if we take the conductors lying between the first + brush and the first − brush positions, we see straight away from the equivalent ring winding that these are the conductors numbered 1, 8, 3, 10, 5, and 12. The poles influencing these conductors are the first north and south poles. Similarly with the other three parallel paths, the conductors in them are influenced by two poles only. It therefore follows that if the magnetic fluxes entering the armature from each of the poles are not exactly equal, the E.M.F.s induced in the various parallel paths will not be equal. Since the armature winding forms a closed circuit independent of any external connections to the brushes, and since, with respect to the armature itself, these paths are *in series*, we see that the E.M.F.s set up in the armature will not exactly balance out. Hence, a parasitic current will be set up in the armature winding, this current being independent of, and superposed on, the current due to the external load. In order to illustrate this point, the equivalent ring winding of the lap winding previously considered is reproduced in a modified form in Fig. 63. The E.M.F.s in the four paths are indicated as E_1, E_2, E_3, and E_4, their directions being indicated by the arrows. The resultant E.M.F. acting round the winding in a given direction, say, clockwise, is

$$(E_2 + E_4) - (E_1 + E_3)$$

If these four E.M.F.s are exactly equal, their resultant will, of

course, be zero, and no parasitic currents will be set up round the winding. If they are not exactly equal, but have a resultant equal to, say, e, then a circulating current will be set up, its magnitude being equal to

$$i = \frac{e}{R}$$

where R is the resistance of all the armature conductors and their end-connections reckoned in series, and not the armature resistance

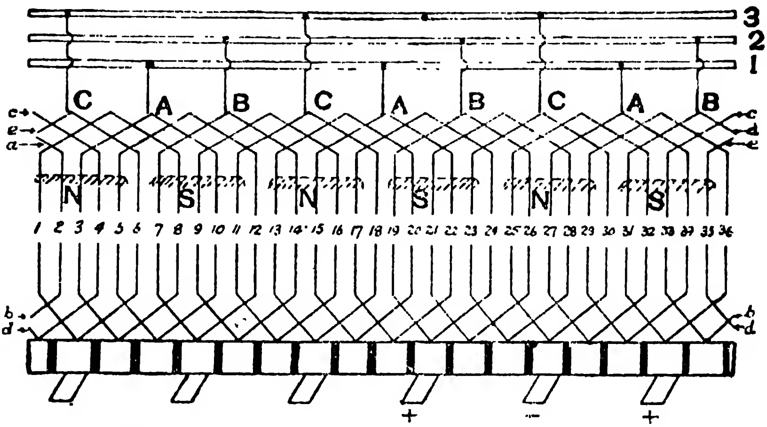


Fig. 66. Six-pole Lap Winding, showing Connections to Equalizer Rings, 36 Conductors

as measured from brush to brush. This current will be quite independent of the external load on the machine, in fact, it will flow even when the brushes are lifted from the commutator.

When the brushes are in place and the machine is delivering current to an external load, the conditions are not quite so simple. In the simplified scheme of Fig. 63 it appears that since the four E.M.F.s, E_1, E_2, E_3, and E_4, are not equal, the two positive brushes can be at slightly different potentials, and also the two negative brushes. Actually such differences of potential between brushes of like polarity cannot exist, because such brushes are joined together externally by copper straps of negligible resistance. These connections are shown in Fig. 64, in which the winding is again represented as a ring winding for simplicity. It is obvious that currents will flow along these connections, and since their magnitudes are only limited by the resistance of the armature and the brush-contact resistances, they may assume very large proportions. In the figure they are indicated as 10 and 15 amp., while the load current is represented as 50 amp. Now the load current divides into two halves of 25 amp. each at the brush connections, and these halves divide

again into quarters of $12\frac{1}{2}$ amp. each, through each armature path, these currents all being indicated in the figure. Hence, when we superpose the circulating current through the brush connections on to the true load currents, we find that the currents collected by the four brush sets are 40, 35, 10, and 15 amp., as indicated by the numbers in small circles. Actually each brush should collect 25 amp. Obviously the overloaded brushes will give trouble; their temperature rise will become much too high, and they will probably spark badly, no matter how their position may be adjusted. It is to be noted, however, that in the case of lap (not wave) windings, any very excessive unbalancing of the current in the different armature paths is to some extent automatically checked by the effect of armature reaction, to be considered later.

It will be seen that the conditions represented by Figs. 63 and 64 are very different. Without the connections between brushes of the same polarity the brush potentials are incorrect, but when these connections are made, as in Fig. 64, the brush potentials are corrected, but at the expense of incorrect proportioning of the current between the various brush sets. Now, these connections put the potentials right, and therefore if similar connections could be made without involving the brushes at all, the potentials and current distribution *at the brushes* would be corrected. The clue is given by the fact that the brushes joined together are all at the same potential, and therefore all that is necessary is to join together by a permanent connection (i.e. not involving the brushes) points on the winding which should be at the same potential. Such points are very easy to find from the wiring diagram. Thus, in Fig. 59 conductors 1 and 13 occupy exactly the same positions with respect to the two north poles, so that their potentials are the same. Also, conductors 8 and 20 are at the same potential, and since conductors 1 and 8 form one loop, while 13 and 20 form another loop, the middle points of their end connections, namely, the apexes of the end connections joining 1 to 8 and 13 to 20, could be connected together. Similarly other pairs which could be joined together can be easily found. Actually the number of points to be joined by any one connection is equal to half the number of poles. Such connections are called equalizing connections, and they are usually carried out by rings joined to tappings taken off from the back of the armature. Fig. 65 shows the elementary scheme of equalizer connections, the number of rings in this case being three. A very large armature may have as many as 8 or 10 equalizer rings. A more complete diagram of equalizer connections is given in Fig. 66. Notice that there are three tappings to each equalizer, three being half the number of poles.

10. **Selective Commutation.** Since each path in a wave-wound armature consists of conductors distributed under all the poles, inequalities in the fluxes from the poles do not produce inequalities

in the total E.M.F.s induced in each of the two paths. As a result, equalizer connections are not required with a simple wave winding. If more than two brush sets are used, then trouble with the brushes may be produced by what is called "selective commutation." The use of more than two brushes does not divide the armature into more than two parallel paths, but the current collected from the armature is divided between the brushes of like polarity. If there are differences in contact resistance, then it is easy to see from the diagrams in Fig. 67 that the currents picked off by the individual brush sets may be different. Thus the two brushes A and B should each collect a current I, where $2I$ is the load current, but if

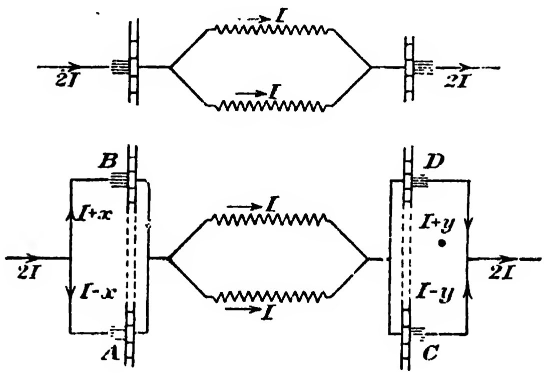


Fig. 67

Selective Commutation

the contact resistance of A is greater than that of B, A will obviously collect less current than B, so that B will be overloaded. Similarly with brushes C and D. The brushes must therefore be adjusted very carefully when more than two brush sets are used with a wave winding. The advantage of using the maximum number of brush sets is that the current per brush set is reduced, and therefore the axial length of the commutator is reduced. This gives a cheaper machine.

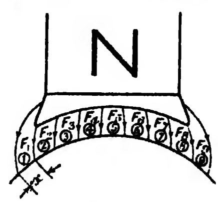


Fig. 68

11. **E.M.F. Equation.** Let Φ be the flux per pole, Z the number of armature conductors, and N the speed in r.p.m. Divide the flux Φ into portions Φ_1, Φ_2, Φ_3, etc., embracing the various conductors, as in Fig. 68. Let x be the spacing of the conductors and t the time in seconds required to move the distance x.

$\therefore$ E.M.F. induced in conductor 1

$$= \frac{\Phi_1}{t} \times 10^{-8} \text{ volts}$$

E.M.F. induced in conductor 2

$$= \frac{\Phi_2}{t} \times 10^{-8} \text{ volts}$$

and so on.

$\therefore$ E.M.F. induced in all the conductors under one pole

$$= \frac{\Phi_1}{t} \times 10^{-8} + \frac{\Phi_2}{t} \times 10^{-8} + \ldots\ldots$$

$$= \frac{1}{t} \times 10^{-8} (\Phi_1 + \Phi_2 + \Phi_3 + \ldots\ldots$$

$$= \frac{\Phi}{t} \times 10^{-8}$$

Now the time of one revolution $= \frac{1}{N}$ minute

$$= \frac{60}{N} \text{ seconds}$$

$\therefore$ Time taken to move the distance x, which is $1/Z$ of the periphery,

$$t = \frac{60}{NZ} \text{ seconds}$$

$\therefore$ E.M.F. induced in all the conductors under one pole

$$= \frac{\Phi ZN}{60} \times 10^{-8} \text{ volts}$$

$\therefore$ E.M.F. induced in all the conductors under all the P poles, *if all the conductors were in series*

$$= \frac{\Phi ZN}{60} \times P \times 10^{-8} \text{ volts}$$

But there are, say, A parallel paths through the winding, and the

E.M.F. induced in the winding is the same as that in one parallel path, namely,

$$E = \frac{\Phi ZN}{60} \times \frac{P}{A} \times 10^{-8} \text{ volts}$$

Remember that for a lap winding $A = P$, but for a wave winding $A = 2$.

Example. A two pole generator has an armature 12 in. long, 12 in. diameter, and the radial depth of the iron in the core is 3 in. Each pole subtends an angle of 120° and the flux density in the air gap is 5,000 lines per sq. cm. If there are 200 conductors and the armature rotates at 1,000 r.p.m., find (*a*) the E.M.F. induced; (*b*) the flux density in the core.

Each pole subtends one-third of the periphery,

$$\therefore \text{ Area of air gap} = 12 \times \frac{\pi \times 12}{3} \times 2{\cdot}54^2 \text{ sq. cm.}$$

$$= 975 \text{ sq. cm.}$$

$$\therefore \text{ Flux per pole } \Phi = 975 \times 5{,}000 = 4{,}875{,}000 \text{ lines.}$$

Since $P = 2$, A also must be 2.

$$\therefore E = \frac{\Phi ZN}{60} \times \frac{P}{A} \times 10^{-8} = \frac{4{,}875{,}000 \times 200 \times 1{,}000}{60} \times \frac{2}{2} \times 10^{-8}$$

$$= 162 \text{ volts.}$$

The flux through the cross section of the core is one-half the useful flux, i.e. 2,440,000 lines.

Cross section of core = axial length × radial depth
= 36 sq. in. or 233 sq. cm. gross.

A portion of this will be taken up by insulation between core discs, and if 10 per cent is allowed for this the net section will be

$$233 \times {\cdot}9 = 210 \text{ sq. cm.}$$

$$\therefore \text{ Flux density in core} = \frac{2{,}440{,}000}{210} = 11{,}600 \text{ lines per sq. cm.}$$

Examples on Chapter V.

(1) The two-circuit armature of a four pole generator has 51 slots, each slot *containing 20 conductors. What will be the voltage generated in the machine when driven at 1,500 r.p.m., assuming the useful flux per pole to be* 0·7 megalines? (*London Univ., 1915.*)

Ans.—*357 volts.*

(2) State Faraday's Law of magneto-electric induction, and find *from first principles* the voltage of a four-pole lap-wound continuous current generator having a pole flux of 2×10^6 lines, and 560 conductors running at a speed of 980 r.p.m. How is the terminal voltage affected by load? (London Univ., 1911.)

***Ans.*—183 volts.**

(3) Calculate the resistance of an armature from the following data—

Number of slots = 150.
Conductors per slot = 8.
Mean length of one turn = 250 cm.
Cross section of each conductor = $1\cdot0 \times 0\cdot25$ cm.
Number of parallel paths through armature = 6.
Specific resistance of copper at working temperature = 2×10^{-6} ohm per cm. cube.

(If the resistance of one parallel path is calculated, the actual resistance will be one-sixth of this.)

Ans.—·033 ohm.

(4) A four-pole armature is built up of 400 core discs each $\frac{1}{40}$ in. thick; the discs are 12 in. external and 6 in. internal diameter. There are 600 conductors and the armature is wave wound. If the flux density in the armature is 10,000 lines per sq. cm., and the speed is 1,000 r.p.m., what will be the E.M.F. generated?

Ans.—778 volts.

(5) The exciting winding of a dynamo has 2,000 turns per pole and the exciting current is 2 amperes. The armature, which is lap wound, has 500 conductors, rotates at 1,200 r.p.m., and gives 100 volts. Calculate the reluctance of the magnetic circuit.

Ans.—The reluctance of an individual magnetic path is 0·005.

(6) A 40 h.p., 500-volt traction motor has four poles. There are only two brush arms. Assuming that there are 41 slots and 123 commutator bars, make a diagram sufficient to explain how the armature winding may be arranged and connected to the commutator. It is sufficient to show the connections of three coils. (London Univ.)

(7) What is the object of equalizing connections on a c.c. generator, and to what class of winding are they applied? Give a diagram to show the way in which these equalizers are connected up to the armature winding. (London Univ., 1909.)

CHAPTER VI

ARMATURE REACTION AND COMMUTATION IN D.C. GENERATORS

1. **Armature Reaction.** Armature reaction is the effect of the magnetic field set up by the armature currents on the distribution of flux under the main poles. Consider an armature rotating in a clockwise direction in a bi-polar field, as shown in Fig. 69. Let the brushes make contact with those conductors which lie in the geometric neutral plane (G.N.P.). Then all the conductors under the N. pole carry currents whose directions are inwards, while the currents in the conductors under the S. pole are outwards. This distribution of armature currents is magnetically equivalent to a solenoid carrying current, and we therefore see that the armature sets up a M.M.F. directed along the brush axis, in this case from right to left. The main M.M.F. is downwards, and if we represent these two M.M.F.s by the vectors OA and OB respectively we have OC for the resultant M.M.F. The magnetic neutral plane (M.N.P.) is perpendicular to OC, but on no load, when there is no armature reaction, M.N.P. is coincident with G.N.P. The effect of armature reaction is thus to shift the M.N.P. round in the direction of rotation. The magnitude of the shift obviously depends upon the length of OA, and therefore on the magnitude of the armature current. The lines of force produced by the armature reaction take the path across the pole faces, as shown by the dotted lines. Each line of force crosses the air gap twice, and, comparing the directions of the armature and main fluxes, we see that the field strength in the gaps is weakened under the leading pole tips and strengthened under the trailing pole tips. The distributions of gap flux density on load and on no load are shown in Fig. 70, curves I and II. If the armature teeth are not saturated the weakening of the field under the leading pole tips will be equal to the strengthening under the trailing pole tips and the total flux will be unaltered. The areas

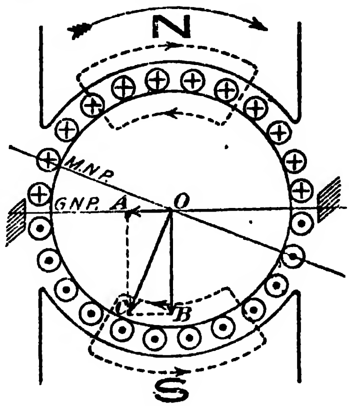


FIG. 69

ARMATURE REACTION IN A GENERATOR

under the two curves in Fig. 70 will thus be equal. If the teeth are strongly saturated on no load the ampere-turns producing the armature cross magnetizing field, as it is called, will not be able to increase the flux through the teeth under the trailing half of the pole to the extent that they are able to demagnetize the teeth under the leading half. The result is that, on the whole, there will be a certain amount of demagnetizing action as well as distortion of the main field

2. Position of Brushes. Since the meeting and separating points of two E.M.F.s occur at the *magnetic* neutral planes, and the brushes are placed at these points, it is obvious that the brushes must be given a "forward lead" in order to bring them into the

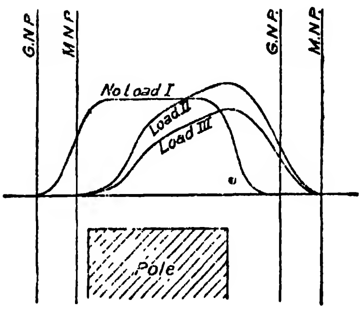


Fig 70

Modification of Air-gap Density due to Armature Reaction

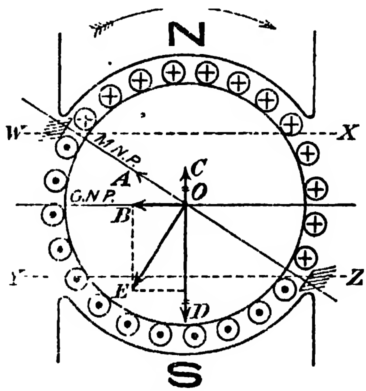


Fig. 71

Armature Reaction in a Generator

resultant M.N.P. This condition is shown in Fig. 71. The total armature M.M.F., OA, still acts along the brush axis and it can now be resolved into two components, the cross magnetizing component, OB, acting along the G.N.P., and a demagnetizing component, OC, acting in opposition to the main M.M.F., OD. The resultant, OE, of all these M.M.F.s is of course perpendicular to the M.N.P. We see that when the brushes are given a forward lead the armature exerts a definite demagnetizing as well as distorting effect. The armature distorting, or cross magnetizing ampere-turns are those lying above the horizontal WX and below the horizontal YZ; the demagnetizing ampere-turns are those lying between WX and YZ, and they are comprised within an angular distance of twice the brush lead. The distribution of air gap density is now represented by curve III in Fig. 70.

3. Armature Ampere-turns. It is obvious that the armature demagnetizing ampere-turns must be neutralized by adding extra

ampere-turns on the main field winding. It is, therefore, necessary to be able to calculate the armature ampere-turns per pole. Fig. 72 shows a portion of a multipolar armature the brushes of which have been given a forward lead of θ. There are thus $\frac{2\theta}{360} \cdot Z$ demagnetizing conductors in each interpolar gap, that is $\frac{\theta}{360} \times Z$ on either side of each pole tending to demagnetize it. These will act as though they were joined together as shown, and they will produce $\frac{\theta}{360} \times Z$ demagnetizing *turns* per pole. Hence, if I is the current flowing in each conductor, demagnetizing ampere-turns per pole,

$$AT_d = \frac{\theta}{360} \times ZI$$

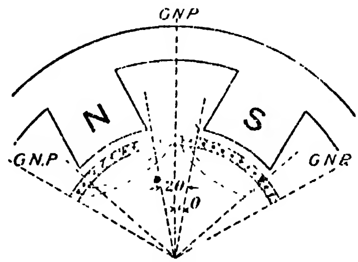


FIG. 72

CALCULATION OF ARMATURE REACTION

The total ampere-conductors per pole, both cross and demagnetizing,

$$= \frac{ZI}{2p}$$

$\therefore$ Total armature ampere-turns per pole

$$= \frac{ZI}{4p}$$

$\therefore$ Cross magnetizing ampere-turns per pole,

$$AT_c = ZI\left(\frac{1}{4p} - \frac{\theta}{360}\right)$$

Example. A four-pole generator has a wave wound armature with 722 conductors, which delivers 50 amperes on full load. If the brush lead is 8°, calculate the armature demagnetizing and cross magnetizing ampere-turns per pole.

$$\text{Current per conductor } I = \frac{I_a}{2} = \frac{50}{2} = 25 \text{ amperes}$$

$$AT_d = \frac{8}{360} \times 722 \times 25$$

$$= 400$$

$$AT_c = 722 \times 25\left(\frac{1}{8} - \frac{8}{360}\right)$$

$$= 1850.$$

4. Commutation. Whenever a brush spans two commutator segments the winding element connected to those segments is short-circuited. By commutation we mean the changes that take place in a winding element during the period of short circuit by a brush. These changes are illustrated in Fig. 73, the winding elements being drawn as portions of a ring winding for simplicity. In (*a*) the element *B* is on the point of being short-circuited and it is carrying, in a direction from left to right, half the current delivered by the armature to the brush; (*b*) shows the element in the middle of its short circuit period, and it will be seen that it is possible for the currents to reach the brush without passing through this element; in (*c*) the same element *B* is shown immediately after short circuit and in this position it is, *or should be*, carrying the full current in a direction from right to left. We thus see that during the period of short circuit by a brush the current in a short-circuited element should be reversed *and brought up to its full value in the reversed direction.* If the current in *B* has not attained its full value in the position shown in (*c*), then, since element *C* is carrying the full current, the difference between the currents through elements *C* and *B* has to jump from the commutator bar *b* to the brush in the form of a spark. Thus the cause of sparking at the commutator is the failure of the current in the short-circuited elements to reach the full value in the reversed direction by the end of short circuit. The curve of current against time in such a case is shown in Fig. 74, curve I; what is required is a curve of current similar to curve II.

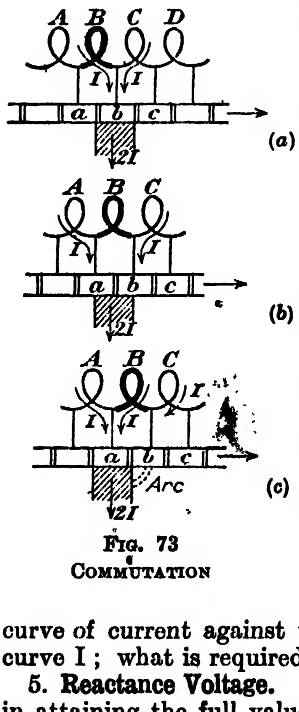


FIG. 73
COMMUTATION

5. Reactance Voltage. The difficulty experienced by the current in attaining the full value in the reversed direction by the end of short circuit is due to the fact that the rate of change of current is so great that the self-induction of the coil sets up a back E.M.F. which opposes the reversal. Since the current in the coil has to change from $+I$ to $-I$, the total change is $2I$. If t is the time of short circuit and L the self-induction of the coil, then the average value of the self-induced E.M.F. is

$$L \times \frac{2I}{t}.$$

This is called the "reactance voltage." The self-induction, L, is

here a composite quantity, being made up of the true self-induction of the short-circuited coil and the mutual inductions of its neighbouring coils. An approximate value for L can be determined by Hobart's Rule. Fig. 75 shows an armature coil ; the sides AB and CD are embedded in iron, whereas the triangular ends are free.

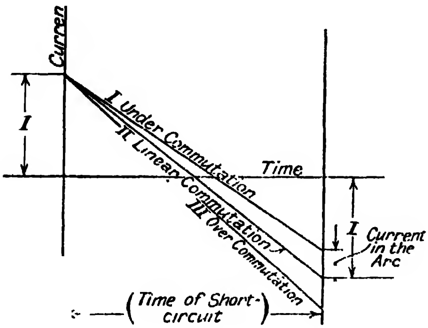


FIG. 74

CURRENT CHANGES IN A SHORT-CIRCUITED ELEMENT

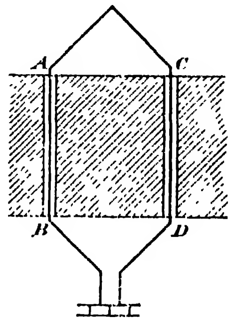


FIG. 75

TO ILLUSTRATE HOBART'S RULE

For slots of normal proportions, Hobart has found by experiment that 1 amp. flowing through a coil of a single turn produces a flux of 4 lines per cm. of embedded length, and 0·8 line per cm. of free length. Hence, l_1 and l_2 are the total embedded and free lengths respectively, flux ampere due to the whole coil

$$= 4l_1 + 0{\cdot}8l_2 \text{ for a single turn coil}$$

and $\quad 4nl_1 + 0{\cdot}8nl_2$ for a coil of n turns.

The self-induction of the coil is therefore

$$\text{Flux per ampere} \times \text{No. of turns} \times 10^{-8}$$

$$= n^2(4l_1 + 0{\cdot}8l_2) \times 10^{-8} \text{ henrys.}$$

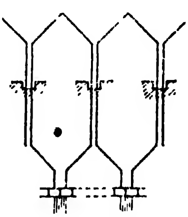


FIG. 76

SHORT-CIRCUITED ELEMENTS IN A LAP WINDING

The effect of neighbouring coils can be taken into account by writing for the effective coefficient of self-induction of a short-circuited coil

$$L = \begin{pmatrix}\text{Total flux through coil per ampere,}\\ \text{due to self and neighbouring coils}\end{pmatrix} \times \text{No. of turns} \times 10^{-8}$$

Fig. 76 shows the disposition of the short-circuited coils in a lap winding. The flux per ampere through one coil due to the embedded length is twice that which would exist if the coil had no neighbours, but the flux due to the free length is not influenced by the neighbours.

∴ Total flux per ampere through a coil of n turns

$$= 8nl_1 + 0{\cdot}8nl_2$$

and the effective coefficient of self-induction is

$$L = n^2(8l_1 + 0{\cdot}8l_2) \times 10^{-8} \text{ henrys.}$$

This applies to the case of a brush short-circuiting only one coil at a time. If the brush width is such that it short-circuits m coils at a time, then, since these m coils will probably lie in the same slots, the flux through a coil will be increased m times. We then have

$$L = mn^2(8l_1 + 0{\cdot}8l_2) \times 10^{-8} \text{ henrys.}$$

This expression shows that L is proportional to the square of the number of turns in an armature coil. For good commutation it is, therefore, advisable to use single turn coils. This can be done in large machines where the large number of commutator bars required by such a winding does not make the bars too narrow, but it is obviously impossible with small machines.

It is obvious from Fig. 75 that the period of short circuit is the time taken for a point on the commutator to move a distance equal to the brush width, less the thickness of the mica insulation between segments. Calling these w_b and w_m respectively, and the peripheral velocity of the commutator in cm. per second, v, we have

$$\text{Period of short circuit } t = \frac{w_b - w_m}{v} \text{ seconds.}$$

It is, of course, impossible to give maximum allowable values for the reactance voltage to cover all cases. In general, it can be taken that it should not exceed 0·6 or 0·7 volt in non-interpole machines. For a large interpole machine it may be as high as 15 volts.

6. **Interpoles.** There are two methods of making the current in the short-circuited element attain its full value in the reversed direction by the end of short circuit. They are known as E.M.F. and resistance commutation respectively. In E.M.F. commutation the short-circuited coil has a voltage induced in it which neutralizes the reactance voltage. This induced voltage must therefore be in the same direction as the final direction of the current. The magnetic field required to induce this E.M.F. is called the commutating field. One method is to shift the brushes so that they lie, not in the M.N.P., but in the fringe of the field produced by the next main pole farther ahead. This field will induce an E.M.F. in the required direction in the short-circuited coils, with the result that if the brushes are advanced sufficiently beyond the M.N.P., sparkless commutation will be obtained. This method is now obsolete because, since the effect of armature reaction is to weaken the field under the leading pole tips, and this weakening

increases as the load increases, a very large brush shift is required when the load is heavy. Also the brush position has to be adjusted when the load varies, an obvious disadvantage. A better method of providing the commutating field is to make use of interpoles. These are small auxiliary poles placed in the geometric neutral planes, that is, midway between the main poles. Their polarity must, in the case of a generator, be that of the next main pole farther ahead, as shown in Fig. 77. Since the commutating field produced by them has to be proportional to the armature current, they are series excited (*see* Chap. VII). The neutralization of the

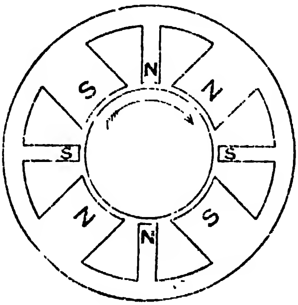


FIG. 77

FIELD FRAME WITH INTERPOLES

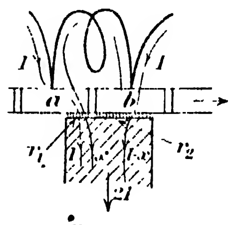


FIG 78

EFFECT OF BRUSH CONTACT RESISTANCE

reactance voltage is thus rendered automatic when interpoles are used, and modern machines will operate between no load and 20 or 25 per cent overload with fixed brush position without appreciable sparking. The most important advantage of the use of interpoles is that they raise the sparking limit of a machine to about the same value as the heating limit, so that for a given output an interpole machine can be made smaller, and therefore cheaper, than a non-interpole machine.

It must be noticed that for a given armature current there is a proper value for the commutating field, and that it is possible for this field to be too strong. In such a case the reversed current in the short-circuited coil is forced to too high a value by the end of short circuit, and sparking at the commutator takes place in the reversed direction. This is called over-commutation and is illustrated graphically by curve III, Fig. 74.

7. **High Resistance Brushes.** The second method of obtaining good commutation is to use high resistance brushes. When the current flowing from the right (Fig. 78) reaches the commutator bar b, there are two parallel paths open to it. The first is straight across the bar b to the brush, the second, round the short-circuited element in a counter-clockwise direction and across the bar a.

With brushes having a low contact resistance there is no inducement for the current to take the second path. With carbon brushes, which have a high contact resistance, more and more of the current flowing to the brush from the right hand will be shunted round the element as the bar b passes the brush, because the area of contact of bar b with the brush is diminishing, and its contact resistance, r_2, increasing, while the area of contact of the brush with bar a is increasing, and its resistance, r_1, therefore decreasing. Carbon brushes have therefore almost entirely replaced copper brushes. The disadvantage of carbon brushes is that they can only be worked at a current density of about 40–50 amp. per sq. in., as compared with 150–200 for copper brushes. This necessitates a larger commutator. The properties of a few grades of carbon brush are shown below—

	Max. Current Density, Amp. per sq. in.	Max. Contact Resistance, Ohms per sq. in.	Pressure on Commutator, Lb. per sq. in.
Copper	200	·003	1·5
Ordinary Carbon	40	·04	2·0
Electrographitic	60	·02	2·0

To illustrate the increase in contact resistance obtained by using carbon instead of copper brushes, take the case of the ordinary carbon brush. For the same area of brush,

$$\frac{\text{Contact resistance of carbon brush}}{\text{Contact resistance of copper brush}} = \frac{\cdot 04}{\cdot 003} = 13$$

But for the same current collected, the area of the carbon brush is $200/40 = 5$ times the area of the copper brush. Hence, since the contact resistance is inversely proportional to the area, we have for the same current collected

$$\frac{\text{Contact resistance of carbon brush}}{\text{Contact resistance of copper brush}} = \frac{13}{5} = 2\cdot 6$$

This is sufficient to give improved commutation. In practice, both E.M.F. and resistance commutation are used together in the same machine.

8. **Compensating Winding.** In the case of large direct current machines subjected to very violent fluctuations in load, e.g. turbo-generators, motors for rolling mills, and colliery winders, it is usual to neutralize the cross-magnetizing effect of armature reaction by providing the field with a compensating winding. Whenever the load changes, the flux density in the air gap changes, as we have seen. A sudden change in load will produce a sudden change in flux threading the armature coils, and this in turn will set up a

statically induced E.M.F. in these coils. The magnitude of this induced E.M.F. will depend upon the rapidity of this change of load, and it will be very high if the change of load is very great and takes place almost instantaneously. This induced E.M.F. will appear as a voltage between consecutive commutator bars, and if it is great enough it will give rise to a flash over, that is, an arc completely encircling the commutator, and thereby short-circuiting the whole armature. Since the distortion of the field which produces this phenomenon is caused by the armature conductors which lie under the pole faces, it can be eliminated by neutralizing the magnetic effect of those conductors. This is done

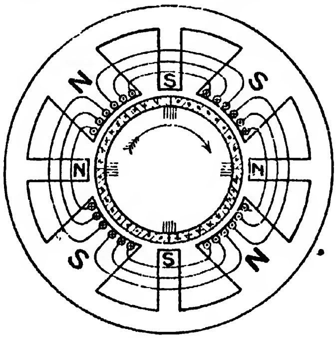


Fig. 79
Multipolar Field with Compensating Windings

by housing the compensating winding in the pole faces and connecting it in series with the armature in such a way that the currents in the armature and compensating windings are in opposite directions. The arrangement of these windings on a four-pole interpole generator is shown in Fig. 79.

9. **Special Armature Windings.** The methods discussed above for the prevention of sparking at the brushes are methods which are external to the armature itself. Obviously, if it is at all practicable, the armature itself should be wound in such a manner that its reactance voltage shall be as small as possible. This voltage is proportional to the self-induction of the coil undergoing commutation, and is therefore proportional to the square of the number of turns in the coil. Thus, the fewer the turns per coil the smaller will be the reactance voltage, for which reason it is preferable to use only single-turn coils whenever possible. Again, when a brush spans more than two segments, two or more neighbouring coils may be undergoing commutation at the same instant, in which case they will induce E.M.F.s in one another by the process of mutual induction. The total reactance voltage in any coil is thus the sum of its

own self-induced E.M.F. and the mutually induced E.M.F.s set up in it in virtue of the changes of current taking place in neighbouring coils. It is, therefore, desirable that these mutually induced E.M.F.s, as well as the self-induced E.M.F., shall be made as small as possible. One solution is to design the winding so that if adjacent conductors are undergoing commutation at the same instant, they do not lie side

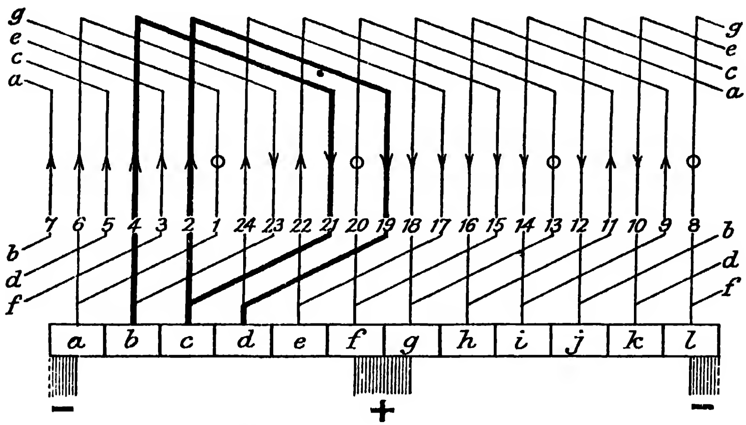


Fig. 80. Example of "Short-cord" Winding

by side, but are separated by a small band of conductors carrying the full current in opposite directions. This can be done by using what is called a "short-chord" winding, that is, a winding in which the rear pitch, i.e. the coil width, is somewhat less than the pole pitch. Fig. 80 shows such a winding. In this winding conductors 1 and 20 are undergoing commutation at the same time, but they are separated by the band of conductors numbered 24, 23, 22, and 21. Similarly with conductors 13 and 8.

CHAPTER VII

CHARACTERISTICS OF D.C. GENERATORS

1. **Types of Generators.** Generators are usually named according to the manner in which the field or exciting current is produced. Not including generators with permanent magnet fields, e.g. magnetos, there are four types, as illustrated in Fig. 81. These are the series wound, separately excited, shunt wound, and compound wound. In the series generator the field winding consists of relatively few turns of heavy cable or copper strip connected in series with the armature and the external load ; the separately excited generator has its field supplied from an independent source ; the shunt generator has its field connected across the armature terminals ; and the compound generator has both series and shunt excitation.

2. **Series-Wound Generator.** The current which flows through the armature also flows through the field winding, and this fact determines the nature of the characteristics. If the flux per pole is plotted against the current, the armature being driven at constant speed, the magnetic characteristic is obtained, its shape being given by curve I, Fig. 82. This curve starts a little way up the voltage axis due to the residual magnetism when the current is zero. If it were not for the demagnetizing effect of the armature reaction this curve would give the total flux when the machine is delivering current to the external load, but because of the armature reaction the actual curve lies somewhat below this. (Curve II.) This curve gives to a different scale the E.M.F. generated in the armature, because this E.M.F. is proportional to the flux. It is called the internal or total characteristic. The terminal voltage of the generator is equal to the E.M.F. generated, E, less the ohmic drop of volts in the machine. If R_{se} and R_a are the resistances of the series field and armature respectively, then, since each carries the load current I, the ohmic drop is

$$(R_a + R_{se})I$$

and for the terminal voltage, E_t, we have

$$E_t = E - I(R_a + R_{se})$$

The ohmic drop is represented graphically by a straight line through the origin, and deducting ordinates of this line from corresponding ordinates of curve II, we obtain the curve of terminal voltage, curve III. This is called the external characteristic. We see that as the current taken from a series generator is increased, the terminal voltage first increases according to an approximately linear

law, reaches a maximum, and finally decreases. If it is worked on the initial straight portion of the characteristic it will give a voltage approximately proportional to the current. The booster (Chap. X) is a series dynamo of this kind. If worked on the drooping portion, and if so designed that this portion is nearly vertical, the dynamo will give approximately constant current, independent of the resistance of the external circuit. Series dynamos of this kind used to be employed for lighting a number of arc lamps in series, but they are now obsolete.

If a straight line through the origin, such as OP, is drawn on the diagram, its gradient

$$\tan\theta = PM/OM = \frac{\text{Terminal voltage}}{\text{current}}$$
$$= \text{Resistance of external circuit.}$$

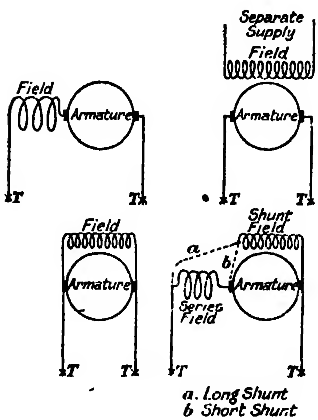


FIG. 81

METHODS OF EXCITATION

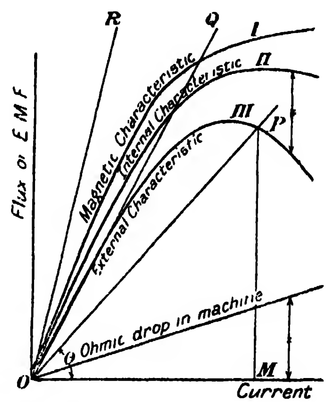


FIG. 82

CHARACTERISTICS OF SERIES GENERATOR

Thus, if a series of such lines is drawn, the gradient of each corresponding to a definite resistance, the points of intersection with the external characteristic will give the terminal voltage and current which will be obtained when the external circuit has the corresponding resistance. Neglecting the initial ordinate due to residual magnetism we can draw a straight line OQ which is tangential to the characteristic. The resistance represented by this line is called the *critical resistance*, since it is the greatest resistance with which the generator will be able to build up its magnetism. If the external resistance is greater than this, as represented by OR, then since OR has no point of intersection with the characteristic, the generator will not excite, and it will deliver no current.

3. **Separately-Excited Generator.** The excitation is here independent of the load current, so that if there were no armature reaction the flux would be constant as indicated by curve I, Fig. 83. Because of armature reaction the curve of actual flux is slightly drooping (curve II). This second curve gives to a different scale the E.M.F. induced in the armature, and it is therefore the total characteristic. The terminal voltage on load is the total E.M.F. generated, less the ohmic drop in the armature (not in the field in this case) ; hence, if we draw the curve of armature drop, R_aI, curve III, and deduct its ordinates from those of curve II, we

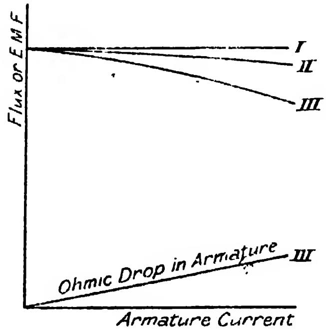


FIG. 83

CHARACTERISTICS OF SEPARATELY EXCITED GENERATOR

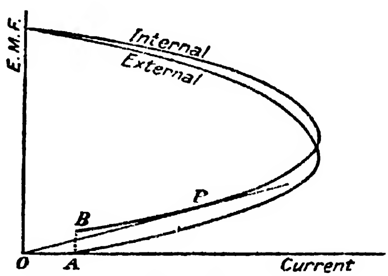


FIG. 84

CHARACTERISTICS OF SHUNT GENERATOR

obtain the external characteristic, upper curve III. We see that in this case there is a slight decrease in terminal voltage as the load increases. This decrease can be easily neutralized by a slight increase in exciting current.

4. **Shunt-Wound Generator.** When the external circuit is open the field winding can be regarded as being in series with the armature. The machine will therefore build up its own magnetism, and will give full voltage on no load if the resistance of the field winding is less than the critical resistance. If the voltage across the shunt field did not fall as the external load increased, this generator would have a characteristic similar to that of a separately-excited generator, but since a fall in terminal voltage causes a decrease in exciting current, the total decrease in voltage is greater than if the machine were separately excited. The external characteristic therefore droops rather more than if the machine were separately excited. If the load on the machine is gradually increased by decreasing the resistance of the external circuit, a decrease in resistance when the current is small will cause an increase in current. At the same time the increased

current will lower the terminal voltage, this, of course, tending to decrease the current. At first the effect of the decreased resistance predominates over the effect of decreased terminal voltage, but when the current has reached a certain value (much greater than the normal full load in modern machines) the load resistance shunts the field winding to such an extent that the terminal voltage decreases more rapidly than the load resistance. Then a further decrease in external resistance actually causes a decrease in current. The characteristic thus turns back, and when the armature is actually short-circuited it cuts the current axis at some point A (Fig. 84). The same thing happens to the total characteristic, except that in this case it stops at some point B, the ordinate AB representing the voltage due to residual magnetism. Actually AB will, in such a case, be very small, because the magnetism will be almost completely neutralized by armature reaction: it may even be reversed. It is for this reason that shunt generators often fail to excite after they have been shut down through a severe short circuit. If a tangent line OP to the total characteristic is drawn, the resistance represented by the gradient of this line gives the minimum external resistance for which the generator will excite if it is made to excite on load. If the external resistance is less than that represented by the line OP it will fail to excite, and therefore deliver no current. *There are thus two critical resistances for a shunt generator,* one for the field and the other for the external circuit.

If the terminal voltage on load is plotted against the load current the external characteristic is, of course, obtained. In order to be able to determine the total characteristic experimentally it is necessary to draw the curve of drop of volts in the armature, R_aI, and the curve of shunt current. If R_{sh} is the shunt resistance, then shunt current

$$I_{sh} = \frac{E_t}{R_{sh}}$$

and the required curve is obtained by plotting I_{sh} horizontally against E_t; it is, of course, a straight line through the origin, but because of the high resistance of the shunt field it has a very steep gradient, as shown in Fig. 85. Now the armature supplies both load and shunt currents, so that for the armature current we have

$$I_a = (I + I_{sh})$$

where I is the load current.

If we take any point P on the external characteristic and draw the perpendicular PM, then for the given terminal voltage, $OM = I$. Draw PA horizontally, then $AB = I_{sh}$, and if we mark off $MN = AB$ then

$$ON = OM + MN = (I + I_{sh}) = I_a$$

Hence, the vertical RN is equal to the drop in the armature, and therefore if we produce MP to Q, making $PQ = RN$, the total length QM is the sum of the terminal voltage and total armature drop, that is the total E.M.F. generated. In this way a point Q on the total characteristic is obtained, and if other points are obtained in the same way, the total characteristic can be drawn.

It is useful to be able to pre-determine from the magnetic characteristic what the external characteristic will be. The magnetic characteristic is the curve of open circuit voltage, or of flux, against exciting current, and it can be pre-determined with considerable accuracy from design data, or can be determined experimentally. A resistance line OA (Fig. 86) representing the

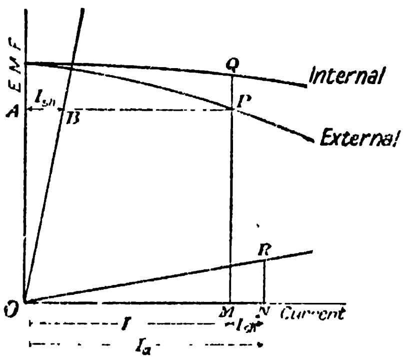


FIG. 85

CHARACTERISTIC OF SHUNT GENERATOR

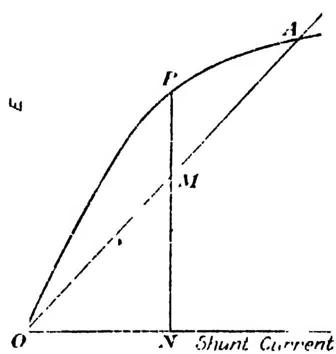


FIG. 86

CHARACTERISTIC OF SHUNT GENERATOR

shunt resistance R_{sh} is drawn. If any ordinate PMN is drawn, then for a shunt current equal to ON we know that

$$\begin{aligned} MN &= \text{voltage across shunt winding} \\ &= \text{voltage across machine terminals} = E_t \\ \text{But} \quad PN &= \text{total voltage generated} \\ \therefore \quad PM &= \text{ohmic drop in the armature} = R_a I_a \\ \therefore \quad PM &\propto I_a \end{aligned}$$

If a series of points is taken on the curve and the resulting values of the length MN plotted against the corresponding values of PM the external characteristic will be obtained. The ordinates will already be in volts and the abscissae can be converted to amperes by the relations

$$I_a = \frac{PM}{R_a}; \quad I_{sh} = \frac{MN}{R_s}$$

External current $I = I_a - I_{sh}$

As an example consider the following. A shunt generator has a total of 50 ohms resistance in the shunt circuit, this including the field winding as well as the regulator. Its terminal voltage is 25 when run at 500 r.p.m., 113 at 1,000 r.p.m., and 200 at 1,500 r.p.m. Draw the magnetization characteristic for 1,000 r.p.m. and hence determine the terminal voltage at 1,000 r.p.m. if the resistance of the shunt circuit is reduced to 40 ohms.

The 50 ohm line is drawn first of all. With 200 volts induced, the shunt current will be $200/50 = 4$ amps., so that this line passes through the origin and the point (4,200) as in Fig. 87. The three points of 25, 113, and 200 volts lie on this line as indicated. Since the induced voltage on no load is proportional to the speed so long as the flux remains constant, the 25 volt and 200 volt points corrected to 1,000 r.p.m. are

$$25 \times \frac{1{,}000}{500} = 50 \text{ volts}$$

and

$$200 \times \frac{1{,}000}{1{,}500} = 133{\cdot}3 \text{ volts.}$$

We thus have three points corresponding to 1,000 r.p.m., viz. A, B, and C, and the origin makes a fourth point, so that we can draw the characteristic as a smooth curve through point O, A, B, and C.

Finally, if the shunt resistance is reduced to 40 ohms, the shunt current at 100 volts will be 100/40 2·5 amps., the 40 ohm line thus passing through the point (2·5, 100). Its intersection with the characteristic is at the point D, showing that at 1,000 r.p.m. a shunt resistance of 40 ohms will give a terminal voltage of 126.

5. Conditions for Self-excitation. From the previous discussions of the characteristics of the various types of dynamo, it will have been realized that there are certain definite conditions which have to be fulfilled before a series or shunt-excited dynamo will build up its magnetism, and thereby generate a voltage. These conditions are as follows—

(1) There must be some residual magnetism in the field magnet. If there is, a voltmeter connected to the armature with the field disconnected will give a small reading.

(2) This magnetism must be in the right direction with regard to the connections of the field winding to the armature. If the machine has been shut down through a short-circuit, the magnetism may have become reversed through excessive armature reaction. In such a case, a moving-coil voltmeter connected to the armature will give a small back reading. As a result, when the small E.M.F. induced in the armature produces a current through the field winding, the field M.M.F. will strengthen the reversed flux and so build up the magnetism, and therefore the voltage, in the wrong direction.

The remedy is to connect the field winding to a separate external source which will set up a new residual magnetization in the right direction.

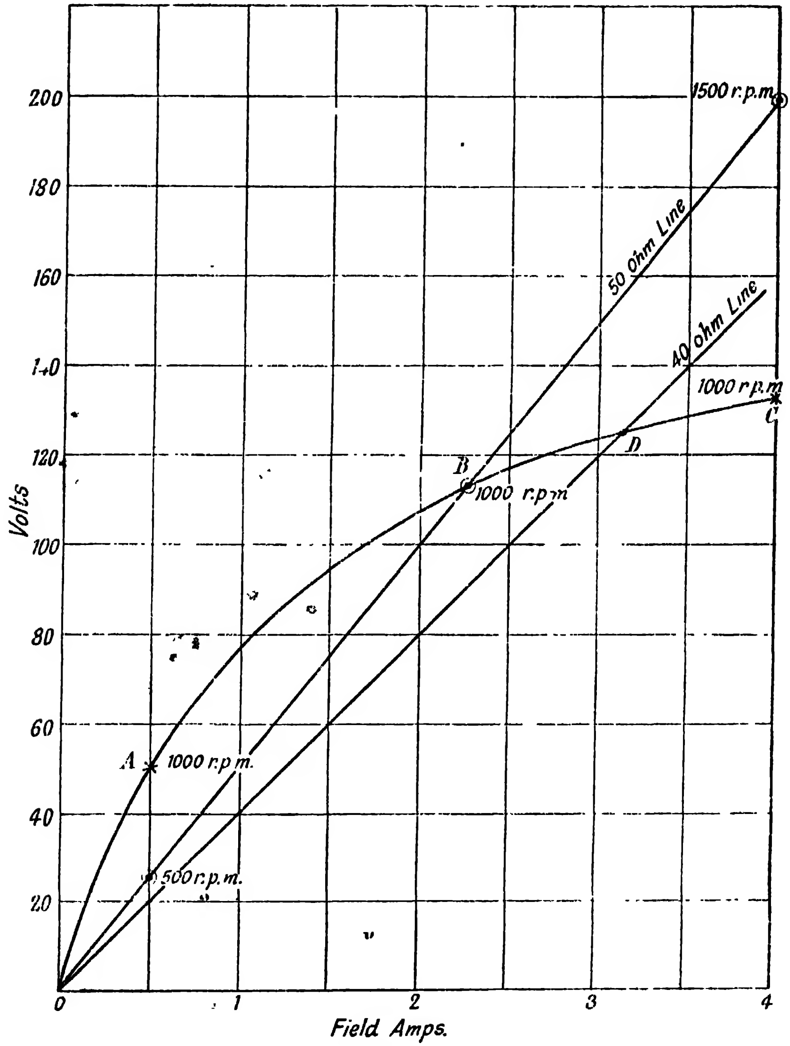


FIG. 87

(3) In the case of a series dynamo, the resistance of the circuit must be less than the critical resistance.

(4) In the case of the shunt dynamo, the field is *in series with the*

armature with respect to the path taken by the exciting current, and consequently there is also a critical resistance for the field circuit of a shunt dynamo. There is almost invariably a shunt regulator in circuit, and failure to excite is often due to too much of this regulator being in circuit.

(5) In the case of the shunt dynamo there is also a lower limit for the resistance of the external load, below which the machine will fail to excite if started up with the load switched on to the armature.

6. **The Compound-Wound Generator.** The shunt generator gives a terminal voltage which falls off somewhat with increase of load. It is usual to include an adjustable resistance called the shunt regulator in the field circuit, and to cut out some of this resistance when the load increases. This can be done by hand, and the terminal voltage kept constant if the fluctuations in voltage are fairly slow, as, for example, with a lighting load. When the fluctuations in load are very rapid, as with a traction load, then hand regulation is impossible and it is necessary to keep the voltage to the required value automatically. Again, it is usual to connect a generator to a pair of feeding points by a cable called a feeder. From these feeding points radiate other cables called distributors, across which the consumers are connected. The voltage across these feeding points is kept constant. Now if each conductor of a feeder has a resistance R, then, when delivering current I, there will be a drop in the feeder of $2IR$. Hence, if the constant voltage at the feeding points is E we have for the generator terminal voltage

$$V_t = E + 2IR$$

an equation which shows that the terminal voltage must *rise* with increase of load. This rise is obtained by providing a shunt generator with additional series excitation, the design being such that, over the working range, the series characteristic does not droop. The characteristics for the shunt and series turns separately are shown in Fig. 88, and that of the compound generator is obtained by adding ordinates of the two curves. The two component curves are somewhat concave downwards, and the resulting curve is therefore the same.

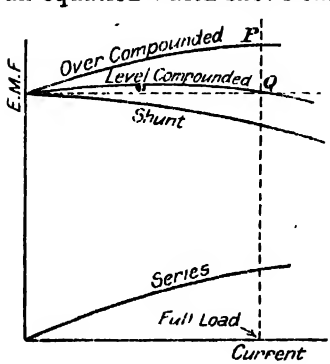


FIG. 88

CHARACTERISTICS OF COMPOUND GENERATOR

If the series excitation is such that the terminal voltage on full load is the same as on no load, the generator is "level" compounded. If the terminal voltage rises with load it is "over" compounded. The length PQ gives this

rise, and if such a generator is to maintain constant voltage at a pair of feeding points we have, obviously,

$$PQ = 2IR$$

The length PQ expressed as a percentage of the voltage E at the feeding points is called the " percentage compounding." Example—

$$E = 500 \text{ volts}, \ R = \cdot 025 \text{ ohm, full load current}$$
$$I = 1{,}000 \text{ amperes}$$

$\therefore$ Drop in feeder

$$2IR = 50 \text{ volts}$$

$\therefore$ Full load terminal voltage at the generator

$$V_t = 500 + 50 = 550$$

$\therefore$ Percentage compounding

$$\frac{V_t - E}{E} \times 100 = 10\%$$

There are two ways of connecting the shunt field in a compound generator, called short and long shunt. These are illustrated by the alternative connections shown dotted in Fig. 81. The short shunt is the more usual arrangement as it gives a somewhat higher voltage due to the fact that the shunt field has the full armature voltage across it. In the long shunt arrangement the voltage across the shunt is the armature voltage less the ohmic drop in the series field.

7. **Efficiency of D.C. Machines.** When calculating the efficiency of electrical machinery it is convenient to express power in electrical units, namely, watts. The efficiency of a generator is then

$$\eta = \frac{\text{Output}}{\text{Intake}} = \frac{\text{Output}}{\text{Output} + \text{Losses}}$$

The losses can be subdivided as follows—

I. Copper Losses.

(*a*) Armature copper loss = $I_a^2 R_a$. About 30 to 40% of the total full load losses.

(*b*) Field copper loss = $I_{sh}^2 R_{sh}$ in a shunt winding; $I_{se}^2 R_{se}$ in a series winding. About 20 to 30% of total losses at full load.

(*c*) The loss due to brush contact resistance. This can conveniently be taken into account by including the brush contact resistance in the armature resistance.

II. Magnetic or Iron Losses.

(*a*) Hysteresis loss. Proportional to $\{(\text{Flux})^{1 \cdot 6} \times \text{Speed}\}$.

(*b*) Eddy current loss. Proportional to $\{(\text{Flux})^2 \times \text{Speed}^2\}$.

These two losses are therefore approximately constant in a machine whose flux is approximately constant, e.g. shunt or compound machines, but variable in the case of a series machine. The two together represent 20 to 30 % of the total losses at full load.

III. Mechanical Losses.

(a) Friction at bearings and commutator.
(b) Windage of rotating armature.
Together 10 to 20% of total losses, independent of the load, and proportional to the speed for small variations in speed.

The magnetic and mechanical losses are often grouped together and called collectively the stray losses.

The efficiencies of generators depend, of course, upon the output, average values for the full load efficiencies varying from 70 per cent for 1 kW machine to 95 per cent for a 1,000 kW machine.

In the case of a shunt or compound machine the shunt copper loss and stray losses are constant. We can then write

$$\text{Total losses} = \text{Armature copper loss} + \text{Constant losses } (W_c)$$
$$= R_a(I + I_{sh})^2 + W_c$$

where I is the load current.

$$\text{Output} = VI$$

$$\therefore\ \eta = \frac{VI}{VI + R_a(I + I_{sh})^2 + W_c} \quad \frac{VI}{VI + R_aI^2 + W_c} \text{ approx.}$$

since I_{sh} is small compared with I

$$\eta = \frac{1}{1 + \left(\frac{R_aI}{V} + \frac{W_c}{VI}\right)}$$

Now the product of the terms in the brackets is a constant since E_t is approximately constant, I being the variable. Hence, their sum is a minimum, and the efficiency, a maximum

$$\text{when} \quad \frac{R_aI}{V} \quad \frac{W_c}{VI}$$

$$\text{when} \quad I = \sqrt{\frac{W_c}{R_a}}$$

Hence, if the efficiency is plotted against the load current, a curve of the form shown in Fig. 89 will be obtained, the position of the maximum being defined by the above value of the load current.

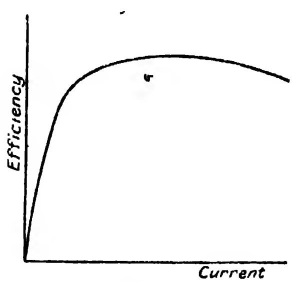


Fig. 89

Efficiency Characteristic

Example. A 400 volt shunt generator has a full load current of 200 amperes; its armature resistance is ·06 ohm, and field resistance, 100 ohms; the stray losses are 2,000 watts. Find the h.p. of its prime mover when it is delivering full load, and find the load for which efficiency of the generator is a maximum.

$$I = 200 \text{ amp. on full load.}, \quad I_{sh} = \frac{400}{R_{sh}} = 4 \text{ amp.}$$

$\therefore \quad I_a = I + I_{sh} = 204$ amp.

$\therefore$ Armature copper loss $I_a{}^2R_a = 204^2 \times \cdot 06 = 2{,}500$ watts

Field copper loss $I_{sh}{}^2R_{sh}$ or $I_{sh}V = 4 \times 400 = 1{,}600$ watts

Constant losses W_c = field copper loss + stray losses
$= 3{,}600$ watts

$\therefore$ Total full load losses $= 2{,}500 + 3{,}600 = 6{,}100$ watts

Output VI $= 400 \times 200 = 80{,}000$ watts

Intake $= 80{,}000 + 6{,}100 = 86{,}100$ watts

$\therefore$ H.P. of engine $= \dfrac{86{,}100}{746} = 115$

$$\eta = \frac{80{,}000}{86{,}100} = \cdot 93, \text{ or } 93\%$$

The load at which η is a maximum is

$$I = \sqrt{\frac{3{,}600}{\cdot 06}} = 245 \text{ amp.}$$

This is greater than the normal load because the armature resistance is on the low side for a 400 volt generator of 80 kW output.

Example. A series generator of total resistance 0·5 ohm is running at 1,000 r.p.m. and delivering 5 kW at a terminal p.d. of 100 volts. If the speed is raised to 1,500 r.p.m., and the load, adjusted to 8 kW, find the new current and terminal p.d.

Assume that the machine is working on the straight portion of the characteristic; then the flux is proportioned to the current, and the E.M.F. generated, proportional to the product of flux and speed.

(*a*) Speed 1,000 r.p.m.

Output 5 kW $= 5{,}000$ watts

$$V = 100 \quad \therefore I = \frac{5{,}000}{100} = 50 \text{ amp.}$$

$\therefore$ Drop in machine $IR = 50 \times 0{\cdot}5 = 25$ volts, and the total E.M.F. generated $E_1 = 100 + 25 = 125$ volts

$\therefore$ 125 $\propto$ speed $\times$ flux
$\propto$ speed $\times$ current, $\propto 1{,}000 \times 50$ . . . (1)

(*b*) Speed 1,500 r.p.m.

Let I = new current

$$\therefore \quad V = \frac{\text{Output}}{I} = \frac{8{,}000}{I}$$

Drop in machine $RI = \cdot 5I$

$\therefore$ E.M.F. generated $E_2 = V + \text{drop} = \dfrac{8{,}000}{I} + \cdot 5I$

Again, $E_2 \propto$ new speed $\times$ new current

$$\therefore \frac{8{,}000}{I} + \cdot 5I \propto 1{,}500 \times I \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

Dividing equation (2) by (1),

$$\frac{\frac{8{,}000}{I} + \cdot 5I}{125} = \frac{1{,}500I}{50{,}000}.$$

$$I = 49 \cdot 6 \text{ amp.}$$

$$V = \frac{8{,}000}{49 \cdot 6} = 161 \text{ volts}$$

EXAMPLES ON CHAPTER VII.

(1) A shunt dynamo gave an external characteristic as follows—

Current	0	4	8	12	16	20
Terminal Volts	89·5	88·5	87	84·8	82·2	79

Its armature and shunt resistances were ·22 and 45 ohms respectively. Plot its total characteristic.

(2) The above dynamo, when connected short-shunt compound, gave the following terminal voltages for the same load currents 89·5, 89·9, 90, 90, 89·5, 88·2. The resistance of its series winding was ·10. Deduce its total characteristic.

(3) If a dynamo is level compounded at 1,500 r.p.m., what will be the nature of its voltage characteristic at 1,000 r.p.m., if the shunt regulator is not altered ?

(4) Given the magnetization curve of a self-excited, shunt-wound generator, show how to determine the open circuit voltage with a given total resistance in the shunt circuit. Show how to estimate the rise in voltage when the speed is increased by 10%, and explain how the external characteristic of the machine may be estimated. (London Univ., 1922.)

(5) Find how many series turns per pole are needed on a 500 kW compounded dynamo required to give 500 volts on no load and 550 volts on full load, the requisite ampere-turns per pole being 7,900 and 11,200 respectively. The shunt winding is designed to give 500 volts at no load when its temperature is 20° C. The final temperature is 60°C. (C. and G., 1923.)

Ans.—4 series turns per pole.

(6) A shunt generator has a full load current of 100 amperes at 110 volts. The stray losses are 500 watts and the shunt resistance is 55 ohms. If the efficiency at full load is 88 per cent, find the armature resistance. Then find the efficiencies at one-half of full load and at 50 per cent overload, assuming that the voltage is kept constant at 110. At what load current will the efficiency be a maximum ?

Ans.—(*a*) ·075 ohm ; (*b*) 86 per cent ; (*c*) 87 per cent ; (*d*) 96 amps.

CHAPTER VIII

THE DIRECT CURRENT MOTOR

1. **General Principles.** Any direct current generator will run as a motor, that is, convert electrical power to mechanical power, if its field and armature are connected to a suitable electric supply. The essential construction of a motor is identical with that of a generator, and when there is any external difference in appearance this is due to the fact that whereas the frames of generators can as a rule be open, those of motors are either partly or totally enclosed because of the rougher (mechanical) usage to which the latter are subjected.

Fig. 90 (A) shows one conductor on the armature of a generator rotating clockwise under a N. pole. Fleming's right-hand rule indicates that the induced E.M.F. is inwards, and therefore the current in that conductor is also inwards. This current sets up a magnetic field, the lines of force of which are concentric circles round the wire and in a clockwise direction (from the corkscrew rule).

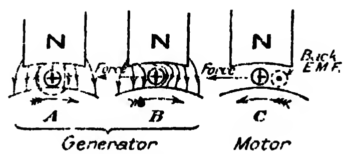


FIG. 90

This field is seen to act in the same direction as the main field on the right-hand side, and in opposition on the left-hand side, the result being that the actual distribution of the lines of force in the resultant field is as indicated in Fig. 90 (B), some of the lines being bent round the conductor. Now magnetic lines of force are always in a state of tension, the bent lines of force setting up a mechanical force on the conductor much in the same way that the bent elastic of a catapult produces a mechanical force on the stone. In the case of the generator it will be noticed that this force is in opposition to the direction of motion. It is therefore called the magnetic drag, and it is this drag acting on all the conductors that the prime mover has to do work against. For a current of I amperes flowing in a conductor of l cm. placed in a field of H c.g.s. units the drag is given by

$$f = \frac{HIl}{10} \text{ dynes}$$

the denominator 10 being introduced because 1 amp. is one-tenth of the c.g.s. unit of current.

We have seen from the previous description of a D.C. armature that the conductors do not lie midway between the pole face and

a smooth armature core, but that they are all housed in slots. The above explanation, although convenient for the calculation of the torque of the armature, is not a correct description of the mechanism of torque production. Hence let the conductors lie in slots as in Fig. 91. In figure *A* the main flux is shown concentrated into tufts which pass into the tops of the teeth, while the armature flux is shown by the dotted fluxes embracing the slots. The directions of these fluxes correspond with those of Fig. 90. We see that the effect of the armature flux is twofold. Firstly, it increases the flux in the left-hand halves of the teeth and reduces it in the right-hand halves, so that the distribution of flux density across the tooth

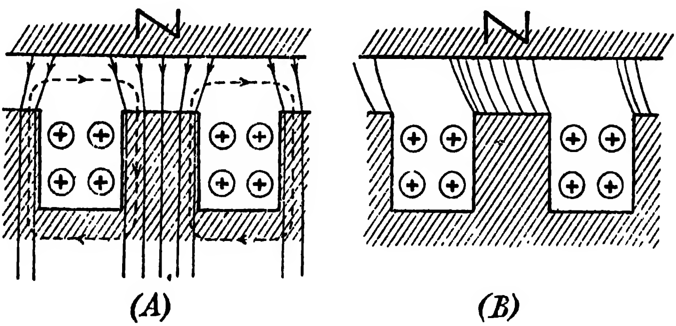


FIG. 91

MECHANISM OF TORQUE PRODUCTION IN A D.C. MACHINE

section is no longer uniform. Secondly, it inclines the directions of the magnetic force in the air gap so that the tufts of lines of force are not radial but are disposed somewhat after the manner of figure *B*. Since the lines of force are in a state of tension the tangential component of this tension will set up a torque on the armature, and from the figure we see that the direction of this torque is counter-clockwise. Hence in an actual machine with slotted armature the torque is not due to mechanical forces on the conductors themselves, but to the tangential component of the magnetic pulls on the teeth.

2. Back E.M.F. Now suppose that the machine is uncoupled from the prime mover and that current is sent through the armature and field from an external source. If the excitation of the field and the directions of the armature currents are the same as before, the magnetic drag will be set up in the same direction as before, and under its influence the armature will rotate. The machine will now be running as a motor. We thus see that for the same direction of armature currents and the same excitation, the direction of rotation of a motor is opposed to that of a generator—Fig. 90 (*C*).

Now when the motor armature rotates the armature conductors

cut the lines of force of the field, and as a result, they have E.M.F.s induced in them. The direction of one such induced E.M.F. in an individual conductor is, of course, given by Fleming's right-hand rule, and applying this rule to the conductor in Fig. 90 (C) we see that the E.M.F. is outwards, that is, *opposed to the current.* This induced E.M.F. in the case of a motor is therefore called the "back E.M.F." The supply E.M.F. has to do work against this back E.M.F. in forcing the current through the armature. If the induced E.M.F. in an armature is opposite in direction to the current, then the machine is motoring; if it is in the same direction as the current, the machine is generating. The magnitude of the back E.M.F. is, of course, given by the expression for the generated E.M.F. Calling the back E.M.F. E_b, we have

$$E_b = \frac{\Phi ZN}{60} \times \frac{P}{A} \times 10^{-8} \text{ volts.}$$

3. **Speed of a D.C. Motor.** The speed of a motor automatically adjusts itself to the load so that the electrical power required to drive the current through the armature is equal to the mechanical power required to drive the load. The load here includes all those losses in the motor which are produced by rotation, namely, the magnetic and mechanical losses.

Hence, $E_b I$ is $\propto$ Speed $\times$ (Total retarding torque).

Now $$E_b = \frac{\Phi ZN}{60} \times \frac{P}{A} \times 10^{-8}$$

$\therefore$ Resultant E.M.F. acting in the armature circuit

$$E - E_b = E - \frac{\Phi ZN}{60} \times \frac{P}{A} \times 10^{-8}$$

where E is the E.M.F. applied to the armature terminals.

$$\therefore \qquad I_a = \frac{E - \frac{\Phi ZN}{60} \times \frac{P}{A} \times 10^{-8}}{R_a}$$

$$\therefore \text{ Speed} \qquad N = \frac{E - R_a I_a}{\Phi Z} \times \frac{60A}{P} \times 10^{8}$$

Now in modern machines the drop of volts in the armature, $R_a I_a$, is small compared with E, so that we have, approximately,

$$N = \frac{1}{\Phi} \times \left(\frac{E}{Z} \times \frac{60A}{P} \times 10^{8}\right)$$

$$\propto \frac{1}{\Phi} \text{ so long as } E \text{ is constant.}$$

Thus the speed of a D.C. motor is inversely proportional to the flux per pole.

4. Conditions for Maximum Power. We have

$$E = E_b + R_a I_a$$

$$\therefore \quad EI_a = E_b I_a + R_a I_a^2$$

This is an equation of power, the various terms having the following meanings—

EI_a = total power supplied to the armature

$E_b I_a$ = power converted into mechanical power

$R_a I_a^2$ = power dissipated as heat due to the armature resistance

Hence, mechanical power of the motor

$$W_m = EI_a - R_a I_a^2$$

$$\therefore \quad \frac{dW_m}{dI_a} = E \quad 2R_a I_a$$

$$= 0$$

and corresponds to a maximum when

$$R_a I_a = E/2 \text{ or } E_b = E/2.$$

Thus a motor develops the maximum mechanical power when the armature current is such that the back E.M.F. is equal to one-half of the applied E.M.F. This is not attained in practice, since it necessitates a current well beyond the normal working range. Also an amount of power equal to the mechanical power developed would be wasted in heating the armature, so that, taking other losses into account, the efficiency would be well below 50 per cent.

5. Torque. If P = no. of poles and A = no. of parallel paths through the armature, the current in each individual conductor is

$$I = \frac{I_a}{A}$$

Consider that the flux per pole Φ is divided into separate fluxes Φ_1, Φ_2, Φ_3, etc., which embrace the individual conductors, as shown in Fig. 92.

Then force on conductor 1 $= \dfrac{HIl}{10}$ dynes

$$= \frac{\Phi_1}{xl} \times \frac{I_a}{10A} \times l = \frac{\Phi_1 I_a}{10xA}$$

force on conductor 2 $\dfrac{\Phi_2 I_a}{10xA}$

force on conductor 3 $= \dfrac{\Phi_3 I_a}{10xA}$

$\therefore$ Total tangential force on all the conductors under one pole

$$= \frac{\Phi_1 I_a}{10xA} + \frac{\Phi_2 I_a}{10xA} + \frac{\Phi_3 I_a}{10xA} + \dots\dots$$

$$= \frac{\Phi I_a}{10xA}$$

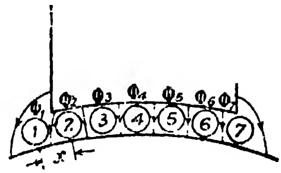


FIG. 92

CALCULATION OF TORQUE OF A MOTOR

$\therefore$ Torque due to this force

$$= \frac{\Phi I_a}{10xA} \times r \text{ dyne cm.}$$

Now the distance x $= \dfrac{2\pi r}{Z}$

$\therefore$ Torque due to all the conductors under one pole

$$= \frac{\Phi I_a}{10A} \times \frac{Z}{2\pi r} \times r = \frac{\Phi I_a Z}{20\pi} \times \frac{1}{A} \text{ dyne cm.}$$

Hence torque due to all the conductors under all the P poles

$$T = \frac{\Phi I_a Z}{20\pi} \times \frac{P}{A} \text{ dyne cm.}$$

$$= \frac{1}{20\pi \times 981 \times 454 \times 2{\cdot}54} \times \Phi I_a Z \times \frac{P}{A} \text{ lb. in}$$

$$= 1{\cdot}41 \times 10^{-8}\, \Phi I_a Z \times \frac{r}{A} \text{ lb. in.}$$

Example. A four-pole armature 50 cm. in diameter has 1,000 conductors, each of active length 25 cm. The pole span is two-thirds of the pole pitch, and the average flux density in the gap is 6,000 lines per sq. cm. Calculate the torque in lb. in. when each conductor carries 10 amp.

No. of active conductors $=$ those in the magnetic field

$= \frac{2}{3}$ of 1,000 $= 667$

Force on each conductor $\dfrac{HIl}{10} = \dfrac{6{,}000 \times 10 \times 25}{10} = 150{,}000$ dynes

$\therefore$ Torque exerted by each conductor

$$= 150{,}000 \times 25 = 3{,}750{,}000 \text{ dyne cm.}$$

$\therefore$ Total torque $= 3{,}750{,}000 \times 667$ dyne cm.

$$= \frac{3{,}750{,}000 \times 667}{981 \times 454 \times 2 \cdot 54} = 2200 \text{ lb. in.}$$

The equation for the torque can also be derived directly from the back E.M.F. equation, as follows. The power required to overcome the back E.M.F. is that which provides the mechanical torque. If the torque is T, the work done per revolution is $T \times 2\pi$, since the work done by a torque is equal to the product of the torque and the angle turned through (in radians). The work done by the torque per second is thus

$$T \times 2\pi \times N/60$$

which must be $E_b I_a$. Therefore

$$T = E_b I_a \div (2\pi N/60)$$

$$= \frac{\Phi Z N}{60} \times \frac{P}{A} \times 10^{-8} \times I_a \times 10^7 \div (2\pi N/60)$$

$$= \frac{\Phi I_a Z}{20\pi} \times \frac{P}{A} \text{ dyne cm.}$$

$$= 1 \cdot 41 \times 10^{-8} \Phi I_a Z \times \frac{P}{A} \text{ lb. in.}$$

$$= 1 \cdot 17 \times 10^{-9} \Phi I_a Z \times \frac{P}{A} \text{ lb. ft.}$$

The multiplier 10^7 is to reduce the joules to ergs, which are dyne centimetres.

6. **Armature Reaction and Commutation.** Suppose that the brushes are placed first of all in the G.N.P. (Fig. 93), then if the armature is rotating clockwise the directions of the armature currents will be outwards under the N. poles, and inwards under the S. poles. The armature therefore acts like a solenoid whose axis coincides with the brush axis, as in the case of a generator, but in this case the armature M.M.F. OA is directed from left to right. OB is the main M.M.F., and OC the resultant M.M.F.

We see that the magnetic neutral plane M.N.P., which is perpendicular to OC, is now shifted backwards, instead of forwards as in the case of the generator. Also on comparing the directions of the lines of force of the armature cross field with the main field, we see that the main field is strengthened under the leading pole tips and weakened under the trailing pole tips.

It is necessary to give the brushes a backward lead in order to bring the brush axis into the M.N.P. We can then divide the armature conductors into two groups, those lying above WX and

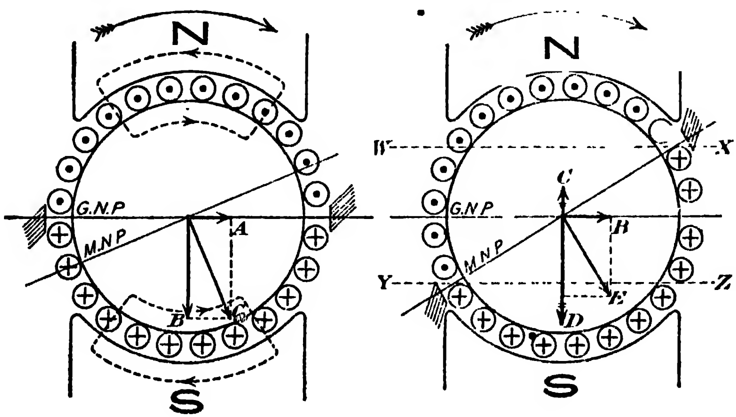


FIG. 93
ARMATURE REACTION OF A MOTOR

below YZ, and those lying between these two planes. The first group produces a cross field OB whose M.M.F. acts from left to right, and the second group produces a M.M.F. OC which is in opposition to the main M.M.F. OD, and therefore sets up a demagnetizing effect. Thus when the brushes in a generator and a motor are set so as to be in the M.N.P., a demagnetizing action is set up by the armature in each case, but the cross magnetizing fields are opposite in direction.

In some cases motors are required to run in both directions without sparking. In such a case it is obvious that the brushes must be placed in the G.N.P. In order to cut down the distortion of the field produced by the cross field to a minimum, the field magnets are designed to be nearly saturated. At the same time the air gap is made rather longer than in the case of a motor running in one direction only. Since each line of force of the cross flux has to cross the same air gap twice, this causes a considerable reduction in the total cross flux, thereby minimizing distortion. It is, of course, necessary to provide the excitation with additional ampere-turns to drive the flux across the long air gaps, and such

machines are said to have "stiff" magnetic fields, the field system being magnetically strong compared with the armature.

D.C. motors, like generators, are commonly fitted with interpoles in order to ensure good commutation. In either case their function is the same, namely, to force the current in a coil short-circuited by the brush to attain its full value in the reversed direction by the end of short circuit. Since the currents in the armature conductors flow in the reverse direction to those of the generator when the directions of rotation are the same, it follows the polarity of the interpoles must be reversed. Their polarity with respect to the polarity of the main poles and to the direction of rotation is, for a motor, as indicated in Fig. 94.

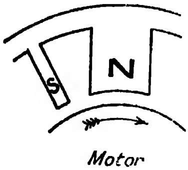


FIG. 94
POLARITY OF INTERPOLES

7. **Characteristics of D.C. Motors.** There are three ways of exciting the field of a D.C. motor; these are series, shunt, and compound, and the characteristics of the motor are determined by the method of excitation. Consider first of all the series motor.

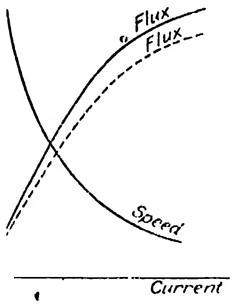


FIG. 95

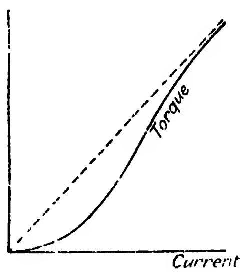


FIG. 96

CHARACTERISTICS OF SERIES MOTOR

The flux varies with the motor current, the relationship between the two being given by the magnetic characteristic (Fig. 95). If the flux due to armature reaction is deducted, the dotted curve in the figure will be obtained. Now we have seen that the speed is inversely proportional to the flux, and therefore, if we plot reciprocals of the flux against current, the speed-current curve will be obtained. From its shape we see that the series motor is essentially a variable speed motor, the speed being low on heavy load and dangerously high on light load. For this reason the series motor is never run without some mechanical load on it, otherwise it might fly to pieces. If the saturation point is not reached, the flux Φ is approximately proportional to the current I.

Now torque $T \propto \Phi \times I$

But $\Phi \propto I$

$\therefore T \propto I^2$

On light loads the torque-current curve is therefore a parabola, but when I is so great that Φ becomes nearly constant, T becomes proportional to I to the first power, and the curve becomes asymptotic to the dotted straight line through the origin, as shown in Fig. 96.

Below the saturation point we have Φ proportional to I

$$\therefore N \propto \frac{1}{\Phi}$$

$$\propto \frac{1}{I}$$

From the above we see that, so long as the iron is not saturated, the series motor exerts a torque proportional to the square of the current. Its starting torque is therefore very high, for which reason it is used in cases where heavy masses have to be accelerated quickly, e.g. for electric traction and for hoist work. The falling off in speed as the load increases is also a considerable advantage for such duties, since it automatically relieves the motor from having to carry excess loads. Thus, if a tram travelling up hill falls to half speed, the load on the motor is only half of what it would be if the tram raced up at full speed.

If the speed-current and torque-current curves are drawn on the same diagram, then by measuring ordinates on the two curves corresponding to a series of values of current, the speed-torque curve can be drawn. This curve is sometimes called the "mechanical characteristic," and for a series motor it is similar in shape to the speed-current curve.

Example. A series motor takes 20 amp. at 400 volts to drive a fan at 250 r.p.m. Its resistance is 1 ohm. If the torque required to drive the fan varies as the square of the speed, find the necessary applied voltage and current to drive the fan at 350 r.p.m.

Assuming a straight line magnetic characteristic,

$$T \propto \Phi \times I \propto I^2. \quad \text{But } T \propto N^2$$

$$\therefore I \propto N. \qquad \text{Hence, new current} = \frac{350}{250} \times 20 = 28 \text{ amp.}$$

Motor back E.M.F. at 250 r.p.m. $= 400 - (20 \times 1) = 380$

Motor back E.M.F. at 350 r.p.m. $= E - (28 \times 1)$

Now back E.M.F. $\propto \Phi \times N$, i.e. $\propto I \times N$

$$\therefore \frac{E-28}{380} = \frac{28 \times 350}{20 \times 250}$$

$$\therefore \quad E = 717 \text{ volts.}$$

8. **The Shunt Wound Motor.** For a constant applied P.D. the exciting current is constant. The flux will therefore have its maximum value at no load, and because of armature reaction will decrease slightly as the load increases. For most purposes this decrease can be neglected, and the flux per pole, Φ, can be regarded as constant. We then have from the equation on page 121, speed

$$N = \frac{E - R_a I_a}{\Phi Z} \times \frac{60A}{P} \times 10^8$$

$$= K(E - R_a I_a) \text{ where } K \text{ is a constant.}$$

Let N_0 and I_0 be the no load speed and armature current, and N and I, the speed and current at any other load. Then

$$N = K(E - R_a I)$$

$$N_0 = K(E - R_a I_0)$$

$$\therefore N = N_0 \times \frac{E - R_a I}{E - R_a I_0} \quad \therefore \frac{N_0 - N}{N_0} = \frac{R_a I - R_a I_0}{E - R_a I_0}$$

This equation shows that since I_0 is small the drop of speed of a shunt motor from no load to any other load is proportional to the drop of volts in the armature at that load. This, of course, assumes zero armature reaction; actually, since the armature exerts a demagnetizing action, Φ will decrease as I increases, and therefore the fall in speed will not be quite so great as indicated by the above expression. Since the full load drop in the armature is small compared with the applied voltage, we see that the speed-current curve of a shunt motor is a slightly drooping curve, as shown in Fig. 97. It is instructive to deduce the drop of speed in another manner. Whatever the load on the motor may be, the back E.M.F. adjusts itself to such a value that sufficient armature current can pass to produce a torque equal to the total opposing torque. Let the line AB represent the applied voltage, and let the

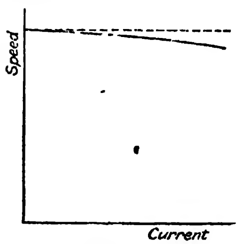


FIG. 97

CHARACTERISTIC OF SHUNT MOTOR

back E.M.F. be AC at a given load. Then BC is the resultant E.M.F. and represents to another scale the armature current.

Since Φ is approximately constant, BC represents to still another scale the opposing torque. If the torque is doubled, C has a new position, C_1, where $BC_1 = 2BC$. The back E.M.F. is now AC_1, and, since the speed is proportional to the back E.M.F., the speed falls from AC to AC_1. If the torque is trebled, C moves to C_2 where $BC_2 = 3BC$, and the new current and speed are represented by BC_2 and AC_2 respectively. Thus, if the lengths of AC are plotted against the lengths BC, the speed-current characteristic is obtained.

Example. A six-pole lap wound shunt motor has poles 20 cm. square and a flux density in the gap of 5,000. The armature is wound with 500 wires having a total length of wire of 24,000 cm. and ·07 sq. cm. area. Find the speed of the motor with 100 volts on the terminals and 120 amp. in the line. (London Univ., 1912.)

$$\Phi = 5{,}000 \times 20^2 = 2 \times 10^6 \text{ lines}$$

Resistance of all the armature wires in series $= \dfrac{2}{10^6} \times \dfrac{24{,}000}{\cdot 07}$

$$= \cdot 7 \text{ ohm.}$$

Hence resistance of armature with 6 parallel paths

$$R_a = \frac{\cdot 7}{6^2} = \cdot 0194$$

Deducting, say, 5 amp. for the shunt current, the armature current will be 115 amp., and the drop in the armature, $\cdot 0194 \times 115 = 2$ volts, roughly.

$\therefore$ Back E.M.F. $\quad E_b = 100 - 2 = 98$ volts.

But $\quad E_b = \dfrac{\Phi Z N}{60} \times \dfrac{P}{A} \times 10^{-8}$

$$\therefore N = \frac{60 \times 98 \times 10^8}{2 \times 10^6 \times 500}$$

$$= 588 \text{ r.p.m.}$$

From the general equation for the torque of a D.C. motor, we see that, since Φ is approximately constant, the torque is proportional to the armature current. Hence, the torque-current characteristic is a straight line through the origin. The mechanical characteristic can be determined from the two electrical characteristics as before. It is a slightly drooping curve similar in shape to the speed-current curve.

Sometimes, when starting a shunt motor, it is found that it takes a very large current and runs at a high speed in the wrong direction. This phenomenon is due to a break in the shunt winding. The demagnetizing effect of the sudden rush of current at starting reverses the residual magnetism in the field, and the only flux present is that due to armature reaction. This flux is comparatively weak and in the reverse direction to the proper main field, thus causing the motor to run at high speed in the reversed direction.

Since the armature drop of volts on full load is proportional to the fall in speed from no load to full load, and since the full load drop is only about 4% of the applied voltage on the average, the shunt motor can be regarded as a constant speed motor. If the drop in speed has to be made up, this can be effected by inserting resistance in the shunt field, thereby reducing the flux. Shunt motors are therefore suitable for driving light machine tools and for all purposes where an approximately constant speed is required.

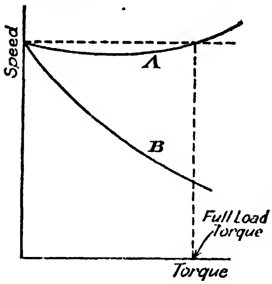


FIG. 98
CHARACTERISTICS OF COMPOUND MOTORS

9. **Compound Wound Motor.** The series field may be connected in the circuit so as to either help the shunt field or oppose it. In the first case the motor is "cumulatively" compounded, and in the second case, "differentially" compounded. If the series field of the differentially compounded motor is so adjusted that the full load decrease in flux produced by it is just sufficient to make the full load speed equal to the speed on no load, then, for any other load within this range, its speed will be approximately constant. Its mechanical characteristic is shown in curve A (Fig. 98). This appears to be an advantage at first sight, but there are two disadvantages. In the first place, when the motor is being started, the shunt field will take some time to build up, and therefore the series field will be established first, owing to the initial rush of current through the armature and series field. The motor will therefore tend to start up the wrong way. When the shunt field is fully established the total field will be so small that there may not be sufficient torque to run the motor, and the armature will take an excessive current from the supply. This difficulty can be overcome by short-circuiting the series field during starting, and only putting it in circuit when the motor is under way. The second disadvantage is that if the motor becomes overloaded, the resulting decrease in flux will tend to force up the speed so that the motor will be very seriously overloaded. In fact, when the current attains a certain value, the motor becomes unstable, and it begins to race like a series motor on no load. The motor is thus dangerous to use unless there is no possibility of the load exceeding the normal full load.

The cumulatively compounded motor has an increase in flux as the load increases, with the result that the speed decreases, but not so rapidly as in the case of the plain series motor, because of the constant flux produced by the shunt field. The mechanical characteristic is therefore between those of the series and shunt motors, its nearness to one or the other depending upon the number of series

turns (Curve B, Fig. 98.). The falling off in speed affords relief to the motor if a sudden load comes on, and the motor is therefore used largely for driving heavy machine tools where sudden deep cuts may be taken. This motor is also suitable for driving continuous running rolling mills, for which duty it is coupled to a heavy flywheel. When a billet is inserted in the rolls the sudden increase in load causes the speed to decrease, with the result that the flywheel gives up some of its stored kinetic energy, thereby taking the strain of the sudden peak load off both motor and supply mains.

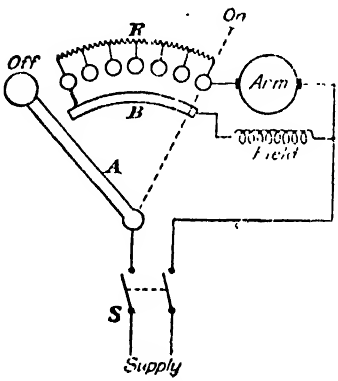


FIG. 99

NON-AUTOMATIC STARTER

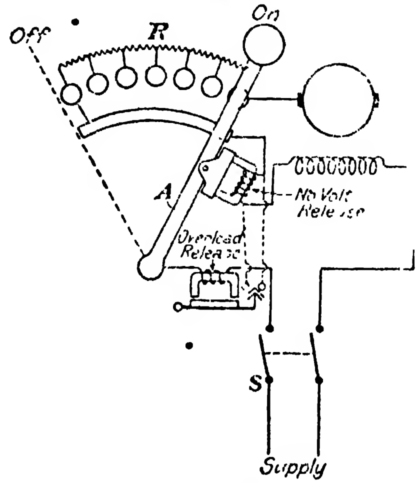


FIG. 100

STARTER WITH NO-VOLT AND OVERLOAD RELEASES

When the load is light, the speed rises and the flywheel acquires the kinetic energy it gave up previously. In this way, provided that the successive peak loads do not come on before the speed has had time to reach the full value again, the flywheel, in conjunction with such a motor, acts as a load equalizer. Its function is to look after all the sudden fluctuations in the load and to leave the motor and supply to cope with the steady average load.

10. **Motor Starters.** For all but fractional h.p. motors, a resistance should be placed in the armature circuit at the moment of starting, and then gradually cut out as the motor speeds up. The simplest arrangement of a starter for a shunt motor is shown in Fig. 99. The arm A makes contact with the studs connected to the starting resistance R, and also to a brass arc, B, by which the connection to the shunt field is made. This arrangement is perfectly satisfactory for starting, but it suffers from the disadvantage that when the main switch S is opened to shut down

the motor, the arm A remains in the "on" position, and if the switch is again closed the armature is put right across the supply. It is thus obvious that when the switch is opened the arm should return to the "off" position. Also, for motors up to about 20 or 30 h.p. it is desirable to incorporate an "overload release" which will disconnect the motor from the supply in the event of an overload. For larger sizes it is better practice to leave this duty to a separate automatic circuit-breaker. A typical starter for a shunt or compound motor is shown in Fig. 100. The arm A is provided with a hinged armature, which is attracted by the coil of a "no-volt" release when in the "on" position. This release is connected in series with the shunt field. When the switch is opened the pull of this coil is removed and the arm is returned to the off position by a spring. The overload release consists of another coil, which carries the full line current. It attracts an armature which, when lifted, bridges two brass studs connected to the ends of the coil of the no-volt release, thereby short-circuiting this. The spring then pulls the arm back and stops the motor. The position of the armature of the overload release can be adjusted by a set screw, and the release thus set to operate at any desired overload.

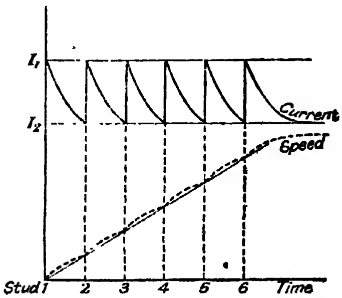


FIG. 101

VARIATIONS IN CURRENT AND SPEED DURING STARTING

The grading of the steps of the starting resistance, and the time taken to move the arm across the row of contacts, should be such that, during the operation of starting, the motor current is kept within upper and lower limits. This is illustrated in Fig. 101. The stepped curve shows the variation of current. As soon as contact is made with any stud the diminution in resistance causes the current to increase initially; the motor then increases in speed with consequent increase in back E.M.F., so that the current decreases while the arm is held on that stud. When the current reaches the minimum value the arm is moved to the next stud, and the current again suddenly increases to the maximum value; and so on. The variations in speed are illustrated by the dotted curve. In order to calculate the various steps of the starting resistance, it is necessary to assume values for the upper and lower limits of

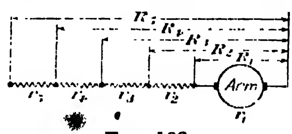


FIG. 102

CALCULATION OF SHUNT MOTOR STARTER

the current; let these be I_1 and I_2 respectively. Let r_1, r_2, r_3, etc., be the various steps, and let R_1, R_2, R_3, etc., be the total resistances between the various studs and the far terminal of the armature, as illustrated in the scheme in Fig. 102. On making contact with any stud the current suddenly jumps to the value I_1, and when leaving any stud the current is I_2. Hence

when leaving stud 5, say, $I_2 = \dfrac{\text{Resultant E.M.F}}{R_5}$

$$= \frac{E - E_b}{R_5}.$$

where E_b is the back E.M.F.

$$= \frac{E - AN}{R_5} \qquad . \quad . \quad . \quad . \quad (1)$$

where A is a constant.

On first making contact with stud 4 the current jumps to I_1, but the speed has not time to change from N,

$$\therefore I_1 = \frac{E - AN}{R_4} \qquad . \quad . \quad . \quad . \quad (2)$$

On the point of leaving stud 4 the speed has increased to say, N_1,

$$\therefore I_2 = \frac{E - AN_1}{R_4} \qquad . \quad . \quad . \quad . \quad (3)$$

On first making contact with stud 3,

$$I_1 = \frac{E - AN_1}{R_3} \qquad . \quad . \quad . \quad . \quad (4)$$

Dividing equation (2) by (1), and (4) by (3), we hav

$$\frac{R_5}{R_4} = \frac{I_1}{I_2} \text{ and } \frac{R_4}{R_3} = \frac{I_1}{I_2}$$

Hence, if we put $\dfrac{I_1}{I_2} = \gamma$, we have, if there are n live studs, i.e. $(n-1)$ sections in the resistance,

$$\frac{R_n}{R_{n-1}} = \frac{R_{n-1}}{R_{n-2}} = \ . \ . \ . \ \frac{R_2}{R_1} = \gamma$$

Multiplying all the terms together, we have

$$\gamma^{n-1} = \frac{R_n}{R_1}$$

$$\therefore \gamma = \sqrt[n-1]{\frac{R_n}{R_1}}$$

Now R_1 is the armature resistance, which is known. On making contact with the first line stud the speed is zero and the current is I_1. Hence

$$R_n = \frac{E}{I_1}$$

Therefore, since γ is known from the required ratio of the maximum and minimum currents during starting, n can be calculated. The various resistances, R_n, R_{n-1}, R_{n-2}, etc., can next be calculated, and the differences of successive pairs give the required resistances, r_1, r_2, r_3, etc.

Example. As an example of the calculation of the resistance steps of a shunt motor-starter, take the case of a 30 h.p., 220-volt motor which takes 113 amp. when running normally on full load. The motor has to start up against full load and the starting current has not to exceed $1\frac{1}{2}$ times the normal full-load value. Resistance of armature circuit 0·02 ohm.

Since the motor has to start against full-load torque, the minimum current during starting will be the full-load current of 113 amp. Hence $I_2 = 113$.

$$I_1 = 1\tfrac{1}{2} \times 113 = 169{\cdot}5 \text{ amp.}$$

and $$\frac{I_1}{I_2} = \gamma = 1{\cdot}5$$

$$R_n = \frac{E}{I_1} = \frac{220}{169{\cdot}5} = 1{\cdot}3 \text{ ohms}$$

and $$R_1 = 0{\cdot}02, \text{ as stated.}$$

Now $$\frac{R_2}{R_1} = \gamma = 1{\cdot}5$$

$$\begin{aligned}
\therefore \quad R_2 &= 1{\cdot}5 \times R_1 = 1{\cdot}5 \times {\cdot}02 = {\cdot}03 \\
\text{Similarly,} \quad R_3 &= 1{\cdot}5 \times R_2 = 1{\cdot}5 \times {\cdot}03 = {\cdot}045 \\
R_4 &= 1{\cdot}5 \times R_3 = 1{\cdot}5 \times {\cdot}045 = {\cdot}068 \\
R_5 &= 1{\cdot}5 \times R_4 = 1{\cdot}5 \times {\cdot}068 = {\cdot}102 \\
R_6 &= 1{\cdot}5 \times R_5 = 1{\cdot}5 \times {\cdot}102 = {\cdot}153 \\
R_7 &= 1{\cdot}5 \times R_6 = 1{\cdot}5 \times {\cdot}153 = {\cdot}23 \\
R_8 &= 1{\cdot}5 \times R_7 = 1{\cdot}5 \times {\cdot}23 = {\cdot}35 \\
R_9 &= 1{\cdot}5 \times R_8 = 1{\cdot}5 \times {\cdot}35 = {\cdot}53 \\
R_{10} &= 1{\cdot}5 \times R_9 = 1{\cdot}5 \times {\cdot}53 = {\cdot}8 \\
R_{11} &= 1{\cdot}5 \times R_{10} = 1{\cdot}5 \times {\cdot}8 = 1{\cdot}2
\end{aligned}$$

The total was to be 1·3, so $R_{11} = 1{\cdot}2$ will be near enough for most purposes, but if it is desired to keep to the value 1·3, the adjustment can be made when the individual steps have been calculated. We have

$$\begin{aligned}
r_2 &= R_2 - R_1 = {\cdot}03 - {\cdot}02 = {\cdot}01 \text{ ohm} \\
r_3 &= R_3 - R_2 = {\cdot}045 - {\cdot}03 = {\cdot}015 \text{ ,,} \\
r_4 &= R_4 - R_3 = {\cdot}068 - {\cdot}045 = {\cdot}023 \text{ ,,}
\end{aligned}$$

$$
\begin{aligned}
&= R_5 - R_4 = \cdot 102 - \cdot 068 = \cdot 034 \text{ ohm} \\
&= R_6 - R_5 = \cdot 153 - \cdot 102 = \cdot 051 \quad ,, \\
r_7 &= R_7 - R_6 = \cdot 23 - \cdot 153 = \cdot 077 \quad ,, \\
r_8 &= R_8 - R_7 = \cdot 35 - \cdot 23 = \cdot 12 \quad ,, \\
r_9 &= R_9 - R_8 = \cdot 53 - \cdot 35 = \cdot 18 \quad ,, \\
r_{10} &= R_{10} - R_9 = \cdot 8 - \cdot 53 = \\
r_{11} &= R_{11} - R_{10} = 1 \cdot 2 - \cdot 8 =
\end{aligned}
$$

Thus there are 10 steps with 11 contacts, and, if desired, the deficit of 0·1 ohm can be divided equally among a few of the steps, say, r_7 to r_{11}, making these ·079, ·14, ·20, ·29, and ·42 ohm respectively.

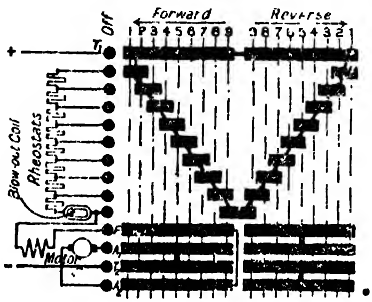


FIG. 103

Finally, for a motor of this size it may be desirable to attain the first rush of current in two steps, the first step allowing 169·5/2, say, 85 amp. to pass. The total resistance in the starter must thus be 220/85 = 2·6 ohms, the extra step therefore being 2·6 − 1·3 = 1·3 ohms.

11. **Starters for Series Motors.** These are usually controllers, that is, they can be left in circuit for any length of time, since they also act as speed regulators. These controllers are in the form of a rotating cylinder carrying segments which make contact with fixed insulated "fingers." A blow-out coil is also provided, its function being to extinguish any arc formed when a segment breaks contact with a finger. This blow-out coil is simply an electromagnet, the lines of force of which cross the space between the fingers and the segments, so that, when an arc is formed, it is acted on by a mechanical force perpendicular to the plane of the current and the flux, that is, in a vertical direction. The arc is thus blown out. The developed view of such a controller is shown in Fig. 103. It is arranged for both forward and reversed directions of rotation.*

* The calculation of a starter for a series motor is beyond the scope of this book. For a complete solution, see Dover, *Electric Traction*, Smith, "Steps in the Starter of a Series Motor," *Jour. I.E.E.*, Vol. 58, p. 645, and Dover, *Jour. I.E.E.*, Vol. 60, p. 867.

12. Time of Acceleration. The time required for the motor to attain full speed can be calculated as follows. It will be seen by the speed curves of Fig. 96 that the speed, and therefore the angular velocity ω, is approximately proportional to the time if the starting handle is moved uniformly. Hence, the angular acceleration $d\omega/dt$ is constant. Let I be the mean value of the current *available for acceleration.* Then accelerating torque

$$T = \left(\frac{\Phi Z}{20\pi} \times \frac{P}{A}\right) \times I \text{ dyne cm.}$$

If K is the moment of inertia of the moving masses in gram. cm.^{2}, then

$$T = K\frac{d\omega}{dt} \text{ or } \frac{d\omega}{dt} = \frac{T}{K} = \left(\frac{\Phi Z}{20\pi} \times \frac{p}{A}\right) \times \frac{I}{K}$$

Hence, the time required to attain the full speed ω,

$$t = \frac{\omega}{\text{angular acceleration}}$$

$$= \left(\frac{20\pi}{\Phi Z} \times \frac{A}{P}\right) \times \frac{K\omega}{I}$$

Example. A 50 h.p., 400 volt, 500 r.p.m. shunt motor takes on full load 110 amp. Its armature has a moment of inertia of 200×10^6 gram. cm.^{2}. Find the time taken to attain full speed if the maximum and minimum currents during starting are 150 and 120 amperes.

Full speed angular velocity $\omega = 52{\cdot}5$ radians per sec.

$$\text{Torque on full load} = \frac{33{,}000 \times \text{h.p.}}{2\pi \times N} = 525 \text{ lb. ft.}$$

$$= 71 \times 10^8 \text{ dyne cm.}$$

The mean starting current is 135 amp: and therefore, since 110 of these are required to produce the load torque, the other 25 amp. are available for acceleration. A current of 110 amp. produces a torque of 71×10^8, hence the current of 25 produces an accelerating torque of

$$T = \frac{25}{110} \times 71 \times 10^8 = 16 \times 10^8 \text{ dyne cm.}$$

$$\frac{d\omega}{dt} = \frac{T}{K} = \frac{16 \times 10^8}{200 \times 10^6} = 8$$

$$t = \frac{\omega}{d\omega/dt} = \frac{52{\cdot}5}{8} = 6{\cdot}6 \text{ sec.}$$

13. Speed Control of D.C. Motors. I. RHEOSTATIC CONTROL. Let an adjustable resistance R be placed in series with the armature making the total resistance in the armature circuit $(R + R_a)$, then the back E.M.F. for any armature current I_a is given by

$$E_b = E - (R + R_a)I_a$$

At no-load, and with no-series resistance R in circuit the back E.M.F. is approximately equal to the applied P.D., E. Since, for constant

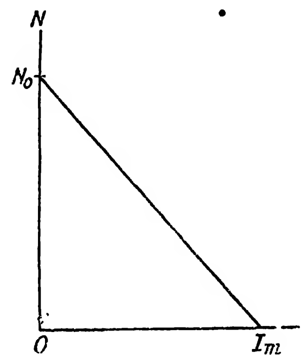


FIG. 104

excitation, the speed is proportional to the back E.M.F., we have, denoting the no-load speed by N_o—

$$\frac{N}{N_o} = \frac{E - (R + R_a)I_a}{E}$$

$$\therefore \qquad N = N_o\left\{1 - \frac{R + R_a}{E} . I_a\right\}$$

Now put $\qquad R + R_a = R_t$

$$\therefore \qquad N = N_o\left(1 - \frac{R_t}{E} . I_a\right)$$

For a given resistance R_t the speed is thus a linear function of the armature current I_a, the graph of N against I_a being a drooping straight line, as indicated in Fig. 104. The amount of the droop obviously depends upon the value of R_t, and therefore upon R, and obviously a whole family of speed-current curves can be drawn, each curve in the family corresponding to a definite value of R.

We also see that for each value of R there will be a certain value, I_m, of the armature current which just stalls the motor. This value is given by

$$0 = N_o\left(1 - \frac{R_t I_m}{E}\right)$$

$$\therefore \qquad I_m = \frac{E}{R_t}$$

Example. A shunt-wound motor runs at 500 r.p.m. on a 200-volt circuit. Its armature resistance is 0·5 ohm, and the current taken is 30 amp. in addition to the field current. What resistance must be placed in series with the armature in order that the speed may be reduced to 300 r.p.m., the current in the armature remaining the same? If the load is changed so that with the inserted resistance the armature current is reduced to 15 amp., what then will be the speed? (London Univ.)

From
$$N = N\left(1 - \frac{R_t}{E} \cdot I_a\right)$$

we have
$$300 = 500\left(1 - \frac{R_t}{200} \cdot 30\right)$$

$$\therefore \qquad R_t = 2{\cdot}667 \text{ ohms}$$

$$\therefore \qquad R = R_t - R_a = 2{\cdot}667 - 0{\cdot}5 = 2{\cdot}167 \text{ ohms}$$

If the current falls to 15 amp., the resistance R_t remaining at 2·667 ohms, we have

$$N = 500\left(1 - \frac{2{\cdot}667}{200} \times 15\right)$$

$$= 400 \text{ r.p.m.}$$

We can also calculate the current which will just stall the motor. Thus with $R_t = 2{\cdot}667$ ohms, we have for this current

$$I_m = \frac{E}{R_t} = \frac{200}{2{\cdot}667} = 75 \text{ amps.}$$

The above example illustrates one of the disadvantages of this method of control, namely, that for a given value of the resistance the speed is not a constant, but is a function of the load current. Thus, with an external resistance of 2·167 ohms, the speed is 300 r.p.m. when the current is 30 amp., but it rises to 400 r.p.m. when the current falls to 15 amps. This means that if the speed is to be kept sensibly constant on a rapidly-changing load, the value of the controller resistance must be varied to suit those changes. For precise speed control it can be said that the speed must be a function of the controller resistance only, and not a function of the load current, and because of this, more elaborate systems of control, such as the Ward-Leonard control described later, have been evolved for drives where precision of control is a necessity.

Another disadvantage of the method is that there will be considerable loss of energy in the controller, particularly at low speeds, since the resistance carries the full armature current. The intake of the

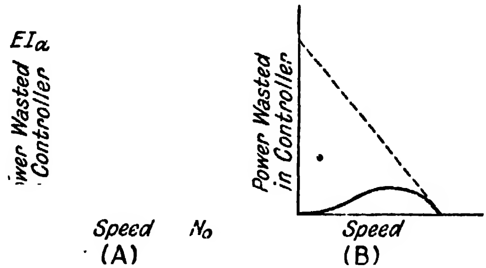


FIG. 105

armature circuit is EI_a watts, while the power converted into mechanical power is E_bI_a watts.

$$\therefore \text{ Efficiency of armature circuit} = \frac{E_bI_a}{EI_a}$$

$$= \frac{E - R_t \, . \, I_a}{E}$$

$$= \left(1 - \frac{R_t}{E} \, . \, I_a\right)$$

$$= \frac{N}{N_o}$$

$$\frac{\text{Power wasted in controller}}{\text{Armature intake}} = \frac{N_o - N}{N_o}$$

In other words, the power wasted is proportional to the reduction of speed. Thus, if the speed is reduced 50 per cent, the power wasted will be 50 per cent, and the efficiency will be 50 per cent. If the speed is brought down to zero, then the whole of the armature intake will be wasted in the controller. The graph of power wasted in the controller against speed is a straight line, as illustrated in Fig. 105 (*A*). The disadvantage of this inefficiency is that it not only entails a considerable loss of energy, but it also necessitates a large and expensive controller with elaborate arrangement for the dissipation of the heat produced in it.

In the above discussion we have assumed that the armature current has remained constant during the change in speed; in other words, that the load has been one of constant torque. In the case

of appliances working on the centrifugal principle, such as fans, the torque is proportional to the square of the speed, and this makes a considerable difference to the efficiency of the rheostatic method.

Let T = torque

and T_m = maximum torque

at any speed $T \propto N^2$

But $I_a \propto T$

$\therefore \quad I_a \propto N^2 = AN^2$, say,

where A is a constant. At full speed N_o the torque is T_m, so that

$$I_m = AN_o^2$$

$$\therefore \quad A = \frac{I_m}{N_o^2}$$

For the speed N at any torque T, we therefore have

$$\frac{N^2}{N_o^2} = \frac{T}{T_m} = \frac{I_a}{I_m}$$

$$\therefore \quad I_a = I_m \times \left(\frac{N}{N_o}\right)^2$$

Now the back E.M.F. is proportional to N, and we have

$$E_b = E \times \frac{N}{N_o}$$

Hence, drop of volts along the controller

$$= E - E_b = E - E \times \frac{N}{N_o}$$

$$= E\left(1 - \frac{N}{N_o}\right)$$

Hence, power wasted in the controller

$$W = \begin{pmatrix}\text{drop along}\\ \text{controller}\end{pmatrix} \times \text{current}$$

$$= E\left(1 - \frac{N}{N_o}\right) \times I_m \times \left(\frac{N}{N_o}\right)$$

$$= EI_m\left(1 - \frac{N}{N_o}\right)\left(\frac{N}{N_o}\right)^2$$

Now write $\dfrac{N}{N_o} = r$

$$\therefore \qquad W = EI_m(1 - r)r^2$$
$$= EI_m(r^2 - r^3)$$

when $\qquad N = N_o,\ r = 1$, and $W = 0$,

when $\qquad N = 0,\quad r = 0$, and $W = 0$.

We thus see that in this case the power wasted in the controller is zero at both extreme limits of speed, and by giving r a series of values between 0 and 1·0 it is easy to see that the graph of power wasted is of the form shown in Fig. 105 (B).

Differentiating the expression for W with respect to r, we have

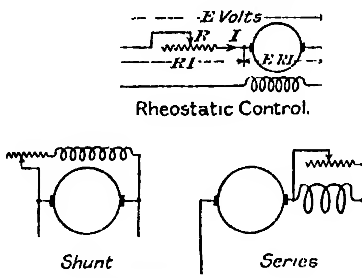
$$\frac{dW}{dr} = 2r - 3r^2$$
$$= 0 \text{ when } r = \tfrac{2}{3}$$

Hence, the maximum loss of power will take place when the speed is two-thirds of full speed. Again, EI_m is the full-load intake of the motor, W_m, say.

$$\therefore \quad \text{Maximum loss} \quad = W_m\left\{\left(\frac{2}{3}\right)^2 - \left(\frac{2}{3}\right)^3\right\}$$
$$= \cdot 148 W_n.$$

as against W_m, when the load is one of constant torque. This shows that when the load is one whose torque is a function of the speed, the rheostatic method is no longer excessively inefficient.

II. Field Control. We have seen that the speed of a D C. motor is inversely proportional to the flux per pole, and hence, if this



Field Control

Fig. 106

Methods of Speed Control

flux is varied, the speed will vary. This is accomplished by means of a shunt regulator in the case of a shunt motor, and a diverter in the case of a series motor, as shown in Fig. 106. This method is both convenient and economical, but obviously it will only give speeds greater than normal. By a combination of methods I and II, speeds below or above normal can be obtained. If a large range of speed by the field control is required, the motor must be fitted

with interpoles, because, owing to the high speed, the reactance voltage will be high, and owing to the fact that the field is weakened, the commutating field, if it is the fringe of the main field, will also be weak. Because of the high reactance voltage at high speeds, the commutating field must be strong if the motor is to run sparklessly.

The necessary variations in shunt resistance to obtain a given change in speed can only be determined when the magnetization

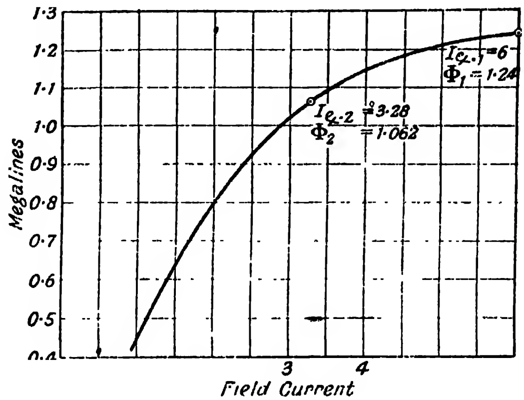


FIG. 107

characteristic of the machine is known. Since the back E.M.F. is proportional to the product of the flux and speed, we have

$$E_{b1} \propto \Phi_1 N_1$$
$$E_{b2} \propto \Phi_2 N_2$$
$$\therefore \qquad \Phi_2 = \Phi_1 \times \frac{E_{b2}}{E_{b1}} \times \frac{N_1}{N_2}$$

The conditions of loading of the motor will enable the back E.M.F.s to be calculated, and the new flux Φ_2 can then be determined from the above equation. A reference to the magnetization characteristic will then give the new excitation.

Example. A 230-volt D.C. shunt motor whose magnetization curve is given by the figures below, runs at no-load at 1,200 r.p.m. The resistance of the field magnet coils is 38·3 ohms. Find what resistance must be placed in series with the field magnet coils to increase the speed to 1,400 r.p.m. at no-load. (London Univ.)

Amperes in magnet coils . . .	1	2	3	4	5	6
Flux per pole in megalines . . .	0·44	0·8	1·02	1·15	1·21	1·24

Since the armature resistance and current are not specified, we can take the back E.M.F. as being equal to the applied P.D., so that the expression simplifies to

$$\Phi_2 = \Phi_1 \times \frac{N_1}{N_2}$$

The exciting current for full excitation is

$$I_{ex.1} = \frac{230}{38 \cdot 3} = 6 \text{ amp.}$$

$$\therefore \qquad \Phi_1 = 1 \cdot 24 \text{ megalines}$$

$$\therefore \qquad \Phi_2 = 1 \cdot 24 \times \frac{1200}{1400} = 1 \cdot 062 \text{ megalines}$$

The magnetization curve is drawn in Fig. 107, from which we see that the new value of the exciting current is

$$I_{ex \cdot 2} = 3 \cdot 28 \text{ amp.}$$

Hence, new total resistance in shunt circuit

$$= \frac{230}{3 \cdot 28} = 70 \cdot 2 \text{ ohms.}$$

∴ Resistance in shunt regulator to increase the speed from 1,200 to 1,400 r.p.m. = 70·2 – 38·3 = 31·9 ohms.

III. SERIES PARALLEL CONTROL. This method, in conjunction with auxiliary rheostatic control, is used for electric traction. It requires two motors mechanically coupled, this coupling, in the case of a tram or locomotive, being supplied by the adhesion of the wheels to the rails. Consider two series motors in parallel, as shown in Fig. 108 (B), and let them be taking a total current I.

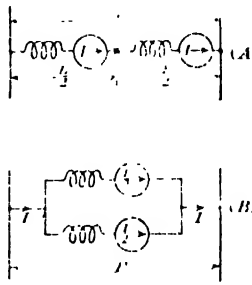


FIG. 108
SERIES PARALLEL CONTROL

$$\text{Then, speed} \propto \text{Back E.M.F./Flux}$$

$$\propto \text{Applied E.M.F./Current per motor}$$

approximately.

$$\propto E \div \frac{I}{2} \text{ or } \frac{2E}{I}$$

$$\text{Again, Torque} \propto (\text{Current} \times \text{Flux})$$

$$\propto \text{Current}^2$$

$$\propto \frac{I^2}{4}$$

Now consider the motors connected in series (Fig. 108, A), and let

the pair take the same total current as before. Then the P.D. across each motor is $E/2$, so that

$$\text{Speed} \propto \frac{\text{Applied E.M.F.}}{\text{Current per motor}}$$

$$\propto \frac{E}{2I}$$

that is, one quarter of the speed attained by the motors in parallel.

Also, $\text{Torque} \propto \text{Current}^2$

$$\propto I^2$$

that is, four times the torque produced by the motors in parallel.

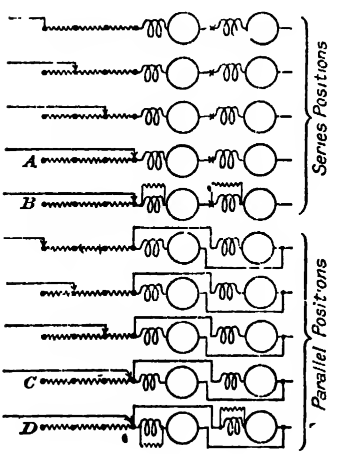


Fig. 109

Connections of Series Parallel Controller for Different Speeds

The various changes in connections as the controller handle is moved up to the full speed position are shown in Fig. 109. At the four running positions, A, B, C, and D, the controller can be left in that position for any length of time, but the other positions have to be passed through fairly quickly, because the resistances cannot carry current for long periods.

IV. Ward-Leonard Control. This method is commonly used where a very delicate speed control over the whole range from zero to full speed is required, as, for example, with colliery winders. The method consists simply in working the motor with a constant excitation and applying to its armature sufficient voltage to give the speed required. A variable voltage supply is therefore required, and it is obtained from a motor-generator, or convertor, set. The scheme is shown in Fig. 110. M is the variable speed motor. A is the motor, and B the generator of the convertor set. The variable voltage of the generator is obtained by varying its excitation by means of the shunt regulator R. Speeds in the reversed direction are obtained by reversing the generator excitation as indicated diagrammatically by the reversing switch (R.S.). The convertor set runs always in the same direction.

When applied to the control of very large motors, e.g. colliery winders or reversing rolling mills, the supply is nearly always an

alternating one, and it is often desirable to incorporate a load equalizing arrangement in order to keep from the supply the violent peak loads which occur in such cases. The necessary modifications are shown in Fig. 111. The supply being alternating the motor A of the convertor set is of course a three-phase motor, and since D.C. is required for the excitation of the variable speed motor M and variable voltage generator B, the convertor set has an exciter E direct coupled to it. The flywheel for load equalizing is also coupled to the convertor set. Now if the flywheel is to keep the fluctuations in load from the mains, it must have its speed reduced when the peak load comes on, and have it raised again in times of light load. The necessary variations in speed of the converter set are obtained automatically by means of a "slip regulator," which puts resistance in the rotor circuit of the motor A when the load increases and cuts it out again when the load decreases. This resistance is generally a liquid resistance R, the electrodes of which balance the torque of a small "torque motor" (T.M.) at normal load This motor is supplied through a current transformer (C.T.), and therefore carries current proportional to that delivered to the motor A. If the load suddenly increases there is a momentary tendency of the current taken by A to increase, and therefore the current in T.M. tends to increase. This increases the torque exerted by T.M., and it therefore lifts up the electrodes,

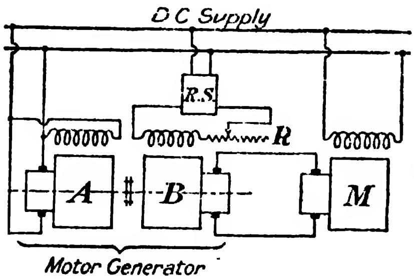


FIG. 110

WARD-LEONARD CONTROL

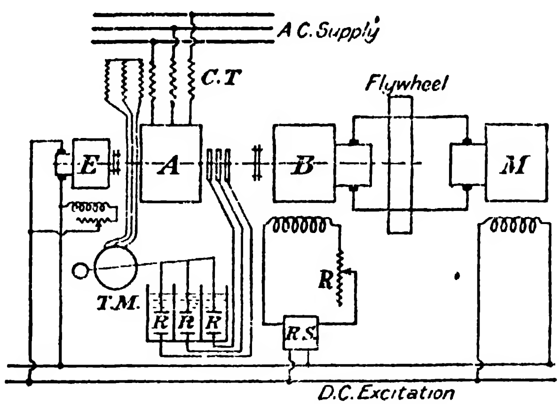


FIG. 111

WARD-LEONARD-ILGNER CONTROL

thereby increasing the resistance in series with the rotor of A and decreasing the speed of the convertor-flywheel set. The reverse action takes place when the load decreases. The Ward-Leonard Control with flywheel load equalizer is known as the *Ward-Leonard-Ilgner Control.*

The Ward-Leonard control can be modified in the following manner. Instead of obtaining the whole of its energy from the generator B, the motor M can have its armature connected in series with that of B, and with the supply, as shown in Fig. 112, with the result that the voltage applied to M is the resultant of the supply voltage and the voltage in B. Suppose that the excitation of B is adjusted so that its voltage is equal and opposite to that of the

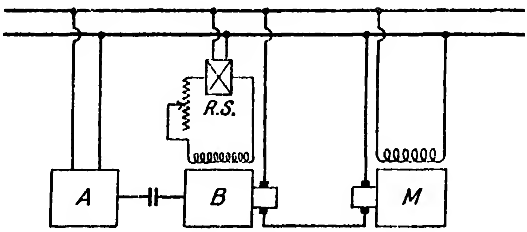


FIG. 112

supply, then the resultant voltage will be zero and the motor will not rotate. As the excitation of B is reduced, the resultant voltage will increase, and the speed of M will therefore increase. With zero excitation of the generator the motor speed will be one-half of full speed. If, now, the excitation of B is reversed the generator will "boost" instead of "buck" the supply voltage, and the motor speed will increase still further. On full excitation when the generator voltage is equal to the supply voltage, the motor will be running at full speed.

This method differs in two important particulars from the Ward-Leonard control. In the first place, the two machines B and M in the Ward-Leonard control are wound for the same maximum voltage, and this voltage is independent of the supply voltage. In the second method the maximum voltage of B is equal to the supply voltage, while that of M is equal to twice the supply voltage. In the second place, the starting conditions for the Ward-Leonard control correspond to zero excitation of the generator B. Now the starting conditions are usually very heavy, which means that the generator has to be designed to fulfil the very difficult conditions of delivering a heavy current on a small excitation. This difficulty is overcome in the modified method, since the starting conditions correspond to full "buck" by B, and therefore to full excitation.

EXAMPLES ON CHAPTER VIII.

(1) Prove the expression for the force on a conductor carrying a steady current in a magnetic field. Hence, find the torque on an armature carrying a total of 50,000 ampere-conductors, the diameter of the core being 36 in., the length, 12 in., the pole arc being 70%, and the flux density in the gap, 4,500. (London Univ., 1910.)

Ans.—1,620 lb.-ft.

(2) The flux in each pole of a four-pole motor is 2×10^6 lines. What total number of ampere-conductors must be carried by the armature to produce a torque of 400 lb. ft. ? Prove any formulae you use for the calculation. (C. and G., 1908.)

Ans.—42,600.

(3) Prove that when a current is taken through a magnetic field, work is done. Hence, find the pole flux of a 50 h.p. four-pole motor having 30,000 ampere-turns on the armature running at 800 r.p.m., the efficiency being ·85. (London Univ., 1911.)

Ans.— 1·28 megalines
(assuming iron and friction losses to be one-third of total losses).

(4) A shunt motor which is supplied with current at 440 volts, runs when unloaded and at atmospheric temperature at a speed of 1,000 r.p.m. After some hours its temperature rises 30°C. Calculate the speed of the motor, assuming the voltage of the circuit falls to 435, that the armature current is 85 amp., and that the armature resistance is 0·04 ohm. The temperature coefficient of the conductors may be taken as 0·44% per °C. (London Univ., 1914.)

Ans.—1,130 r.p.m.

(5) A 220 volt shunt motor running light takes 4 amp. The field resistance is 220 ohms, and the resistance of the armature (hot) is ·5 ohm. What will be the armature current when the motor is giving 6 h.p. ? Explain what assumptions are made in calculating this result, and what errors these assumptions are likely to involve. (London Univ., 1922.)

Ans.—About 24·5 amp.

(6) Show that a shunt motor when allowed to speed up with constant field without altering the resistance of the armature, will increase in speed according to an exponential law. (London Univ., 1922.)

(7) A 220 volt shunt motor takes 3·4 amp. when running light. The field current is 1·0 amp. When the armature is at rest it requires 6·2 volts to pass 20 amp. through it. Find the approximate output and efficiency of the motor when the armature current is (*a*) 20 amp., (*b*) 40 amp. (London Univ., 1916.)

Ans.—(*a*) 5 h.p., 81 per cent ; (*b*) 10·4 h.p., 86·2 per cent.

(8) State clearly what information you would need to enable you to calculate a metallic starter with a given number of steps for a series motor. Calculate the resistance steps for a starter of a 500 volt shunt motor, given number of steps 12, maximum current during starting 20 amp., resistance between armature terminals 1 ohm. (C. and G., 1921.)

Ans.—5·9, 4·5, 3·45, 2·6, 2·0, 1·54, 1·18, 0·9, 0·69, 0·52, 0·41, 0·31.

(9) A compound-wound continuous current generator is used as a motor without any change of connections. Discuss its speed characteristic on a moderate fluctuating load and its behaviour on very large loads. For what purpose would such a machine be suited? Would there be any advantage in fitting it with a flywheel? (London Univ., 1911.)

(10) Find an expression for the force on a conductor carrying a steady current in a magnetic field. Hence find the torque on an armature carrying a total of 25,000 ampere conductors, the diameter of the core being 36 in., the length 12 in., the polar arc 70 per cent, and the flux-density in the gap 6,000. (London Univ., 1912.)

Ans.—1,078 lb.-ft.

(11) What is meant by the "mechanical characteristic" of an electric motor? Sketch the form it takes in the case of a shunt-wound direct-current machine connected to mains which are kept at a constant voltage. A 20 h.p. direct-current shunt motor has a full load efficiency of 88% when supplied with power at 200 volts, its armature resistance being 0·06 ohm, and its shunt resistance 80 ohms. Find approximately, indicating the assumptions made, the percentage change in speed from no load to full load, the voltage of the supply mains being kept constant at 200 volts (London Univ., 1924.)

Ans.—2·5 per cent.

(12) A four-pole series motor has a magnetization characteristic given by the following figures—

Amp. through field magnet—	10	20	30	40	50	60	70	80	90
Flux per pole (megalines)—	0·29	0·55	0·74	0·85	0·92	0·97	1·01	1·06	1·08

If the armature has 820 conductors and is wave wound, and the resistance of the armature and field magnet windings is 0·15 ohm, estimate the speed at which the motor will run when supplied from a constant P.D. of 230 volts, when taking a current of (*a*) 45 amp., (*b*) 85 amp. Calculate also the torque in lb.-ft. which the motor will exert when carrying these currents. (London Univ., 1924.)

Ans.—(*a*) 920 r.p.m., 76 lb.-ft.; (*b*) 744 r.p.m., 175 lb.-ft.

CHAPTER IX

OPERATION OF D.C. GENERATORS IN SERIES AND IN PARALLEL

1. **Series and Parallel Running.** It is usual to have several generators in a power station, so that the number of generators in operation at any time can be varied to suit the magnitude of the load on the station. There are two ways in which the generators can be connected to the load, namely, in parallel or in series with it. In the former, which is the method almost universally adopted, all the machines work at the same voltage, and the load on any individual machine is proportional to the current delivered by it. In the series arrangement, all the generators deliver the same current, and the load on any machine is proportional to its voltage.

2. **Series Generators in Series.** No difficulties are experienced in working any number of these machines in series (Fig. 113). The method is only rarely employed, the most important example being the constant current transmission system which supplies Lyons (France). In this system there are three generating stations in series, namely, Bozel, 54,000 volts; Moutiers, 30,000 volts; and Fond de France, 36,000 volts; the total voltage being therefore 120,000. The current is kept constant at 150 amp., and if the load varies, the generated voltage is varied by rocking the brushes round the commutator. In times of light load the generators not required for service are taken out of action by short-circuiting them. The constant current series system is known as the Thury System.

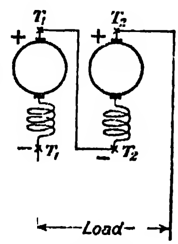


FIG. 113
SERIES GENERATORS IN SERIES

3. **Series Generators in Parallel.** Let E_1 and E_2 be the E.M.F.s induced in the two machines (Fig. 114), and let R be the resistance of either. Then they will share the load equally so long as E_1 and E_2 are equal. Suppose that E_1 becomes slightly greater than E_2, which, of course, is quite probable. Then a current given by

$$i = \frac{E_1 - E_2}{2R}$$

will circulate in a clockwise direction round the local path shown by the dotted line.* The total current delivered by machine I will now be greater than that delivered by II, and therefore, because of the rising characteristics of series machines, the voltage E_1 will increase, while E_2 will decrease. The difference $(E_1 - E_2)$ will thus increase, and so also will the circulating current. The effect

is thus cumulative, so that eventually, if there were no fuses or automatic switches in the circuit, the total current through II would be reversed. This would reverse E_2, and the resultant E.M.F. acting round the local path would be $(E_1 + E_2)$, and the circulating current $(E_1 + E_2)/2R$. The machines would then burn out. To prevent the possibility of reversal of either machine, it is obvious that the circulating current produced by inequalities in voltage must not be allowed to flow through the field windings ; this is done by connecting a heavy copper bar of negligible resistance across the two machines, as shown in the second figure. The circulating current is now confined to the armatures and equalizing bar. This arrangement is used in the "braking" position of a tramcar controller. The two motors are disconnected from the line and are connected in parallel to the resistance. They continue to run because of the kinetic energy of the car, and they transform this kinetic energy into electrical energy which, in turn, is dissipated as heat in the resistances. The motors thus function as generators in the braking positions, and since they are series machines, it is necessary to inter-connect them by an equalizing bar.

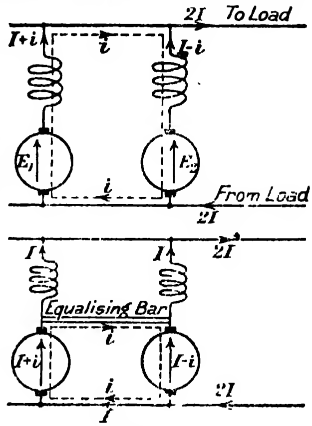


FIG. 114
SERIES GENERATORS OPERATING IN PARALLEL

4. Shunt Generators in Series. When a high voltage is required it is fairly common practice to use two shunt generators, both mechanically coupled and electrically connected in series (Fig. 115). They then carry the same current, and they will share the load equally only so long as each gives the same voltage. If each shunt field is connected to its own armature, as shown in the first figure, any slight differences in voltage will cause differences in excitation, which, in turn, will increase the difference in voltage. There will thus be a tendency to throw more than a fair share of the load on to one machine. This is overcome very easily by connecting the two fields in series and applying the full voltage to them, as shown in the second figure. The excitations are now equal, and, since the speeds are equal, the machines being

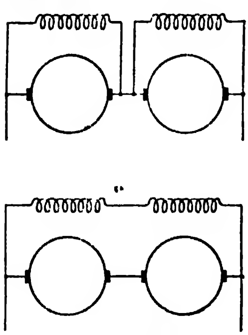


FIG. 115
SHUNT GENERATORS IN SERIES

mechanically coupled, the voltages are equal and therefore the loads on the two machines are equal.

5. **Shunt Generators in Parallel.** When a number of generators work in parallel, all their + terminals and all their − terminals are connected to two heavy copper bars called bus-bars. In the case of D.C. plant these bars are placed behind the switch-board, and they may be regarded as the + and − terminals for the whole station. The generators are connected to these bars through main switches, S_1 and S_2, as shown in Fig. 116. Suppose that machine I is supplying the load and it is necessary to bring up II to share the load with it. The procedure is as follows. The incoming machine II is brought up to speed and its voltage then adjusted by means of its shunt regulator R_2 so as to be equal to, or 1 or 2 volts greater than, the bus-bar voltage. The polarity must also be the same as the polarity of the bus-bars. This adjustment is made by means of a moving coil paralleling voltmeter V, which can be connected either to the incoming machine or to the bus-bars by means of the plug arrangement shown.

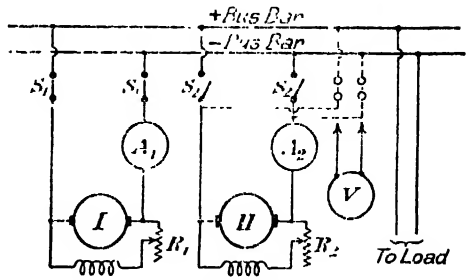


FIG. 116
PARALLEL OPERATION OF SHUNT GENERATORS

When the adjustment is made, the switch S_2 is closed. The machine II will now either motor or generate, according as its induced E.M.F. is less than, or greater than the bus-bar voltage. Since the speed is kept constant by the engine governor, the induced E.M.F. depends on the excitation only. In order to make the machine generate, its excitation must therefore be such that the induced E.M.F. is greater than the bus-bar voltage. After closing S_2 the excitation of machine II is therefore gradually increased and during this operation the load will be gradually transferred from I to II, the ammeter reading A_1 decreasing while A_2 increases. Thus, by adjusting the excitations, either machine can be made to take any desired share of the total load. If it is desired to shut down one machine, say, No. I, it is not good practice to open its main switch, partly because of the violent sparking which will take place at the switch contacts, and also because the whole of the load will be suddenly thrown on to No. II. The excitation in No. I is gradually reduced until the whole of the current is taken by No. II; then the main switch, which is now carrying no current, is opened, and steam is finally shut off. The machine then slows down gradually, and its exciting current

as gradually dies away. It is thus unnecessary to open the field circuit while it is carrying current. If this is done, excessively high voltages will be set up because of the very high self-induction of the field winding of a large machine.

The slightly drooping character of the voltage characteristic of shunt generators renders the proper division of load between them when working in parallel, automatic. The voltage characteristics of two machines are shown in Fig. 117, the point A being the origin for curve I, and B the origin for curve II. The base AB represents the total current delivered by the two machines. Since the machines are in parallel, they must have the same terminal voltage, and this is obviously given by the length PM, where P is the point of intersection of the two curves. Hence, machine I delivers current AM and II delivers current BM. Suppose that, for some reason, the induced E.M.F. of machine I increases slightly, then its current will increase, say, to AM_1, and that taken by machine II will fall to BM_1. Now for this new condition to be stable, machine I should experience an increase in voltage, while II experiences a decrease; but just the reverse takes place, the voltage of I falling to QM_1 and that of II rising to RM_1. Such a state of affairs is thus impossible, and the machines automatically return to the condition defined by their point of intersection P.

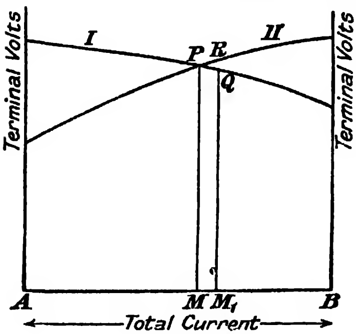


Fig. 117

Operation of Shunt Generators in Parallel

Example. A shunt generator which gives a terminal voltage of 400 at no load, and 360 when delivering 100 amp., is working in parallel with one which gives 400 volts on no load and 350 volts when delivering 100 amp. If the voltage characteristics of each machine are approximately straight lines, find the common terminal voltage and the current in each when they are sharing a total load of 100 amp.

If the two characteristics are drawn as in Fig. 113, and their equations are deduced, *both referred to A as origin*, we have for machine I,

$$V = 400 - k_1 I \text{ where } k_1 \text{ is a constant}$$

$$\therefore 360 = 400 - k_1 \times 100\,; \quad k_1 = \cdot 4$$

For machine II,

$$V = 350 + k_2 I$$

$$\therefore 400 = 350 + k_2 \times 100\,; \quad k_2 = \cdot 5$$

Now let V be the common voltage and I the current in machine. Then,

$$V = 400 - \cdot 4I$$

$$V = 350 + \cdot 5I$$

Subtracting $0 = 50 - \cdot 9I \quad \therefore I = 55\cdot 6$ amp.

$$\therefore V = 377\cdot 8 \text{ volts}$$

and current in machine II $= 100 - 55\cdot 6 = 44\cdot 4$ amp.

6. **Compound Generators in Parallel.** In all cases where generators having any series excitation have to be connected in parallel, an equalizing bar has to be used, since otherwise there is the danger that a circulating current through the bus-bars and machines may cause a reversal of one machine. Fig. 118 is diagrammatic; it shows the essential connections without any switchgear or instruments, while in Fig. 119 the actual arrangement is shown. The main switch is a three-pole switch, and all three poles

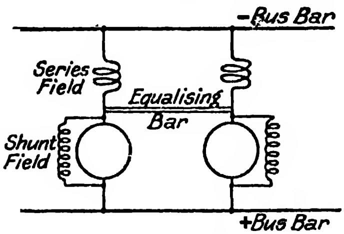


FIG. 118

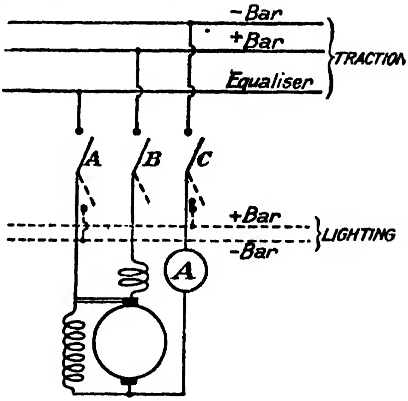


FIG. 119

COMPOUND GENERATORS IN PARALLEL

can be linked together so as to be closed or opened simultaneously, or the equalizer switch can be separate. In the latter case the equalizer switch is closed first and opened last. In stations supplying a combined traction and lighting load it is usual to have separate bus-bars for the two loads. Also, for the lighting load, the compounding is not required, the series fields therefore being left out of circuit. The bus-bars and connections for the lighting load are shown dotted, and it will be noticed that it is not the equalizer switch A, but the middle pole B, which is idle when the machine is thrown on to the lighting bars.

If the generators are provided with interpoles it is absolutely essential that the windings of these poles shall carry the same current as the armature. Hence, if compound interpole machines are worked in parallel, the equalizing bar is connected to the

junction of the series and interpole fields, as in Fig. 120, so that, in the event of the armatures carrying any circulating current, this current will flow through the interpole windings and not be diverted from them by the equalizer bar.

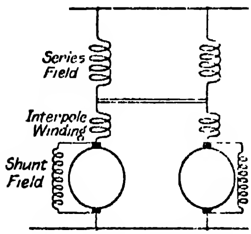


FIG. 120

GENERATOR AND BATTERY. When a battery is used to supply a load, either altogether, or as a standby, arrangements have to be made for charging, either when the voltage has fallen to 1·8 volt per cell, or, more economically, while the load is actually on. Also, owing to the gradual fall in voltage during discharge, it is necessary to arrange that the number of cells in circuit can be gradually increased, so as to make up from time to time for this fall in voltage.

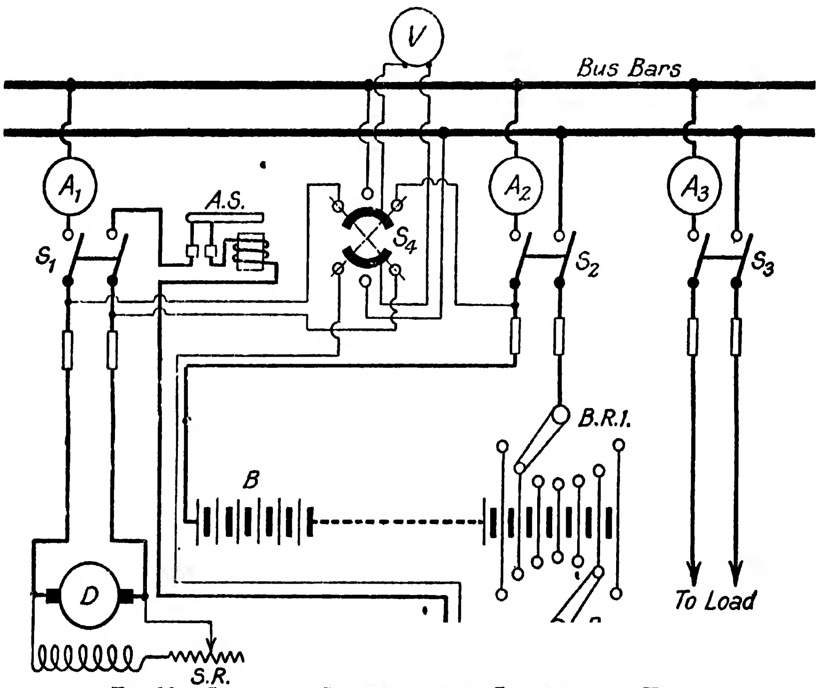


FIG. 121. SCHEME OF CONNECTIONS FOR BATTERY WITH HAND REGULATION DURING CHARGE AND DISCHARGE

The connections for an installation of this type are given in Fig. 121, the plant consisting of charging switch S_1, and discharge switch S_2, automatic switch AS for the prevention of a reversed current flowing

into the generator, voltmeter switch S_4 for paralleling the battery on to the generator, and the battery-regulating switch *BR*. This is a double selector switch, the bottom arm of which controls the numbers

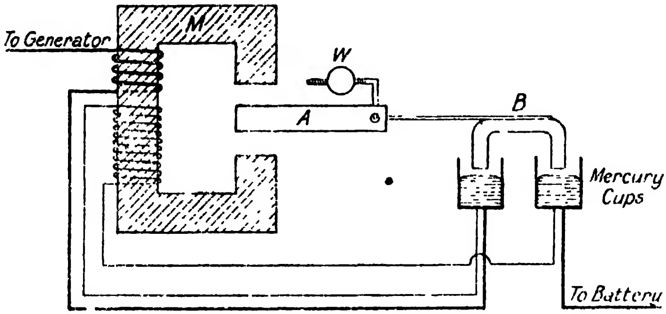


Fig. 122. Automatic Battery Switch

of cells in circuit during charge, and the top arm the number of cells during discharge.

The functions of the automatic switch are (1) to prevent the battery from discharging back into the generator, as explained

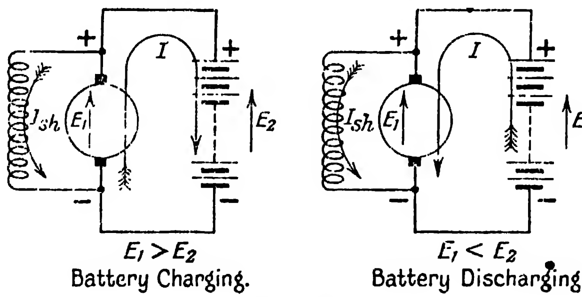


Fig. 123

above, and (2) to ensure that the battery and generator cannot be put in parallel until the generator voltage is in excess of the battery voltage by a small amount, usually 2 or 3 per cent. There are several switches of this type, and Fig. 122 shows the Neville type. The switch portion of this consists of a copper fork which dips into two mercury cups. This forms one extremity of a pivoted beam, the other end of which is a magnetized armature. one pole of which can move in the air gap of a magnet wound with two coils. One coil is a series winding and the other a shunt winding, this latter being excited by the difference between the battery and generator voltages. The scheme is shown in Fig. 121. If the battery voltage is in excess of the generator voltage, the current through the fine wire flows in

such a direction as to hold the switch open. As soon as the generator voltage has exceeded the battery voltage by the required amount the reversed current in the shunt coil is of sufficient magnitude to close the switch, and the flow of current in the series coil then holds the switch closed. At the same time the shunt coil becomes short-circuited, thus preventing a wastage of energy in it. It will be seen that any failure of the current in the series winding releases the arm, and the counterweight lifts the copper fork out of the cups.

In connection with the charging of a battery of accumulators, it is necessary to note that the generator must be shunt wound, since this type only is immune from the risk of a reversal of polarity in the event of the direction of the current in the battery and armature circuit becoming reversed. This point is made clear by Fig. 123, which shows that no matter what the direction of the main current, the shunt current, and therefore the machine polarity, does not reverse.

EXAMPLES ON CHAPTER IX.

(1) Explain how two shunt dynamos work in parallel, and how they share the load. A dynamo gives 200 volts on open circuit and 190 volts when delivering 50 amp. It is in parallel with a battery of cells which, when discharging gives an E.M.F. of 196. The resistance of the battery is ·1 ohm. If the voltage characteristic of the dynamo is approximately a straight line, find how a current of 100 amp. will be shared between dynamo and battery, and what the common terminal P.D. will be.

Ans.—Dynamo current 30 amp., battery current 20 amp., terminal P.D. 194 volts.

(2) Investigate the effect of the resistance of the connections from compound generators to the equalizing bar, on the correct division of load between the machines.

(3) Describe briefly the essential parts of a self-exciting dynamo. A given machine, when running, refuses to develop the required voltage at its terminals. Enumerate possible causes of failure. (Inst. Elec. Eng., 1924.)

CHAPTER X

DISTRIBUTION

1. **System of Distribution.** A low-tension distribution consists of a network of cables which can be divided into three categories, viz. (*a*) the "feeders," which are the cables supplying power in bulk to a selected number of points called feeding points; (*b*) the "distributors," from which current is tapped for the various consumers, these cables generally having the main street for their route; (*c*) the "service mains," which are the small cables teed off from the distributors and taken into the premises of the various consumers.

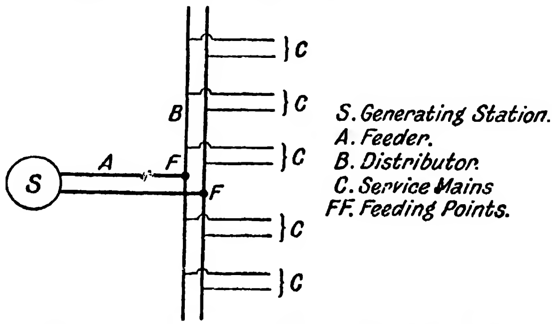


FIG. 124 ELEMENTARY DISTRIBUTION SCHEME

A very elementary scheme employing one feeder, one distributor, and a number of service mains is given in Fig. 124. In this simple scheme the distributor is fed at one point only, but in actual schemes a long distributor may be fed at two points by means of two separate feeders, or alternatively the distributor may be arranged to form a closed circuit, and fed at any desired number of feeding points by means of a number of feeders radiating from the generating station. It will be shown in paragraph 10 that these arrangements give a much smaller drop of volts than a distributor fed at one end only.

2. **Effect of Supply Voltage on the Size of Distributors and Feeders.** By the size of a cable we always mean the cross section of the core carrying the current, and not the overall cross section. Consider first of all the case of a feeder. Since there are no consumers tapped off from a feeder, it is immaterial what the drop of volts along the feeder will be, provided that it is not outside the range of compounding of the generators in the power station. A feeder can thus be designed from the point of view of current-carrying capacity, with drop of volts as a secondary consideration. The allowable

current density for a given type of cable laid in a given manner is not a constant, but decreases somewhat as the cable size increases; this is because the cooling facilities improve as the size is reduced. As a rough approximation we can, however, assume that the current density is a constant. Now, suppose that the voltage of the system is increased n fold, then for a given power delivered the current is reduced to $\frac{1}{n}$th, and therefore the size of the cable is reduced to $\frac{1}{n}$th. There is thus a very appreciable saving in the price of the copper in the feeders when the supply voltage is increased.

Now take the case of a distributor. There is a Board of Trade regulation which states that the variation of voltage at a consumer's

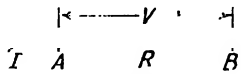


Fig. 125

terminals must not exceed $\pm$ 4 per cent of the declared pressure. Allowing, say, 1 % volt drop along the service mains, this means that the variation in voltage at any point in a distributor must not exceed $\pm$ 3 per cent of the declared pressure. We see from this that whereas a feeder can be designed from the point of view of current-carrying capacity, a distributor must be designed from the point of view of drop of volts. Consider, then, a length AB of distributor, Fig. 125, let the current be I, and let the drop along it be V. Then its resistance is $R = \frac{V}{I}$. Now, suppose that the voltage is increased n fold, so that for the same amount of power delivered, the current is reduced to I/n. In a distributor designed for a definite percentage drop, this percentage will be the same if the drop along the length AB is increased to nV. The resistance of the same length of distributor will now be

$$R^1 = \frac{nV}{I/n} = n^2R$$

Now, for a given length the cross section of a cable is inversely proportional to its resistance, showing that an increase in the working voltage of n times reduces the cross section, and therefore the weight of copper required to n^{-2}.

3. Effect of Pressure on the Efficiency of Transmission.

Let E_1 = voltage at station bus-bars
E_2 = voltage at feeding points
I = current in feeders
R = resistance of each side of a feeder,
i.e. of both + and − sides.

Then, Power put into feeder $= E_1I$

Power taken from feeder $= E_2I$

$\therefore$ Efficiency of transmission $= \dfrac{E_2I}{E_1I} = \dfrac{E_2}{E_1}$

Losses in feeder $= 2I^2R$

Suppose that 1,000 kW have to be transmitted 1 mile, the feeder being worked at a current density of 1,000 amp. per sq. in. Suppose that the consumer's voltage E_2 is 100. Then,

$$I = \frac{\text{Power delivered}}{E_2} = \frac{1{,}000{,}000}{100} = 10{,}000 \text{ amp.}$$

Hence, cross section of copper in either + or − side of the cables

$$a = \frac{10{,}000}{1{,}000} = 10 \text{ sq. in.}$$

Distance l = 1 mile = 5,280 × 12 in.

$\therefore$ Resistance of one side

$$R = \rho \times \frac{l}{a} = \frac{2}{3 \times 10^6} \times \frac{5{,}280 \times 12}{10}$$
$$= \cdot 0042 \text{ ohm}$$

$\therefore$ Drop in volts in feeders

$$= 2IR = 2 \times \cdot 0042 \times 10{,}000 = 84 \text{ volts}$$

$\therefore$ Station voltage

$$E_1 = 100 + 84 = 184$$

$\therefore$ Efficiency of transmission

$$= \frac{100}{184} = 54 \cdot 4\%$$

Now suppose that the voltage at the consumer's terminals is 1,000 instead of 100. Then $E_2 = 1{,}000$ and $I = 1{,}000$ amp.

$\therefore$ Cross section of copper in cables $a = 1$ sq. in.

$\therefore$ $R = \cdot 042$, and drop in feeders $2IR = 84$ volts, as before.

$\therefore$ $E_1 = 1{,}000 + 84 = 1{,}084$, and efficiency $= \dfrac{1{,}000}{1{,}084} = 92\%$

There is thus a considerable increase in efficiency when the voltage is raised. Also, the cross section of copper in the cables is

diminished in proportion to the increase in voltage, and this diminishes the cost of the copper in the cables in proportion. For these reasons it is desirable to transmit at a high voltage. Now in a D.C. system it is not possible to change the voltage between the feeders and distributors without using rotating machinery, and as this is undesirable, the feeders and distributors are worked at the same voltage. This means that the consumer's voltage is the same as the voltage at the feeding points. Suppose, for example, that the load consists of lamps. Then since 200 volts is the highest suitable voltage for this purpose, the feeders must work at this voltage; 200 volts is very low if a large amount of power has to be dealt with, especially if the feeders are long ones, and consequently with a simple two-wire system the efficiency of transmission will be low.

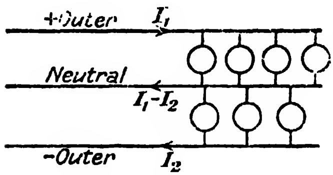


FIG. 126

4. Three-wire System. The above difficulty is largely overcome by using the three-wire system, which consists of two "outers" and an earthed middle wire (Fig. 126). The consumer's apparatus or lamps are connected between one outer and the neutral, the result being that the transmitting voltage, i.e. the voltage between the outers, is twice that at the consumer's terminals. This results in a considerable increase in efficiency, and economy in copper. The middle wire or neutral has half the cross section of either outer. If the loads on the two sides are unequal, then the difference current, called the "out of balance" current, flows in the neutral. Thus if the + and − outers carry I_1 and I_2 amp. respectively, the neutral carries $(I_1 - I_2)$. If a three-wire and a two-wire feeder both transmit the same amount of power the same distance with the same voltage at the consumer's terminals, and with the same efficiency, then the three-wire feeder, if balanced, requires only five-sixteenths as much copper as the two-wire feeder. This is proved as follows.

Let W = power delivered to consumer in watts

E_1 = voltage at station

and E_2 = voltage at feeding points

Then, in the two-wire system, current $I = \dfrac{W}{E_2}$

∴ Drop in both + and − sides,

$$2IR_1 = \frac{2WR_1}{E_2}$$

and $$E_1 = E_2 + \text{drop} = E_2 + \frac{2WR_1}{E_2}$$

where R_1 is the resistance of one conductor.

$$\therefore \text{ Efficiency} = \frac{E_2}{E_2 + \dfrac{2WR_1}{E_2}}$$

In the three-wire system, the total voltage is doubled, and therefore, each outer will carry a current of $W/2E_2$ so long as the currents on the two sides are equal. Hence, if R_2 is the resistance of each outer,

$$\text{Efficiency} = \frac{2E_2}{2E_2 + \dfrac{2WR_2}{2E_2}}$$

Since the efficiencies are equal we have

$$\frac{E_2}{E_2 + \dfrac{2WR_1}{E_2}} = \frac{2E_2}{2E_2 + \dfrac{WR_2}{E_2}} \text{ or } \frac{E_2}{E_2 + \dfrac{WR_2}{2E_2}}$$

Hence $$R_2 = 4R_1$$

Now the cross section, and therefore volume, of a conductor of given length is inversely proportional to its resistance. If we represent the volume of the copper in the two-wire system by 100 we have 50 in each wire.

$\therefore$ Volume of each outer in the three-wire feeder $= \dfrac{50}{4} = 12{\cdot}5$

and volume of middle wire $= \dfrac{12{\cdot}5}{2} = 6{\cdot}25$

$\therefore$ Total volume of copper in three-wire feeder

$$= 12{\cdot}5 + 6{\cdot}25 + 12{\cdot}5 = 31{\cdot}25$$

$$\therefore \frac{\text{Copper in 3-wire feeder}}{\text{Copper in 2-wire feeder}} = \frac{31{\cdot}25}{100} = \frac{5}{16}$$

If the system is not balanced it is most convenient to compare the amounts of copper in the two cases by taking a numerical example.

Example. A three-wire feeder whose outers have a cross section of 0·2 sq. in., is 1 mile long. It carries 220 amp. in the + and 200 in the − outer, and the voltages across the two sides at the feeding points are 200. What size of two-wire feeder will transmit 420 amp. at 200 volts with the same total loss? Compare the amounts of copper in the two feeders.

Taking the specific resistance of copper as $\frac{2}{3}$ microhm per in. cube we have for the resistances of the cores of the three-wire feeder

$$+ \text{ side, } R = \rho \frac{l}{a} = \frac{2}{3 \times 10^6} \times \frac{5{,}280 \times 12}{0{\cdot}2} = {\cdot}21\,;$$

$$\text{side, } R = {\cdot}21\,; \quad \text{neutral } R = {\cdot}42.$$

Hence, loss in feeder

$$= 220^2 \times {\cdot}21 + 200^2 \times {\cdot}21 + 20^2 \times {\cdot}42$$
$$= 18{,}770 \text{ watts.}$$

Let $\qquad R$ = resistance of each side of the two-wire feeder

$$\therefore\; 2 \times 420^2 \times R_1 = 18{,}770, \quad R_1 = {\cdot}053 \text{ ohm}$$

Now the amount of copper in a feeder of given length is inversely proportional to its resistance; hence, if we call the total amount of copper in the two-wire feeder 100, i.e. 50 per cent in each side, we have for the three-wire feeder

$$\text{Copper in + outer} = \frac{{\cdot}053}{{\cdot}21} \times 50 = \quad 12{\cdot}6\%$$

$$\text{Copper in} - \text{outer} = \qquad\qquad 12{\cdot}6\%$$

$$\text{Copper in neutral} = \frac{{\cdot}053}{{\cdot}42} \times 50 = \quad 6{\cdot}3\%$$

$$\text{Total} \quad = \quad 31{\cdot}5\%$$

5. **Balancers.** The generators supplying a three-wire feeder are all connected in parallel across the outers, and it is therefore necessary to fix the potential of the middle wire mid-way between that of the outers. If this is not done the voltages across the two sides will not be equal, unless the currents taken from the outers are equal.

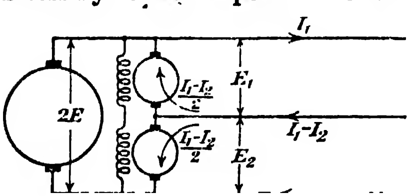


FIG. 127
BALANCE FOR THREE-WIRE SYSTEM

The commonest form of balancer consists of two identical shunt machines mechanically coupled, and having their armatures and fields connected in series across the outers. The middle wire is connected to the junction of the armatures. Consider the arrangement shown in Fig. 127. Let R_a be the resistance of each balancer armature. If the current I_1 is greater than I_2, then the + side will be the more heavily loaded

and the pressure on that side will tend to decrease. The armature of the balancer on the + side will therefore act as a generator, while the armature on the less heavily loaded side, the –, will motor. Thus the effect of the balancer is to convey power from the lightly loaded to the heavily loaded side.

Since the balancer armature on the + side is generating, its terminal voltage is

$$E_1 = E - R_a\left(\frac{I_1 - I_2}{2}\right)$$

Since the balancer armature on the – side is motoring, its terminal voltage is

$$E_2 = E + R_a\left(I_1 - I_2\right.$$

where $2E$ is the total voltage across the outers, and therefore twice the voltage *induced* in each armature

∴ Difference of voltage between the two sides of the system,

$$E_2 - E_1 = R_a(I_1 - I_2)$$

Hence, this difference is proportional to; first, the "out of balance" current $(I_1 - I_2)$; second, the armature resistance of the balancer.

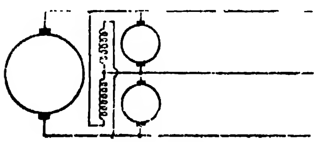


FIG. 128

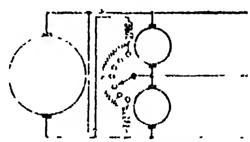


FIG. 129

For this reason balancers are designed with a very small armature resistance, and steps are always taken to arrange the loads on the two sides, so that the out of balance current will be as small as possible.

The difference $(E_2 - E_1)$ can be reduced by cross connecting the balancer fields, as shown in Fig. 128. This causes the generating machine to draw its excitation from the lightly loaded side, which is at the higher voltage, and the motoring machine to draw its excitation from the heavily loaded side.

Hence, the induced E.M.F. in the generating armature is greater than E, while that in the motoring armature is less than E. This increases E_1 and decreases E_2, the difference $(E_2 - E_1)$ being therefore decreased. Additional hand regulation of the voltage can be obtained by connecting an adjustable regulator in series with the balancer fields, as shown in Fig. 129

6. Boosters. A booster is a generator whose voltage is added to, or injected into, a circuit, usually to compensate for a variation

in voltage, as, for example, the drop of volts in a feeder due to its resistance. A booster which is used to compensate for the drop in a feeder is called a "feeder booster." Obviously it is a low voltage, heavy current machine. The booster in this case is a series generator connected in series with the feeder and driven a constant speed by a shunt motor. Since the drop in volts in feeder is proportional to the current, the voltage injected by the booster must also be proportional to the current. Hence, the booster must work on the straight line portion of its voltage characteristic. The method of connecting the booster in the circuit is shown in Fig. 130. The advantage of using a booster instead of, or in addition to, over-compounding the generators is that each feeder can be regulated independently, a great advantage if the feeders are of different lengths. The arrangement of a three-wire station supplying one set of short and one set of long feeders is shown in Fig. 131. Notice that since there is a middle wire whose potential must be kept midway between the outers, it is necessary to have a booster on both the + and – sides for the regulation of the long feeders. The shaded rectangles in Fig. 131 represent the various bus-bars.

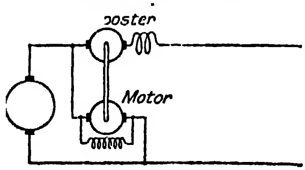


FIG. 130

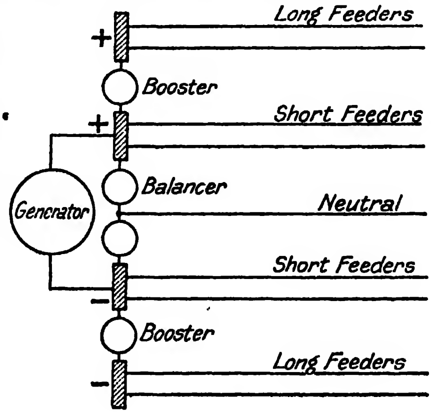


FIG. 131

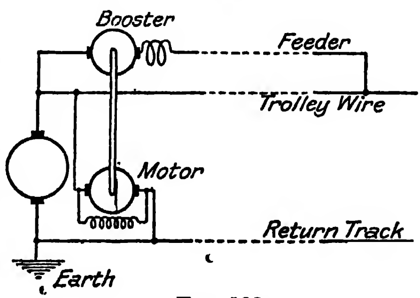


FIG. 132
BOOSTER CONNECTIONS

For some cases, e.g. a tramway system, it is desirable to raise the voltage of the line at some distant point instead of merely at the generator end. This can be done by running a special feeder to the desired point and injecting a voltage into this feeder by means of a booster connected in series with it, as shown in Fig. 132.

7. **Negative Booster.** The negative booster subtracts instead of adds its voltage to the system. It is used in traction systems to prevent the potential of the return rail from becoming too high relative to the earth. A Board of Trade regulation limits this potential to 4·2 volts because, if it becomes much greater than this, the earth currents set up produce electrolysis which causes great damage to any iron pipes in the vicinity. The negative bus-bar at the station is always earthed, the traction system having no middle wire, and therefore, the potential of the rail will be positive relative to earth, and of increasing magnitude as the distance from the station is increased. It is kept within the prescribed limits by means of a booster connected to the rail, as shown in Fig. 129. A second Board of Trade regulation limits the potential difference between any pair of points on the return rail to 7 volts, and to comply with regulation it is generally necessary to use several negative boosters. These negative boosters may be series excited, as in the case of feeder boosters, or they may be excited by having their fields connected across a resistance R in series with one of the feeders. The drop of volts in such a resistance will be proportional to the feeder current, and therefore, the excitation and the induced voltage of the booster will also be proportional to the feeder current. It is this second arrangement of exciting the booster field which is shown in Fig. 133.

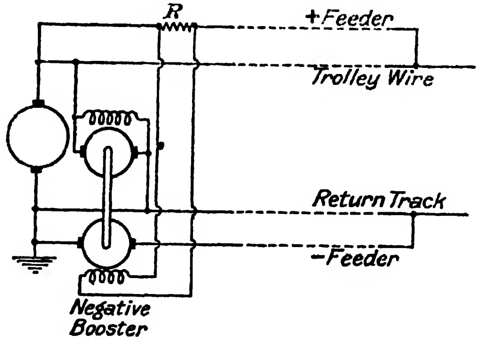


FIG. 133

NEGATIVE BOOSTER CONNECTIONS

8. **Battery Boosters.** In the case of D.C. systems subjected to violent variations in load, as, for example, a traction system, it is common practice to "float" a battery of accumulators across the bus-bars as shown in Fig. 134. Let E_1 and E_2 be the generator and battery terminal voltages, and R the resistance of the circuit through generator and battery. Then generator current

$$I = \frac{E_1 - E_2}{R}$$

and if E_2 were constant, I would be constant, irrespective of any changes in the load. The battery would then constitute a perfect load equalizer, and the generators, working at constant load would be working under ideal conditions. In times of heavy load the

battery would discharge into the line by an amount equal to the difference between the line current and the current I, above. On light load the generator would charge the battery, the charging current being the difference between I and the line current. Unfortunately, the above simple arrangement is not practicable, because the battery terminal voltage E_2 is far from constant. Thus, when charging a battery it is necessary to apply about 2·5 volts per cell, whereas when a battery is discharged it may not give more than 1·8 volts per cell, so that there may be a 40% variation in the terminal voltage of the battery. Suppose that the E.M.F. of a battery on no load is E; then when it is being charged, its E.M.F. will rise to, say, $E + e_1$. Hence, if I is the charging current, and R the battery resistance, the charging voltage to be applied to the battery will be

$$(E_2)_{charge} = E + e_1 + RI$$

On discharge the E.M.F. will fall to, say, $E - e_2$, and the terminal voltage for the same current I will be

$$(E_2)_{discharge} = E - e_2 - RI$$

Hence, $(E_2)_{charge} - (E_2)_{discharge} = (e_1 + e_2) + 2RI$

Now, assume that $e_1 = K_1 I$ and $e_2 = K_2 I$, then the difference between the terminal voltages on charge and discharge becomes

$$(K_1 + K_2 + 2R)I$$

that is, proportional to I. Hence, if a voltage is injected into the battery circuit and this voltage *reverses* with the battery current

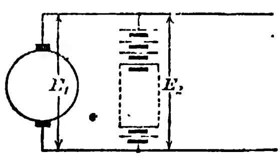


FIG. 134

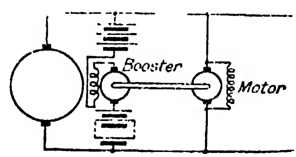


FIG. 135

and is proportional to it, the terminal voltage will be kept constant. The booster which injects this voltage is called a "reversible battery booster." If the excess voltages e_1 and e_2 were rigorously proportional to I, and the characteristic of the booster were a perfect straight line through the origin, then the simple arrangement shown in Fig. 135 would give the necessary constant battery voltage E_2. Unfortunately, neither assumption is strictly true, and therefore the boosters used in practice have to be much more complicated.

There are several boosters which fulfil the necessary requirements, one form, the Entz booster, being taken as an example. The arrangement of the Entz booster is shown in Fig. 136. The

most important part of this booster is the regulator, which consists of two piles of carbon plates pressed down by a lever L. This lever is pulled down on one side by a spring, W, and on the other side by a solenoid, S, in series with the line The booster is separately excited, its field being connected direct to the exciter armature, while the exciter field is connected to the middle point of the regulator and the middle point of the battery. When the line current is equal to the normal full load generator current, no help is needed from the battery, and the spring is adjusted so that, with this current, the resistances of the two carbon piles are equal, thus producing zero current in the exciter field and giving no voltage in the booster armature. If the line current increases, the solenoid is pulled down on the right-hand side, thus reducing the resistance of the right-hand pile. The exciter field now carries a current, and the E.M.F. induced in the booster armature enables the battery to discharge into the line. When the line current is less than normal, the pull on the solenoid relaxes and the left-hand pile has the smaller resistance. This reverses the direction of the current in the exciter field and so reverses the direction of the E.M.F. induced in the booster armature. The battery is then charged by the generator.

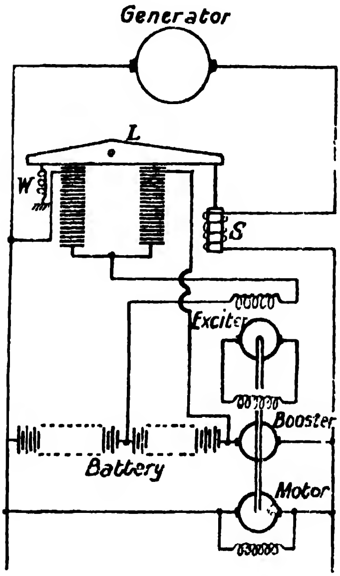


Fig. 136
Entz Booster

In the case of all boosters which have to carry a violently fluctuating current it is necessary that the voltage they produce shall follow, without any time lag, the variations in current. It is therefore necessary to laminate the whole of the field systems of such machines. If this were not done the rapid fluctuations in flux would induce eddy currents in the solid iron, which, apart from producing heat and loss of efficiency, would prevent the necessary rapid change in flux with current, and thus cause the voltage variations to lag behind those of the current.

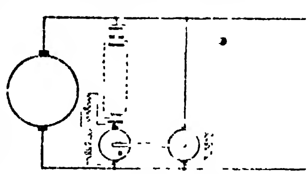


Fig. 137
Irreversible Battery Booster

If a booster is used merely to charge a battery it supplies current

in one direction only, for which reason it is called an "Irreversible Battery Booster." The arrangement of such a booster is shown in Fig. 137. It will be seen that the booster adds its voltage to the bus-bar voltage, and its voltage range must therefore cover the variation of battery voltage between fully charged and discharged conditions. The booster in this case is shunt or separately excited, the necessary regulation being performed by hand. Sometimes a booster is used to charge individual cells if these have become discharged more than the other cells in the battery. Such a booster is called a "Milking Booster."

9. **Calculation of Feeders.** KELVIN'S ECONOMY LAW. We have seen that the drop of volts is not a primary factor in the design of a feeder. A feeder can, therefore, be designed on the basis of current-carrying capacity and, where practicable, of minimum financial loss. The financial loss per annum occasioned by conductors of electricity is made up of (*a*) interest on the capital cost of the conductors, plus an allowance for depreciation, (*b*) the cost of the energy wasted in virtue of the ohmic resistance of the conductor. For a route of given length the weight, and therefore the cost, of the copper is proportional to the area of cross section. Hence, the annual value of the combined interest and depreciation is also proportional to the cross section, and can be written as £Pa,

where P = a constant,

a = cross section of copper.

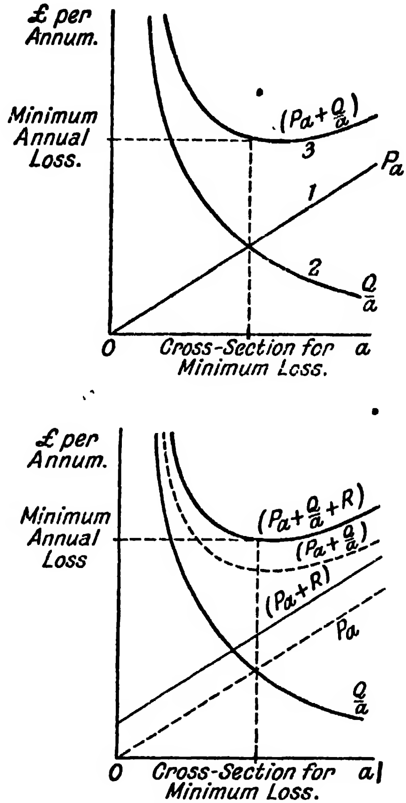
The ohmic resistance is proportional to $1/a$, and therefore for a given curve of demand for a current throughout the year the energy lost in the conductor will be proportional to the resistance, and therefore proportional to $\frac{1}{a}$. The annual value of this energy will, therefore, be proportional to $\frac{1}{a}$, and can be written £ $\frac{Q}{a}$, where Q is another constant.

$$\therefore \qquad \begin{pmatrix}\text{Total annual}\\ \text{financial loss}\end{pmatrix} = £\left(Pa + \frac{Q}{a}\right)$$

If plotted against cross section, the graph of the loss Pa is a straight line, while that of Q/a is a rectangular hyperbola, as shown by curves 1 and 2 in Fig. 138. The graph of total annual loss is given by curve 3, and it exhibits a minimum at that value of a corresponding to the intersection of the two component curves. The above argument has not been concerned with any insulating covering in the case of cables, or of insulators in the case of bare conductors, from which we see that for bare conductors the most economical cross section is that which makes the annual value of

the interest and depreciation on the conductor equal to the annual value of the electrical energy loss due to the resistance of the conductor.

It is to be noticed that a feeder cross section, as calculated from the law, will not always be a practicable one, because it may be too small to carry the current.



FIGS. 138 AND 139. ILLUSTRATE KELVIN'S LAW

Example. If the cost of an overhead line is £600 × A (where A is the sectional area in square inches), and if the interest and depreciation charges on the line are 8 per cent, estimate the most economical current density to use for a transmission requiring full-load current for 60 per cent of the year. The cost of generating electrical energy is 0·5d. per unit. The resistance of a conductor one mile long and 1 sq. in. cross section is 0·43 ohm. (London Univ.)

The value for resistance should obviously be ·043 ohm, and we will use this figure. Consider one mile, and let the full-load current be

I, then in a two-conductor line in which each conductor is of resistance R, power lost at full current

$$= 2I^2R$$

$$= 2I^2 \times \cdot 043 \times \frac{1}{A} \text{ watt}$$

$$= \frac{\cdot 086 I^2}{A} \times 10^{-3} \text{ kW}$$

The annual loss of energy is equal to that produced by the above loss taking place continually for 60 per cent of the year, i.e. for $(\cdot 6 \times 365 \times 24)$ hours. Hence, at £$\frac{1}{480}$ per unit, annual value of the energy lost

$$= £\left(\frac{\cdot 086 I^2}{A} \times 10^{-3} \times \cdot 6 \times 365 \times 24 \times \frac{1}{480}\right)$$

$$= £\ \cdot 00094 \frac{I^2}{A}$$

The annual value of interest and depreciation

$$= 8\% \text{ of } £600A = £48A$$

Hence, for minimum annual loss

$$48A = \cdot 00094 \frac{I^2}{A}$$

$$\therefore \text{ Current density } \frac{I}{A} = \sqrt{\frac{48}{\cdot 00094}} = 226 \text{ amp. per sq. in.}$$

We will now consider the influence of the insulation in the case of a covered cable. For a given type of cable, e.g. with vulcanized india-rubber, or impregnated paper insulation, for a given type of armouring, and for a given working voltage, the cost of the covering does not vary very much with the cross section of the cores, with the result that as a rough approximation we can regard the covering as adding a constant term to the cost of the cable. The annual value of the interest and depreciation can, therefore, be written £$(Pa + R)$, where R is another constant. The effect of this on the graph of interest and depreciation is to raise the graph of £Pa vertically through a distance equal to £R. Similarly, the graph of total annual loss, namely, £$(Pa + Q/a + R)$ is merely the graph of £$(Pa + Q/a)$ raised vertically through a distance £R, without producing any horizontal displacement in the point of minimum cost. This is shown in Fig. 139. It follows from this that the insulation does not have any effect on the value of the cross section which gives the minimum annual loss, so that for an insulated cable the law can be stated as follows. The most economical cross section

is that which makes the annual value of the interest and depreciation, due to the conductor in the cable, equal to the annual value of the energy lost.

Example. A 500-volt, 2-core feeder, two miles long supplies a maximum current of 200 amp., and the demand is such that the copper loss per annum is such as would be produced by the full current flowing for six months. The resistance of a conductor one mile long of 1 sq. in. cross section is ·046 ohm. The cost of the cable including installation is £(6a + 1·2) per yard, and interest and depreciation charges 10 per cent. The cost of energy is 0·5d. per unit. Find the most economical cross section, and plot curves of component and total annual losses.

For one mile, cost of conductors in cable

$$= £6a \times 1760$$
$$= £10560a$$

∴ Interest and depreciation

$$= £1056a$$

Power lost per two cores, per mile, with maximum current flowing

$$= 2I^2R$$
$$= 2 \times (200)^2 \times \frac{\cdot 046}{a} \text{ watts.}$$

∴ Annual value of energy lost

$$= £\left(2 \times (200)^2 \times \frac{\cdot 046}{a} \times 10^{-3} \times \frac{365 \times 24}{2} \times \frac{1}{480}\right.$$

$$= £\frac{33\cdot 58}{a}$$

Hence, for the most economical cross section

$$1056a = \frac{33\cdot 58}{a}$$

$$a = \sqrt{\frac{33\cdot 58}{1056}} = 0\cdot 178 \text{ sq. in.}$$

From the above figures we have for the three constants $P = 1{,}056$, $Q = 33\cdot 58$, and $R = 211\cdot 2$ for each mile of cable, so that the curves are as shown in Fig. 140.

10. Calculation of the Drop of Volts in Distributors. (*a*) DISTRIBUTOR FED FROM ONE END. We have seen that the main consideration in the design of distributors is the drop of volts. This drop depends upon the nature of the current loading of the distributor, and also on whether it is fed from one end only or from both ends. Consider first of all a distributor with concentrated loads, and fed from one end. Let i_1, i_2, i_3, etc., be the currents tapped off, and I_1, I_2, I_3,

etc., the currents in the distributor sections. Let r_1, r_2, r_3, etc., be the resistances of the distributor sections, and R_1, R_2, R_3, etc., the

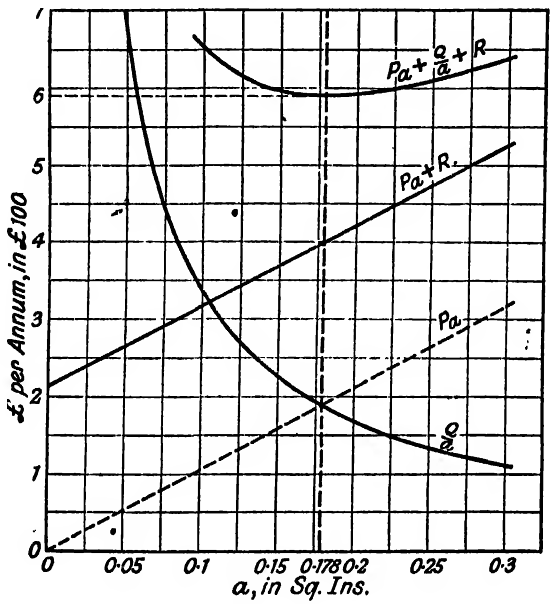


FIG. 140

total resistances from the feeding point F to the successive points (Fig. 141).

Then total drop $= r_1I_1 + r_2I_2 + r_3I_3 + \ldots$

$$\begin{aligned}
&= r_1(i_1 + i_2 + i_3 + \ldots) \\
&\qquad + r_2(i_2 + i_3 + \ldots) \\
&\qquad + r_3(i_3 + i_4 + \ldots) + \ldots \\
&= i_1r_1 + i_2(r_1 + r_2) + i_3(r_1 + r_2 + r_3) + \\
&= i_1R_1 + i_2R_2 + i_3R_3 + \ldots \\
&= \Sigma(iR)
\end{aligned}$$

Thus the drop *at the far end* of a distributor fed at one end is given by the sum of the moments of the various currents tapped off, about the feeding point. The drop at any intermediate point is equal to the sum of the moments of the currents up to that point, plus the moment of all the currents beyond that point assumed to be acting at that point. Thus the drop at the third tapping point will be

$$\begin{aligned}
&i_1R_1 + i_2R_2 + i_3R_3 + (i_4 + i_5 + \ldots)R_3 \\
&= i_1R_1 + i_2R_2 + I_3R_3
\end{aligned}$$

and similarly for any other intermediate point. The total drop, reckoning both conductors, will, of course, be twice the above value.

(*b*) THREE-WIRE DISTRIBUTOR FED FROM ONE END. The drops in the two outers are calculated as above. In the case of the neutral, the currents flowing to it from the positive outer are reckoned positive, while the currents flowing from it to the negative outer are reckoned negative. If the total drop in the neutral is positive it is added to the positive drop and deducted from the negative drop; and *vice versa*.

Example. A three-wire distributor, 300 yd. long, is fed at 230 volts on each side, and is loaded as follows—

Yards from feeding point, + side	30	100	140	160	210	240
Load	20	40	50	10	25	30
Yards from feeding point, − side	60	80	120	180	260	
Load	30	25	35	60	25	

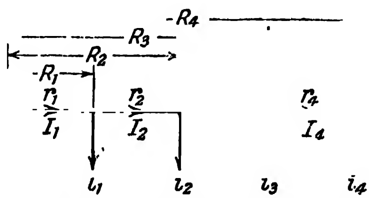


FIG. 141. DISTRIBUTOR FED AT ONE END

The outers each have a resistance of 0·15 ohm, and the neutral has a cross section of one-half each outer. Calculate the P.D. at the far end on each side.

Resistance per yard of outer

$$= \frac{\cdot 15}{300} = 5 \times 10^{-4} \text{ ohms.}$$

∴ Resistance per yard of neutral

$$= 10 \times 10^{-4} \text{ ohms.}$$

Drop in positive outer

$$= (20 \times 30 + 40 \times 100 + \ldots 30 \times 240) \times 5 \times 10^{-4}$$
$$= 25650 \times 5 \times 10^{-4} = 12{\cdot}82 \text{ volts}$$

Drop in negative outer

$$= (30 \times 60 + 25 \times 80 + \ldots 25 \times 260) \times 5 \times 10^{-4}$$
$$= 25300 \times 5 \times 10^{-4} = 12{\cdot}65 \text{ volts}$$

Drop along neutral

$$= \{(25650) - (25300)\} \times 10 \times 10^{-4}$$
$$= + 0{\cdot}35 \text{ volts}$$

$\therefore$ Total drop on positive side

$$= 12{\cdot}82 + 0{\cdot}35$$
$$= 13{\cdot}2 \text{ volts}$$

$\therefore$ P.D. on positive side at far end

$$= 230 - 13{\cdot}2 = 216{\cdot}8 \text{ volts}$$

Total drop on negative side

$$= 12{\cdot}65 - 0{\cdot}35$$
$$= 12{\cdot}3 \text{ volts}$$

$\therefore$ P.D. on negative side at far end

$$= 230 - 12{\cdot}3 = 217{\cdot}7 \text{ volts}$$

(c) DISTRIBUTOR FED AT BOTH ENDS. The potential of the conductor will gradually fall from one feeding point, say, F_1 (Fig. 142),

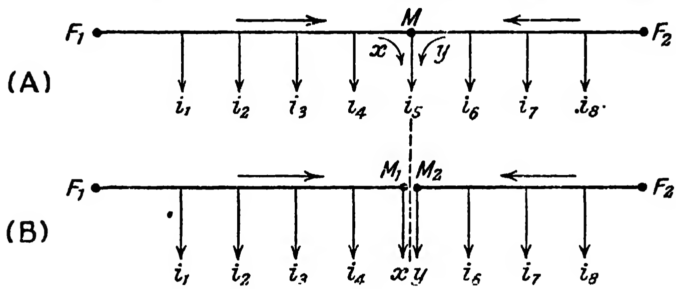


FIG. 142. DISTRIBUTOR FED AT BOTH ENDS

reach a minimum at one of the tapping points, and will rise again as the other feeding point F_2 is approached. There is thus some tapping point M, at which the potential is a minimum. Between F_1 and M all the currents tapped off will be supplied from F_1, while between F_2 and M all the currents tapped off will be supplied from F_2. With regard to the point M itself, the current tapped off there will, in general, come partly from F_1 and partly from F_2, and we can denote these two parts by x and y respectively. If the distributor were cut in two at M, x amperes tapped off at the end served by F_1, and y amperes tapped off at the end served by F_2, then the potential distribution would be unchanged, showing that, from this point of view, we can regard the distributor as consisting of two separate distributors each fed from one end only, as in Fig. 138 (B). In order to calculate the drop we thus require to locate M, and then calculate x and y. This is done by means of the pair of equations

$$\left.\begin{array}{c} x + y = i_5 \\ \text{Drop from } F_1 \text{ to } M_1 = \text{drop from } F_2 \text{ to } M_2 \end{array}\right\}$$

Example. A distributor is fed at both ends. It is a two-core cable half a mile long and of cross-section ·05 sq. in. The loads tapped off are as shown in

Fig. 143, the numbers along the distributor indicating yards. If the two ends F_1 and F_2 are at the same potential, which point will have the lowest potential, and what will be the drop at that point? Take the specific resistance of copper as 0·7 microhm per inch cube.

The best method of locating the point of minimum potential is to take moments about the two ends, and by comparing the sum

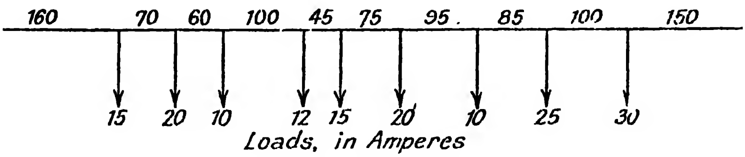


Fig. 143

of these as the calculation proceeds, to make a guess at the point. This is done in the following table—

Moment about F_1 in Ampere-yards.	Sums.	Moment about F_2 in Ampere-yards.	Sums.
15 × 160 = 2,400	2,400	30 × 150 = 4,500	4,500
20 × 230 = 4,600	7,000	25 × 250 = 6,250	10,750
10 × 290 = 2,900	9,900	10 × 335 = 3,350	14,100
12 × 390 = 4,680	14,580	20 × 430 = 8,600	22,700
15 × 435 = 6,525	21,105		

The table indicates that the sixth tapping point from F_1 is the required point. We then have for the two equations

$$x + y = 20$$

$$21105 + 510x = 14100 + 430y$$

$$\therefore \quad x = 1{\cdot}7 \text{ amp. and } y = 18{\cdot}32 \text{ amp.}$$

∴ Drop at M per conductor

$$= 21105 + 510 \times 1{\cdot}7$$

$$= 21973 \text{ ampere-yards}$$

Resistance per yard $= \dfrac{\rho l}{a} = \dfrac{0{\cdot}7 \times 10^{-6} \times 36}{{\cdot}05}$

$$= 50{\cdot}4 \times 10^{-5} \text{ ohms}$$

∴ Drop per conductor

$$= 2{\cdot}1973 \times 10^{4} \times 50{\cdot}4 \times 10^{-5}$$

$$= 11{\cdot}1 \text{ volts}$$

Hence, the total drop, reckoning both conductors, will be 22·2 volts.

We have now to consider the effect of the feeding points F_1 and F_2 not being at the same potential, this condition being, of course, very probable in an actual case. Suppose that there is a difference of v volts between the points F_1 and F_2, F_1 being at the higher potential.

The quickest method is the following. We convert the v volts into ampere-yards; F_2 starts off with this drop below F_1, and this value appears in the column for F_2 as an initial drop. The procedure is then as described above. Thus, in the above example, suppose that v is 5 volts. The resistance per yard is $50 \cdot 4 \times 10^{-5}$ ohms, so that the initial ampere-yards for F_2 are

$$\frac{5}{50 \cdot 4 \times 10^{-5}} = 9920 \text{ ampere-yards}$$

The table then becomes the following—

Moment about F_1.	Sums.	Moment about F_2.	Sums.
		Initial = 9,920	9,920
15 × 160 = 2,400	2,400	30 × 150 = 4,500	14,420
20 × 230 = 4,600	7,000	25 × 250 = 6,250	20,670
10 × 290 = 2,900	9,900	10 × 335 = 3,350	24,020
12 × 390 = 4,680	14,580	20 × 430 = 8,600	32,620
15 × 435 = 6,525	21,105		
20 × 510 = 10,200	31,305		

The dividing point is thus the same. We have

$$x + y = 20$$

and

$$21105 + 510x = 24020 + 430y,$$

giving

$$x = 12 \cdot 2 \text{ and } y \doteq 7 \cdot 8$$

The voltage drops can be calculated as before.

(d) DROP IN A UNIFORMLY-LOADED DISTRIBUTOR.

Let i = current tapped off per unit length

and r = resistance per unit length (Fig. 144)

Drop at distance x from F

$$= \begin{pmatrix}\text{Sum of moments} \\ \text{up to } x\end{pmatrix} + \begin{pmatrix}\text{Moment of the whole load beyond} \\ x\text{, assumed acting at } x\end{pmatrix}$$

$$= \int_0^x idx \, . \, rx + i(l - x) \, . \, rx$$

$$= \tfrac{1}{2}irx^2 + irlx - irx^2$$

$$= irlx - \tfrac{1}{2}irx^2$$

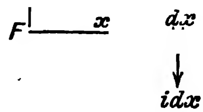


FIG. 144

at the far end $x = l$

$$\therefore \text{ Drop} = \tfrac{1}{2}irl^2$$

$$= \tfrac{1}{2}(il) \times (rl)$$

$$= \tfrac{1}{2}IR$$

where I is the total current, and R the total resistance.

Thus a uniformly-loaded distributor fed at one end gives a total drop equal to that produced by the whole of the load assumed concentrated at the middle point.

Now let the distributor be fed at both ends. Then with both ends at the same potential the point of minimum potential is obviously the middle point. We can thus imagine the distributor cut into two at the middle point, thus giving two uniformly-loaded distributors fed at one end. The resistance of each is $\frac{R}{2}$, and the total load in each is $\frac{I}{2}$.

$$\therefore \text{ Drop at middle point } = \tfrac{1}{2} \times \frac{I}{2} \times \frac{R}{2}$$

$$= \tfrac{1}{8}IR$$

EXAMPLES ON CHAPTER X.

(1) 600 h.p. at 600 volts is to be delivered at a place 2 miles distant from an electrical generating station. Find the power and voltage of the generator required, if the loss in transmission is to be 14% of the power generated. Also determine the cross section of the cable required. (C. and G., 1909.)

Ans.—520 kW at 698 volts. 1·28 sq. in.

(2) A three-wire feeder is half a mile long, the two outers having a cross section of 0·25 sq. in. and the neutral, 0·125 sq. in. If the voltage at the feeding points is to be 200 on each side, and the loads on the positive and negative sides are 130 and 100 amp. respectively, what voltages must be maintained on the two sides at the generating end?

Ans.—216·04 and 203·4 volts.

(3) Calculate the power lost in the above feeder, and the efficiency of transmission. Find the resistance of a two-wire feeder which will transmit 230 amp. at 200 volts and have the same loss as the three-wire feeder. Also compare the amounts of copper in the two- and three-wire feeders.

Ans.—·038 ohm per conductor; 36 to 100.

(4) What is the "time-constant" of an electric circuit? Explain its importance in direct current working, e.g. in reversible boosters, and in alternating current working. How does it vary with the saturation of the iron of e circuit? (London Univ., 1912.)

(5) Describe carefully the conditions under which a reversible battery booster has to work, and the functions it has to perform. Why is it not possible to use a simple series dynamo, driven at a constant speed, as such a booster?

(6) Describe one form of balancer for balancing the voltages on a three-wire D.C. supply system. Make a diagram of connections, showing how it is connected to the mains. (London Univ., 1921.)

(7) Make a diagram of connections showing the necessary machines in the supply station of a three-wire, continuous current system. What is the object of a balancer, and how does it act? If the supply voltage on each side of the three-wire system is kept at the same value at a feeding point, what is the effect on the voltage at terminals of lamps on each side, if the load on one side of the system is greater than on the other? (C. and G., 1914.)

CHAPTER XI

TESTING OF DIRECT CURRENT MACHINES

1. **Types of Tests.** The methods of testing electrical machines can be divided into three classes, direct, indirect, and regenerative.

In the direct method the motor or generator is put on full load and the whole of the power developed by it is wasted. For a generator the load usually consists of a water resistance, the output of the machine being used in heating the resistance. For a small motor a brake can be applied to a water-cooled pulley and the load adjusted until full load current is flowing. Then if W_1 and W_2 are the tensions in lb. on the tight and slack sides of the brake band, r the radius of the pulley in ft., and N the speed in r.p.m.,

$$\text{Motor output} = \frac{2\pi N(W_1 - W_2)r}{33{,}000} \text{ brake h.p.}$$

$$= \frac{2\pi N(W_1 - W_2)r \times 746}{33{,}000} \text{ watts}$$

If I is the total current taken from the line, and E the voltage applied to the motor terminals,

$$\text{Intake} = EI \text{ watts}$$

$$\text{Hence, Efficiency, } \eta = \frac{\text{Output}}{\text{Intake}} = \frac{2\pi N(W_1 - W_2)r}{EI} \times \frac{746}{33{,}000}$$

$$= 17{\cdot}3\, \frac{N(W_1 - W_2)r}{EI} \%$$

If a test of this kind is applied to a series motor the brake must be tight before the motor is switched on to the supply, otherwise the armature may fly to pieces.

A simple brake test like the above is suitable for small machines, but for a large motor it is preferable to direct couple it to a generator, and to load the generator by means of a resistance. The generator is carefully calibrated by determining the various losses in it beforehand, and its efficiency at different loads is thus known. Hence, if the generator output is measured by means of an ammeter and voltmeter, its intake can be calculated. This intake is obviously the motor output.

The testing of large machines by the direct method entails a considerable loss of power; in fact, in the case of a very large machine, there may be no facilities for a direct test.

2. **Indirect Methods.** These consist in determining the losses

and predetermining the efficiency from these. The amount of power required is that to supply the losses only, so that there is no difficulty in applying the method even to very large machines. The disadvantage is that the machine is running light during the test so that, although the efficiency can be calculated with fair accuracy from the results obtained, no indication is given as to the temperature rise on load, or to the commutating qualities of the machine.

The simplest of the indirect tests is the Swinburne test, and it can only be applied to shunt or compound machines. The machine is run light as a motor, and the armature current I_o measured. The armature resistance and field resistance are also measured, and the temperature of these windings observed. Since the machine will be hot when running under normal conditions, the probable hot resistances are calculated, a temperature rise of 50° C. being assumed. Then if R is the measured armature resistance, the resistance after a temperature rise of 50° C. will be

$$R_a = R(1 + 50a)$$

where the value of a, the coefficient of increase of resistance with temperature, will be fixed by the initial temperature.

Let the suffixes 0, 1, and 2 refer to zero temperature, t_1° C., and t_2° C.; then

$$R_1 = R_o(1 + a_o t_1)$$

$$R_2 = R_o(1 + a_o t_2)$$

$$\therefore \; R_2 = R_1\left(\frac{1 + a_o t_2}{1 + a_o t_1}\right)$$

Now let a_1 be the temperature coefficient corresponding to the initial temperature t_1. Then we have

$$R_2 = R_1(1 + a_1 \cdot \overline{t_2 - t_1})$$

$$1 + a_1(t_2 - t_1) = \frac{1 + a_o t_2}{1 + a_o t_1}$$

giving finally

$$a_1 = \frac{a_o}{1 + a_o t_1}$$

Thus, if we take the value $a_o = \cdot 00427$ for pure copper, we have for, say, $t_1 = 40°$ C.

$$a_{40} = \frac{\cdot 00427}{1 + 40 \times \cdot 00427} = \cdot 00364$$

and similarly for any other initial temperature.

If a copper coil of resistance 100 ohms at 40° C. is heated to 100° C., its hot resistance will thus be

$$\begin{aligned} R_{100} &= R_{40}(1 + a_{40} \times 60) \\ &= 100(1 + \cdot 00364 \times 60) \\ &= 121 \cdot 8 \text{ ohms} \end{aligned}$$

Similarly with the field winding. From the *cold* resistance of the armature the copper loss $I_0{}^2R$ is calculated, and from the cold resistance of the shunt field the field copper loss E^2/R_{sh} is calculated. The sum of these two losses deducted from the intake EI_0 gives the friction and iron losses, that is, the stray losses. If now the calculated field copper loss E^2/R_{sh}, using the hot resistance, is added to the stray losses, the result will be the constant losses. Let these be W_c. Then, for any armature current I_a the armature copper loss will be $I_a{}^2R_a$, where R_a is now the hot resistance. Hence, total losses of all kinds

$$W_t = I_a{}^2R_a + W_c$$

The field current will be

$$I_{sh} = \frac{E}{R_{sh}}$$

and the line current will be

$$I_1 = I_a + I_{sh} \text{ if the machine is motoring}$$

$$I_2 = I_a - I_{sh} \text{ if the machine is generating.}$$

Also, EI_1 is the intake when motoring, and EI_2 is the output when generating.

Hence, efficiency when acting as a motor

$$\eta = \frac{\text{Intake} - \text{Losses}}{\text{Intake}} = \frac{EI_1 - W_t}{EI_1}$$

and efficiency when acting as a generator

$$\eta = \frac{\text{Output}}{\text{Output} + \text{Losses}} = \frac{EI_2}{EI_2 + W_t}$$

Thus, if different armature currents I_a are assumed, the efficiency at different loads can be calculated.

If the machine under test is to function as a motor under normal conditions, its probable drop of speed on any load can be determined from the measured no load speed N_0 from the expression

$$N = N_0 \times \frac{E - R_aI_a}{E - R_aI_0}$$

The drop in speed under normal working conditions will probably be somewhat less than this, because armature reaction on load will cause a decrease in total flux, and this will have a tendency to keep the speed up. The speed will also be altered if the brush position is different from that during the test.

3. Separation of Losses. The above method does not give any indication as to how the stray losses are divided between friction, hysteresis, and eddy current losses. These can be separated by the following test. The machine is run light as a motor, and a resistance R is placed in its armature circuit, so that any speed can be obtained. (*See* Fig. 145.) The field is separately excited to the normal value and different voltages applied to the armature, the speed N and the armature intake being observed for each value of the voltage. It is preferable, although not absolutely necessary, to measure the intake by means of a wattmeter, as shown. The armature current I_a is also observed, and the armature copper loss $I_a^2R_a$ deducted from the observed intake. This gives the

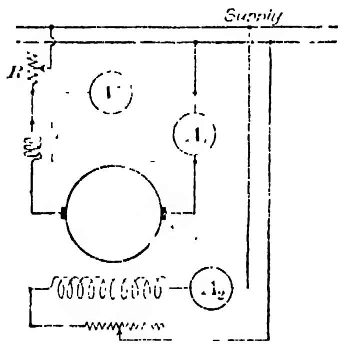


FIG 145

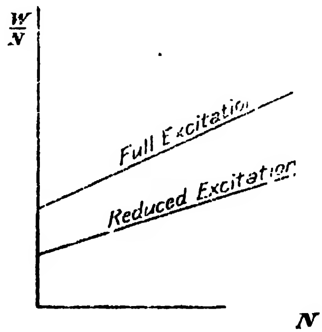


FIG. 146

total stray losses, say W. Now the various components of the stray losses are all functions of the speed, and can be written

Friction loss $= AN + BN^2$

Hysteresis loss $= CN$ so long as the excitation is constant.

Eddy current loss $= DN^2$ so long as the excitation is constant.

A, B, C, and D are all constants. We therefore have

$$W = AN + BN^2 + CN + DN^2$$

or

$$\frac{W}{N} = (A + C) + (B + D)N.$$

Hence, if $\frac{W}{N}$ is plotted against the speed N, a straight line will be obtained (Fig. 146). By taking two points on this line, not necessarily observed results, and substituting the corresponding

values of $\frac{W}{N}$ and N, a simultaneous equation will be formed from which $(A + C)$ and $(B + D)$ can be calculated. The losses for any speed can then be calculated. So far the losses have been separated into those proportional to N, and those proportional to N^2, and such a separation is sufficient for most practical purposes. If it is required to determine each of the constants A, B, C, and D, then it is necessary to repeat the experiment with a reduced excitation. This will not affect the friction loss, but the constants C and D will be different, say, C' and D' respectively. Then the equation to the new curve of $\frac{W}{N}$ against N is

$$\frac{W}{N} = (A + C') + (B + D')N$$

The terms $(A + C')$ and $(B + D')$ can then be determined as before, and by subtracting $(A + C')$ from $(A + C)$ and $(B + D')$ from $(B + D)$, the quantities $(C - C')$ and $(D - D')$ are found. Now C, the coefficient for the hysteresis loss, is proportional to the flux Φ raised to the 1·6 power, and D is proportional to Φ^2. Hence, calling Φ and Φ' the fluxes corresponding to the normal and reduced excitation, we have

$$\frac{C'}{C} = \left(\frac{\Phi'}{\Phi}\right)^{1.6}; \quad \frac{D'}{D} = \left(\frac{\Phi'}{\Phi}\right)^2$$

Again, the ratio Φ'/Φ is given by the ratio of the back E.M.F.s in the two experiments *for any given speed.* Knowing the applied voltage and the armature current, the back E.M.F. can be calculated by deducting the armature drop from the applied voltage. The above ratios can thus be calculated and the constants A, B, C, and D then determined separately.

For works testing it is generally sufficient to separate the stray losses into iron (variable) and friction (constant) losses, without analysing the run loss into its component hysteresis and eddy-current losses. In such a case the connection scheme of Fig. 145 is used, but the procedure is modified as follows: The motor is run at normal speed with normal applied volts and excitation, and when the wattmeter reading has settled down to a steady value readings of armature watts, volts, amps., and speed are taken. Next the excitation is reduced so that the speed increases, and the armature rheostat R is then adjusted to bring the speed back to the same value as before. Readings are again taken and the same procedure repeated over as wide a range of armature voltage as possible. The armature watts (corrected for armature copper loss) are then plotted

against armature volts. Since the speed is kept constant the friction and windage loss will be constant, but since the excitation is gradually reduced, the flux per pole, and therefore the iron loss, will be gradually reduced. If the process could be continued until the armature volts were zero, then the flux would be zero and the iron loss zero. Hence if the graph is produced backwards to meet the watts axis the intercept on this axis will give the friction and windage loss. It is because the graph has to be extended in this way that it is desirable that the experimental results should cover as wide a range of armature volts as possible. If the watts are plotted against volts squared a straight line will be obtained with a shunt motor, and the intercept can then be determined very easily.

If the field current is observed along with the other data then the experiment will also give the magnetization characteristic. The observed volts V corrected for armature drop gives the induced E.M.F.

$$E = V - R_a I_a$$

and this E.M.F. plotted against the exciting current gives the magnetization characteristic. This test is very easy to carry out, except that at very low excitations the response of the motor to changes in the value of R may be very slow, and much time, therefore, taken in bringing the speed to just the right value. The adjustments are greatly facilitated if the speed is observed by a stroboscopic method instead of by means of an ordinary tachometer.

4. The "Retardation" or "Running Down" Test. This is another convenient method of separating the various losses. In this method the machine is run up to a little way beyond normal speed, and the supply is then switched off from the armature. As a result the armature slows down, its kinetic energy being drawn upon to supply the various losses produced by rotation. If I is the moment of inertia of the armature and ω its angular velocity at any instant, then losses due to rotation

$$W = \frac{d}{dt} \text{ of kinetic energy}$$

$$= \frac{d}{dt}\left(\tfrac{1}{2}I\omega^2\right)$$

$$= I\omega\frac{d\omega}{dt}$$

To calculate W it is therefore necessary to determine the curve of ω against t. There are several ways of doing this, the method involving the least amount of special equipment being to connect a very high resistance moving coil voltmeter across the armature

terminals and to use it as a speed indicator. It is calibrated for speed as follows. When motoring at constant speed, the speed, the applied voltage, and armature current are observed. The back E.M.F. is calculated, and the ratio of observed speed to back E.M.F. gives the required constant of the voltmeter when used as a speed indicator. If the machine is a very large one, and takes a long time to slow down, the following procedure can be adopted. A circle of paper is pasted on the dial of a stop clock, and each time an observer watching the voltmeter gives a signal, a second observer marks the position of the seconds hand with a pencil. The first observer gives his signals each time the voltmeter needle passes one of the main divisions as the armature is slowing down. Such a method naturally necessitates a certain amount of skill, but when experience has been gained very consistent results can be obtained. In the case of small machines the fall in speed is too rapid to enable such a method to be adopted.

It is therefore preferable to observe, by means of a stop-watch, the time taken to slow down from a fixed voltage somewhat higher than the normal voltage, to the various main divisions below this on the voltmeter, the motor being brought up to full speed for each observation. Thus, suppose that 200 volts is taken as the reference mark on the voltmeter. The motor is brought up to such a speed that the voltmeter reads somewhat higher than this, and the armature supply is cut off. The voltmeter needle immediately begins to fall, and the time to fall from 200 volts to 190 volts is observed. The speed is then brought up to full value, supply cut off, and the time to fall from 200 volts to 180 volts is observed; and so on. The observed voltages are converted to speeds and the speed time curve plotted as in Fig. 147. To find the gradient dN/dt at any point P, it is usual to draw a tangent to the curve and to measure the intercepts OM and ON. The change of curvature is so small that it is necessary to make several trials, the positions M and N being marked in pencil, but the tangent line not being drawn. If the means of the various lengths OM and ON are taken, we have

$$dN/dt = \frac{OM \text{ (measured in units of } N)}{ON \text{ (measured in units of } t)}$$

Now, if the moment of inertia I is expressed in gram cm.^{2} and ω in radians per sec., then

$$W = I\omega \times \frac{d\omega}{dt} \text{ ergs per sec.}$$

$$= I\omega \times \frac{d\omega}{dt} \times 10^{-7} \text{ watts.}$$

If I is expressed in kilogram-metre^2 units, then, since the kilogram-metre^2 unit is equal to 10^7 gram. cm.^{2} units, the equation for W becomes

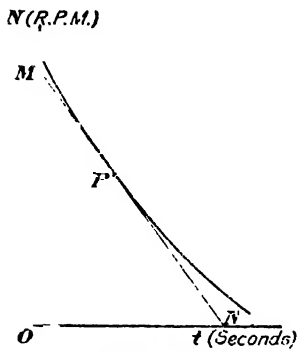


Fig. 147

$$W = I\omega \times \frac{d\omega}{dt}$$

Again, $\omega = 2\pi \times$ rev. per sec

$$= \frac{2\pi}{60} \times N$$

and

$$\frac{d\omega}{dt} = \frac{2\pi}{60} \times \frac{dN}{dt}$$

Hence, finally,

$$W = \left(\frac{2\pi}{60}\right)^2 IN\frac{dN}{dt}$$

$$= 0{\cdot}0109\ IN\frac{dN}{dt}$$

so that, if I is known, the stray losses W can be calculated for any speed N.

There are several ways of finding I, the two most convenient being as follows. When the slowing-down curve has been obtained, a flywheel of known moment of inertia I' is keyed to the shaft and the curve obtained again. Naturally, the time of slowing down will be longer. For any given speed the rates of change (dN/dt) and $(dN/dt)'$ are determined as before. For the same speed the losses will be the same, since a smooth flywheel will not affect these appreciably, and we therefore have

$$0{\cdot}0109\ IN\left(\frac{dN}{dt}\right) = 0{\cdot}0109(I + I')\ N\left(\frac{dN}{dt}\right)'$$

$$\frac{I + I'}{I} = \left(\frac{dN}{dt}\right)\Big/\left(\frac{dN}{dt}\right)'$$

$$\therefore \qquad I = I' \times \frac{\left(\frac{dN}{dt}\right)'}{\left(\frac{dN}{dt}\right) - \left(\frac{dN}{dt}\right)'}$$

The second method consists in observing the time taken to slow down, say, 5 per cent in speed, then applying a known retarding torque and observing again the time taken to slow down the same amount. The retarding torque can be applied by means of a brake, but it is more accurate to apply it electrically, the method of

doing so being indicated in Fig. 148. The armature is controlled by a double throw switch S which is opened when the slowing down under normal conditions is to be observed. When applying an additional retarding torque, S is thrown right over so as to connect the armature to a non-inductive resistance as shown. The current I_a through this resistance produces an additional loss of $I_a^2(R + R_a)$. The current I_a will of course fall as the speed falls, but with a change in speed of not more than 5 per cent the

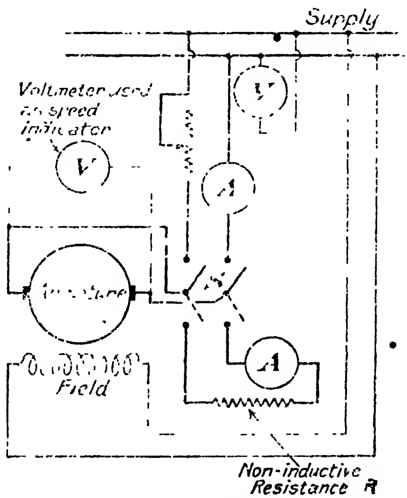


FIG. 148

change in I_a will be so small that it will be sufficiently accurate if a second observer takes its initial and final values, the mean value being used in the expression. Call this additional load W' watts and let the losses corresponding to the mean speed be W. Then, if δN is the change in speed and t and t' the times of slowing down without and with the extra load respectively,

$$W = 0{\cdot}0109 IN \frac{\delta N}{t}$$

$$W + W' = 0{\cdot}0109\, IN \frac{\delta N}{t'}$$

$$\therefore \quad \frac{W + W'}{W} = \frac{t}{t'}$$

$$W = W' \times \frac{t'}{t - t'}$$

Example. On breaking the armature circuit of a separately excited motor, the E.M.F. induced in the armature falls from 200 volts to 190 volts in 30 sec. If a current of 10 amp. is taken from the armature by connecting it to a resistance immediately after disconnecting it from the supply, the same fall in E.M.F. takes 20 sec. Find the stray losses.

We have $N = kE$, where k is a constant

$$\therefore \quad W = 0{\cdot}0109\, IN\frac{\delta N}{t}$$

$$= 0{\cdot}0109\, k^2 IE\frac{\delta E}{t}$$

The mean value of E is 195 volts, and without the additional load $\delta E/t$ is $10/30 = \frac{1}{3}$.

$$\therefore \quad W = 0{\cdot}0109k^2 \times \tfrac{1}{3} \times 195 \times I$$

The mean value of the additional load is $195 \times 10 = 1950$ watts, and the corresponding rate of fall $\delta E/t$ is $\frac{1}{2}$.

$$\therefore \quad W + 1950 = 0{\cdot}0109k^2 \times \tfrac{1}{2} \times 195 \times I$$

$\therefore$ Subtracting,

$$1950 = 0{\cdot}0109k^2 \times \tfrac{1}{6} \times 195 \times I$$

or
$$k^2 I = \frac{6 \times 1950}{0{\cdot}0109 \times 195}$$

$$\therefore \quad W = 0{\cdot}0109 \times \tfrac{1}{3} \times 195 \times \frac{6 \times 1950}{0{\cdot}0109 \times 195}$$

$$= 3900 \text{ watts.}$$

5. **Regenerative Methods.** These methods require two machines, preferably identical. These are mechanically coupled, and are so adjusted electrically that one of them acts as a motor and the other as a generator. The motor supplies the mechanical power to drive the generator, while the electrical power developed in the generator is utilized in the motor. Thus, two machines of any size can be tested under full load conditions, and the power taken from the supply will be that required to overcome the losses only. The method is therefore invaluable where tests of long duration under full load conditions have to be made on very large machines. Such tests are called "heat runs," because the object of the test is to determine the final temperature rise of the machine. Regenerative tests were first introduced by Hopkinson, for which reason they are often called *Hopkinson Tests*.

The method of procedure in the case of shunt machines is indicated in Fig. 149. Machine *I* is run up to full speed by means of the starting resistance, not shown, the main switch of machine *II*

being open. This machine will therefore generate, and when its voltage has been adjusted equal to, and of the same polarity as, the bus-bar voltage, this adjustment being made by means of a paralleling voltmeter V, it can be switched on to the supply. Any required load can now be thrown on to the machines by adjusting the respective shunt regulators. The machine with the smaller excitation will act as a motor, because its back E.M.F. being less than that of the other, will admit a greater armature current, and therefore produce a greater torque

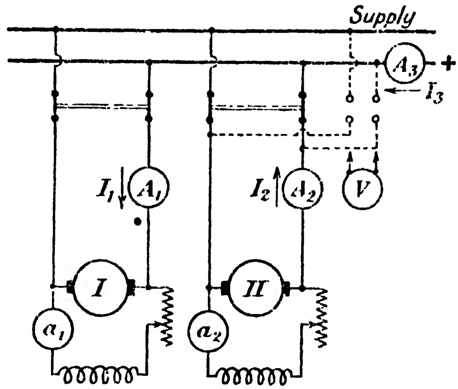


FIG. 149. HOPKINSON TEST

Assuming, first of all, that the shunt losses are the same in both machines, and that the armature copper losses and also the stray losses can be equally divided between them, we have

$$\text{Motor intake} = EI_1$$

$$\text{Generator output} = EI_2$$

Current from supply I_3, say, $= I_1 - I_2$

$$\therefore \text{ Motor intake} = E(I_2 + I_3)$$

$$\therefore \text{ Motor output} = \eta_1 E(I_2 + I_3) \text{ where } \eta_1 = \text{motor efficiency}$$

Again, generator output $= EI_2$

$$\therefore \text{ Generator intake} = EI_2/\eta_2 \text{ where } \eta_2 = \text{generator efficiency}$$

But the generator intake is equal to the motor output, so that

$$\eta_1\eta_2 = \frac{I_2}{I_2 + I_3}$$

By making the assumption of equal division of losses we also assume equal efficiencies. Hence, if we express either efficiency by η, we have

$$\eta = \sqrt{\frac{I_2}{I_2 + I_3}}$$

The error introduced by these assumptions will not be very important in the case of large machines, because the armature currents will not be very different, and the difference in the

excitations required to give the necessary circulating current through the armatures will not greatly affect the iron losses. In the case of small machines the armature currents and the excitations are very different in the two machines, and the efficiency, as calculated above, is far from accurate. The greatest error is made in assuming that the armature copper losses are equal, because these losses are proportional to the square of the current. By calculating these losses separately, but assuming an equal division of stray losses, the error is reduced to a very small amount. The exciting currents are first adjusted till each of the ammeters A_1 and A_2 reads one-half of the ammeter A_3. The machines are now both motoring light, and if I_0 is the current taken from the line, i.e. the reading of A_3, the total excitation and stray losses in the two machines are given by EI_0. We thus have for one machine

$$W_1 = \frac{EI_0}{2}$$

these losses being assumed constant. If the excitations are different the ammeters A_1 and A_2 will carry different currents. Calling these currents A_1 and A_2, and the shunt currents, a_1 and a_2, then

Current in motor armature $I_M = (A_1 - a_1)$

$\therefore$ Motor armature copper loss $R_a I_M^2 = R_a(A_1 - a_1)^2$

$\therefore$ Total motor losses $W_M = W_1 + R_a(A_1 - a_1)^2$

Again, motor intake $= EA_1$

$$\therefore \eta_M = \frac{EA_1 - W_M}{EA_1}$$

In the case of the generator the armature current is

$$I_G = (A_2 + a_2)$$

$\therefore$ Generator armature copper loss

$$R_a I_G^2 = R_a(A_2 + a_2)^2$$

$\therefore$ Total generator losses $W_G = W_1 + R_a(A_2 + a_2)^2$

Again, generator output $= EA_2$

$$\therefore \eta_G = \frac{EA_2}{EA_2 + W_G}$$

In testing compound machines, the series coils are very often left out of circuit; they can be kept in circuit if that on the motoring machine is reversed.

Example. Two shunt motors loaded for the Hopkinson test take 15 amp. at 200 volts from the supply. The motor current is 100 amp. and the shunt currents are 3 amp. and 2·5 amp. If the resistance of each armature is 0·05 ohm, calculate the efficiency of each machine for its particular conditions of loading.

The motoring machine has the smaller shunt current, so that the distribution of current is as given in Fig. 150.

Total intake of set	$= 15 \times 200$	$= 3{,}000$ watts
Copper loss in motor armature	$= (97 \cdot 5)^2 \times \cdot 05$	$= 475$,,
,, ,, generator armature	$= (88)^2 \times \cdot 05$	$= 387$,,
,, ,, motor field	$= 2 \cdot 5 \times 200$	$= 500$,,
,, ,, generator field	$= 3 \times 200$	$= 600$,,
∴ Total copper losses		$= 1{,}962$ watts

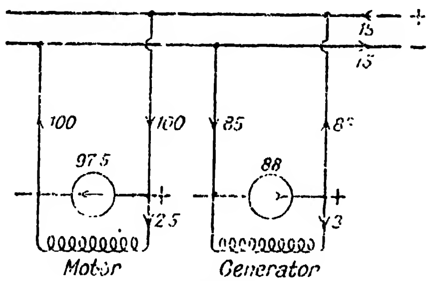


FIG. 150

∴ Total stray losses $= 3{,}000 - 1{,}962$
$= 1{,}040$, say

∴ Stray losses per machine $= 520$ watts

Generator efficiency—

Output	$= 200 \times 85 = 17{,}000$ watts
Field copper loss	$= 600$ watts
Armature copper loss	$= 387$,,
Stray losses	$= 520$,,
∴ Total losses	$= 1{,}507$,,
∴ Intake	$= 17{,}000 + 1{,}507 = 18{,}507$ watts

$$\therefore \quad \eta_G = \frac{17{,}000}{18{,}507} \times 100 = 92\%$$

Motor efficiency—

Intake	$= 200 \times 100 = 20{,}000$ watts
Field copper loss	$= 500$ watts
Armature copper loss	$= 475$,,
Stray losses	$= 520$,,
∴ Total losses	$= 1{,}495$,,
∴ Output	$= 20{,}000 - 1{,}495 = 18{,}505$ watts

$$\therefore \quad \eta_M = \frac{18{,}505}{20{,}000} \times 100 = 92 \cdot 55\%$$

6. **Series Motor Tests.*** Because of the large variation in excitation of a series motor during normal working conditions, tests made with a fixed excitation are of little value. Again, no load tests are impossible because of the dangerous speed attained. For these reasons the testing of series motors is much more difficult than that of shunt motors. If the motor is not too large, a brake test can be made on it, but in the case of large machines it is preferable to make a combined test with two machines. There is, as a rule, little difficulty in obtaining a pair of identical motors, since most series motors are used for traction work, and each tramcar has at least two. When a pair of machines is tested it is preferable to ensure that the iron losses are equal, if this is possible. This necessitates equal excitations. This condition is fulfilled in "Field's Test" by connecting the two fields in series. This arrangement of the test is shown in Fig. 151. Machine I runs as a motor and drives machine II as a generator, the output of the latter being wasted in the adjustable load R. The connection of the generator and motor fields in series ensures that the stray losses in each machine will be the same. In making the test it is preferable to include no switchgear in the connection between generator armature and load, since with this precaution there is no danger of the load being thrown off by mistake.

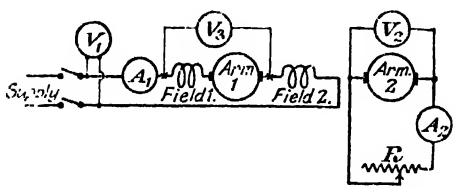


FIG. 151
CONNECTIONS FOR FIELD'S TEST

Let E_1 = supply voltage

I_1 = motor current

E_2 = voltage at terminals of load R

I_2 = load current

$\therefore$ Intake of whole set $= E_1 I_1$

Output $= E_2 I_2$

$\therefore$ Losses of all kinds $W_T = E_1 I_1 - E_2 I_2$

Total copper losses $W_c = \{(R_a + 2R_{se})I_1^2 + R_a I_2^2\}$

$\therefore$ Total stray losses $2W_s = W_T - W_c$

$\therefore$ Stray losses in one machine

$$W_s = \frac{W_T - W_c}{2}$$

* More complete information on the testing of series motors is given in *Electric Traction*, by Dover.

The motor efficiency can now be determined as follows: the generator is working under such abnormal conditions that there is no object in calculating its efficiency.

Motor intake $= E_3 I_1$ where E_3 is the reading of V_3

$$\text{Total motor losses} = (R_a + R_{se})I_1^2 + W_s$$
$$= W_M, \text{ say}$$
$$\therefore \quad \eta_M = \frac{E_3 I_1 - W_M}{E_3 I_1}$$

It is to be noted that the above test is not a regenerative test, since the output of the generator is wasted. A regenerative test on series motors is rather difficult to carry out, the best method being as follows. (*See* Fig. 152.) The generator during the test is loaded back on the supply, the amount of this loading being regulated by a load booster, this booster being a small low voltage generator whose voltage can be varied between zero and a maximum. At the moment of starting it is necessary to have a resistance load, R, across the armature as in Field's Test, but when the set is running this can be cut out and the load looked after by the booster. The generator and motor fields are in series, so that the stray losses in the two machines are the same. In order to keep the voltage across the motor terminals AB constant, an adjustable resistance can be connected in series with the motor, but if, as is usually the case, the supply is at normal motor voltage, it will be necessary to boost up the voltage in order to compensate for the drop in the generator field F_G. This is automatically carried out by a second booster, called the line booster, and this second booster is series excited.

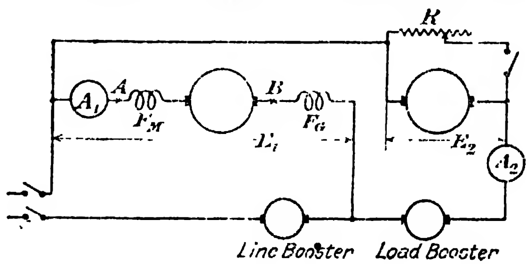


FIG. 152

REGENERATIVE TEST

Let E_1 = supply voltage and I_1 = motor current.

Let E_2 = generator voltage and I_2 = current loaded back on the supply.

$$\therefore \text{ Total losses in the two machines} = E_1 I_1 - E_2 I_2$$
$$\text{Total copper losses} = (R_a + 2R_{se})I_1^2 + R_a I_2^2$$

The calculation of the motor efficiency is thus the same as in Field's test and need not be proceeded with any further.

It will be realized that, by the use of boosters, the motoring machine in Field's test can be worked under normal conditions. In the Hopkinson test the motoring machine has its field weakened, and therefore neither machine is working under quite normal conditions. This can be avoided by keeping the normal voltage across the field of the motoring machine, and increasing the field of the generating machine by means of a field booster.

Where only a single-series motor is available for test the losses can be determined by methods similar to the Swinburne method as applied to a shunt motor. If there is a convenient source available, such as a battery or a low-voltage booster, the field can be separately excited and the losses then determined by running the armature on no load. In the case of the series motor it is necessary to make tests at a series of values of field current covering the range of the machine during normal operation, and the P.D. to be applied to the armature will, in each case, be the rated voltage less the drop of volts in the series winding.

By modifying the test as indicated in the connection diagram of Fig. 153 it is possible to dispense with the separate supply for the field winding. The motor is controlled by means of a rheostat R_2, and some of the intake current is diverted from the armature by means of the rheostat R_1. It is advisable to take a series of values for the field current and to calculate for each value the P.D. to be applied to the armature, this P.D. being given by the expression $(E - R_{se}I_{se})$. This P.D. is then plotted against the current. When starting up it is advisable to make R_1 small so that the whole of R_2 can be cut out without reaching a dangerous speed. The field current is now observed and R_2 is adjusted to give a reading of V corresponding to that on the curve. This may require one or two slight adjustments, after which the two currents, the volts and the speed, are observed.

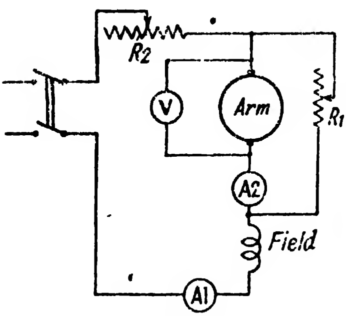


FIG. 153

Next, the resistance R_2 is increased, and the shunting resistance R_1 is adjusted until V again corresponds to the new value of the field current. Proceeding in this manner the whole range of operation of the motor is covered. The results are finally plotted so as to give curves of field current and armature watts, both against speed, as abscissae. Since the field resistance is known the field copper loss $I_{se}^2 R_{se}$ can be determined, and also the armature copper loss I_{se}^2

R_a, since during normal operation the series field and armature carry the same current. Thus the curve of total copper loss can be added, thereby providing all the data for the predetermination of the efficiency at any operating speed.

Examples on Chapter XI.

(1) Describe in detail all the adjustments and tests you would make in determining the efficiency of a continuous current shunt motor by the "loss" method at a given output. Deduce the formula for the current corresponding with a load of W watts, supplied at V volts, assuming the loss tests have furnished the requisite data. (London Univ., 1910.)

(2) A shunt motor running light at 480 volts, takes a current of 2·5 amp. The resistance of its field winding is 800 ohm and of its armature 0·6 ohm. Determine the efficiency of the motor when loaded so that the current is 40 amp., the terminal voltage being maintained at 480 volts. (London Univ., 1913.)

Ans.—88·9%.

(3) If you had to determine the efficiency of a very large motor, say, 200 h.p., and the only supply available was one from which about 20 h.p. could be taken, what procedure would you adopt ?

(4) You are required to determine the no load magnetization curve of a shunt-excited 500 volt, eight pole, continuous current motor intended to be run at 1,000 r.p.m. The source of excitation available is only 250 volts. The voltmeter available only reads to 250 volts. Indicate how you would carry out the test, and give a diagram of connections. (London Univ., 1920.)

(5) Two direct current machines are separately excited and mechanically coupled for the Hopkinson test. The driving power is supplied electrically by a small dynamo, all three armatures being connected in series. If the voltages across the armatures of the machines under test are 200 and 160, calculate the efficiency of each machine, assuming that they are the same.

Ans.—89·5 per cent.

(6) In performing the Hopkinson test on two shunt machines, the following results were obtained. Line voltage 200, line current 50, motor current 500. Shunt currents 6·0 and 5·0. Armature resistance of each machine ·012 ohm. Calculate the efficiencies of the motoring and generating machines as accurately as the data will allow.

Ans.—Each 94·9 per cent approx.

(7) A separately excited motor ran at 1000 r.p.m. with 100 volts applied to the armature. On switching off the armature supply the volts fell from 100 to 90 in 3 sec. A band brake giving 3 ft.-lb. was next applied and the same fall took place in 1·3 sec. Calculate the stray losses.

Ans.—325 watts.

(8) What are the difficulties in applying the regenerative test to two identical series motors ? Describe the test in detail and state how these difficulties are overcome.

CHAPTER XII

ELECTROLYSIS—SECONDARY CELLS—PRIMARY CELLS

1. **Electrolysis.** A liquid conductor which undergoes chemical change on the passage of an electric current is called an *electrolyte.* The immersed conductors forming the terminals of the cell are called *electrodes* ; the positive electrode is also called the *anode*, and the negative electrode the *cathode.* The products of the decomposition which takes place in the cell appear at the electrodes. It appears probable that the splitting up of the molecules of the electrolyte takes place independently of the passage of current, the constituents wandering about haphazard in the solution. These constituent parts of the molecule are called *ions.* The electrical conductivity of such a liquid is due to the fact that the ions carry electric charges, some positive, some negative. On the application of a potential difference to the electrodes the positively charged ions move down the potential gradient towards the cathode, and they are therefore called *cations.* The negatively charged ions move towards the anode and are called *anions.* When an ion reaches an electrode it " gives up its charge " and ceases to be an ion, the subsequent changes it undergoes depending on its chemical constitution.

The metals and hydrogen are electro-positive in character, that is, they form positively charged ions, or cations, when in solution. The non-metals of the chlorine group, e.g. Cl, Br, I, and Fl, and also the acid radicals such as SO_4, NO_3, PO_4, etc., form anions when in solution.

It is well known that an acid radical such as SO_4 cannot exist alone, but must always be associated with the chemically equivalent number of hydrogen or of metal atoms. This does not apply to a radical in solution, because it is then associated with a negative charge, and in this condition appears to be stable.

2. **The Products of Electrolysis** depend upon the nature of the electrolyte, i.e. whether base, acid, or a metallic salt. Consider a solution of a base such as NaOH. The caustic soda molecule splits up in solution into Na and OH, which now exist as a sodium ion, Na associated with a positive charge, and a hydroxyl ion, OH, associated with a negative charge. On applying a P.D. to the electrodes the sodium ions move to the cathode, give up their charge, and become free sodium atoms. They immediately combine with water to form sodium hydroxide, this change being accompanied by the liberation of hydrogen at the cathode. The hydroxyl ion moves to the anode, gives up its charge, and becomes a free OH group. This group cannot exist alone unless it is associated

with a negative charge, and it therefore combines with hydrogen from the water, thereby liberating oxygen at the anode. It will thus be seen that the products of electrolysis are hydrogen and oxygen, from which it would appear that the water only is decomposed. It is obvious, however, that the Na and OH ions play an indispensable part in the process, for without them the liquid would be a non-conductor.

With an acid solution, the products of electrolysis depend upon the nature of the electrodes. Take the case of a solution of sulphuric acid. The molecules of acid split up into hydrogen and sulphion, SO_4, ions, which on the application of a P.D. move towards the cathode and anode respectively. The hydrogen ion gives up its charge and is liberated as hydrogen gas. The SO_4 ion gives up its charge at the anode and must now either combine with two hydrogen atoms or the chemical equivalent of metal atoms to form sulphuric acid or a metallic salt respectively. With, say, a platinum anode, the SO_4 is compelled to take the necessary hydrogen atoms from the water, thus liberating oxygen at the anode. With, say, a copper anode, the SO_4 combines with the copper to form copper sulphate, which passes into solution. There is then no liberation of oxygen at the anode, but the anode itself is gradually dissolved away.

As an example of the decomposition of a salt solution, take the case of a solution of copper sulphate with copper electrodes. The ions are Cu with a positive charge, and SO_4 with a negative charge. The copper ions move to the cathode, give up their charge, and are then deposited as metallic copper on the cathode. The SO_4 ions move to the anode, give up their charge, and then combine with it to re-form copper sulphate. The final result of such an electrolysis is thus the transference of copper from the anode to the cathode.

3. **Quantitative Laws of Electrolysis.** The phenomena of electrolysis were thoroughly investigated by Faraday, who found that the weight of an element or radical deposited at an electrode is proportional to the quantity of electricity which has passed. Hence, if

$w =$ weight of the deposit in grammes
$I =$ current in amperes
$t =$ time in seconds
$A =$ a constant for a given ion

then $\quad w = AIt$

The constant A is composite, its value depending upon the atomic weight and the valency It is obvious that for the electrolytic deposition of metals, or the liberation of gases, w will be proportional to the atomic weight, because each ion carries a definite quantity of electricity. Now the charge carried by an ion is proportional to its valency, for consider molecules of sulphuric acid and copper sulphate. The sulphion in the acid molecule is combined with

two hydrogen atoms, whereas it is only combined with one copper atom in the copper sulphate molecule. From this it follows that the copper atom, which is divalent, has, when in the form of a copper ion, twice the charge of a hydrogen ion, which is univalent. In other words, in order to carry a given quantity of electricity, only one-half as many divalent copper ions as univalent hydrogen ions will be required. Hence, the greater the valency of the ion, the smaller the number of ions required to carry a given quantity of electricity, from which it follows that the weight deposited is inversely proportional to the valency. The equation can therefore be written

$$w = B \cdot \frac{a}{v} It$$

where a = atomic wt.; v = valency; B = another constant.

The constant B is equal to the number of grammes of hydrogen deposited by the passage of 1 coulomb of electricity, namely, to 0·00001038 gramme per coulomb. The quantity Ba, or $a \times 0{\cdot}00001038$, gives the number of grammes of a univalent substance of atomic weight a deposited by 1 coulomb. Thus, in the case of silver, the atomic weight is 107·6 and the weight per coulomb is $107{\cdot}6 \times 0{\cdot}00001038 = 0{\cdot}001118$. The quantity Ba/v gives the number of grammes of a substance of any valency v deposited by 1 coulomb. Thus for divalent copper $a = 63{\cdot}2$, $v = 2$, and therefore, $Ba/v = 0{\cdot}000328$ gramme per coulomb.

The quantity Ba/v is called the *electro-chemical equivalent*; hence, denoting it by z, we have

$$w = zIt$$

The quantity a/v is the *chemical equivalent.*

4. Polarization. The relationship between the current I through an electrolytic cell, and the applied P.D., E, is given by an equation of the form

$$E = \varepsilon + RI$$

where ε is the back E.M.F. or *polarization* E.M.F. of the cell, and RI is the ohmic drop due to its internal resistance. In the case of a cell in which the electrolyte is a salt of the metal of the electrodes, e.g. $CuSO_4$ with copper electrodes, the back E.M.F. is set up by concentrations of electrolyte round the electrodes. If the electrodes are not acted on by the electrolyte, e.g. platinum electrodes in a solution of H_2SO_4, it is due to deposits of gas. These concentrations of electrolyte, or deposits of gas, prevent the fresh arrivals of ions from reaching the electrodes, the result being that the ions are unable to give up their charges, and so set up a back E.M.F.

Multiplying both sides of the above equation by I, we have the energy equation

$$EI = \varepsilon I + RI^2$$

The term RI^2 represents the power used in producing heat, and the term εI represents the power required to effect the chemical changes taking place. The energy absorbed in effecting the chemical changes during a time t is, therefore, εIt, if I has remained constant. Otherwise, $\varepsilon\int Idt$. Consider the case of a dilute sulphuric acid electrolyte with platinum electrodes. Mixed oxygen and hydrogen will be liberated, and the potential energy of this mixture will of necessity be equal to the energy used in liberating it, e.g. to εIt. Now when 1 gramme of these mixed gases is exploded, 3,780 calories of heat are evolved. The equivalent in watt-seconds or joules is $3{,}780 \times 4{\cdot}2 = 15{,}870$. Again, the passage of 1 coulomb of electricity liberates $\cdot 00001038$ gm. of hydrogen and $\cdot 00001038 \times \frac{16}{2}$ gm. of oxygen; that is, $9{\cdot}35 \times 10^{-5}$ gm. of mixed gases. The quantity of electricity required to liberate 1 gramme of mixed gases is therefore $10^5/9{\cdot}35 = 1{\cdot}07 \times 10^4$ coulombs. But we saw that the product εIt was equal to 15,870. Hence, for the back E.M.F. ε, in the case of such a cell, we have

$$\varepsilon = \frac{15{,}870}{1{\cdot}07 \times 10^4} = 1{\cdot}47 \text{ volts.}$$

Hence, to electrolyze dilute sulphuric acid, or any electrolyte in which mixed oxygen and hydrogen are evolved, an applied P.D. greater than 1·47 volts is necessary.

5. Accumulators. In the previous section we saw that an electrolytic cell could set up a back E.M.F. if there was a concentration of electrolyte round the electrodes, or if there was an accumulation of gas on the electrodes. This back E.M.F. can also exist if the electrodes themselves are changed chemically. The most striking example of this phenomenon occurs with lead electrodes in a dilute solution of sulphuric acid. The acid molecules split up into hydrogen and sulphion ions, associated with positive and negative charges respectively. On the application of a potential difference to the electrodes the hydrogen ions travel towards the cathode, where they give up their charges and are liberated as free hydrogen. The sulphion ions give up their charges at the anode, and they then have a choice of combining with either the lead of the anode, or the hydrogen of the water. Both reactions take place at the same time, but the most important is the combination with hydrogen of the water, so as to liberate oxygen. This oxygen at the moment of liberation is in the nascent, i.e. atomic form, and it therefore attacks the lead of the anode and forms in time a brown deposit of lead peroxide, PbO_2. On stopping the current and connecting a voltmeter to the electrodes it will now be found that the cell has an E.M.F. of 2 volts. If the electrodes are joined by a wire the cell will produce a current which will flow through the

electrolyte in the reverse direction to the previous current, and the cell will discharge. The following changes then take place. The hydrogen ions, carrying positive charges, now move towards the positive plate, which, as we have seen, is coated with PbO_2. After liberation of the charge, the hydrogen atoms combine with the oxygen of the PbO_2, while the acid attacks the lead, with the formation of $PbSO_4$.

$$PbO_2 + H_2 + H_2SO_4 = PbSO_4 + 2H_2O.$$

At the negative plate, the sulphion ions, after losing their charge, combine with the lead to form $PbSO_4$.

$$Pb + SO_4 = PbSO_4.$$

Thus if the cell is completely discharged, both plates will be coated with a whitish coating of lead sulphate. If the discharged cell is now charged again by passing current from an external source through it from positive to negative plate, the following changes will take place—

At the negative plate $PbSO_4 + H_2 = Pb + H_2SO_4$

At the positive plate $PbSO_4 + SO_4 + 2H_2O = PbO_2 + 2H_2SO_4$

From this it will be seen that on re-charging, the positive plate will re-acquire its brown coating of lead peroxide, while the negative plate will be reduced to grey metallic lead.

The capacity of such a cell, that is, the number of ampere-hours that can be obtained from it when discharging, will naturally be very small, because of the small surface exposed to the acid. With repeated charging and discharging, the lead plates will become "spongy," and will present a much greater effective surface to the acid, the result being that the capacity will be greatly increased. This process of producing accumulator plates is called "forming"; it is no longer used because of its high cost. A cheaper method is to use plates in the form of cast lead grids, the holes in which are filled with a paste of lead oxide. If red lead, Pb_3O_4, is used for the positive, and litharge, PbO, for the negative, both plates can be formed at the same time by immersing them in dilute sulphuric acid and passing current. The positive plate becomes further oxidized to PbO_2, while the litharge is reduced to metallic lead. Such plates are not mechanically strong, and the paste is liable to fall out if the cell is severely used. In more modern processes the plates are formed by a very quick process. Thus, boiling in nitric acid to oxidize the lead, and then reducing the oxide electrically are possible. It is usual to perform these two operations together by adding "quick formation agents" to the electrolytic bath. These agents, of which nitrates, chlorates, and acetates are examples, form unstable compounds with the lead, these compounds being immediately formed into PbO_2 by the action of the current.

Various types of formed and pasted plates are illustrated in Fig. 154.

When a cell is charged the voltage gradually rises, remains in the neighbourhood of 2·3 volts for some time, and then rises again,

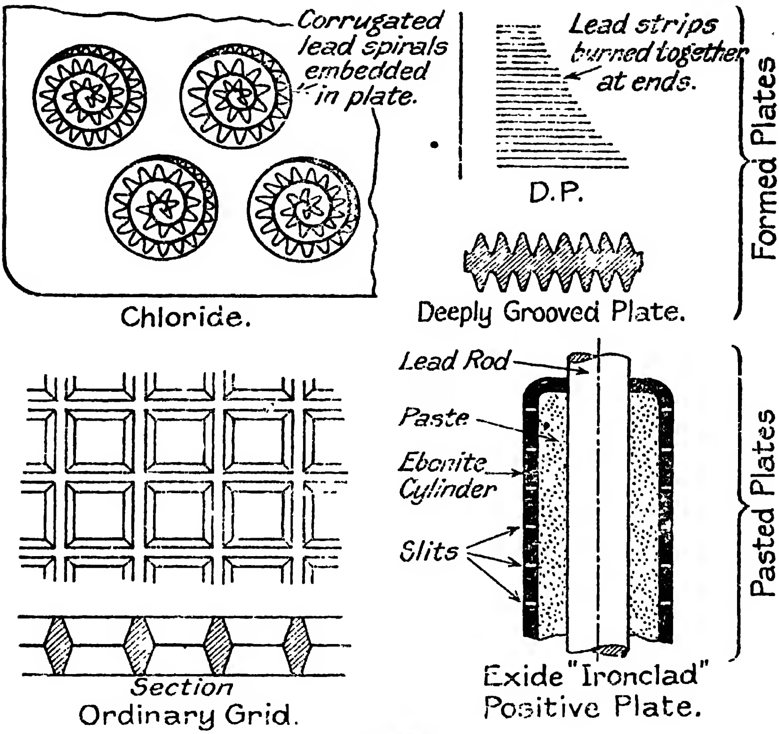


Fig. 154

Types of Accumulator Plates

as shown by Fig. 155. At the same time a large quantity of gas is evolved. The final rise is due to a concentration of sulphuric acid in the pores of the positive plates. If the charged cell is now left standing this strong acid diffuses into the rest of the solution, and the voltage of the cell falls to about 2·2. If now the cell is discharged, the P.D. at the terminals remains roughly constant for a time, and then begins to fall. The final fall is very rapid if the cell is allowed to discharge completely, because of the dilution of the acid in the pores of the plates. In practice a cell should not

be allowed to discharge below 1·8 volts, otherwise there will be an excessive formation of sulphate which will be difficult to remove. Also, the plates may tend to buckle and the paste become loosened. For this latter reason the charge and discharge currents should not be excessive, since too rapid chemical action is the main cause of buckling.

The density of the acid is 1·21 when fully charged, corresponding to 28 per cent acid by weight. When discharged to 1·8 volts the density is much less, about 1·18, corresponding to 25 per cent acid. With varying density, the conductivity of the acid varies, as shown in Fig. 156. It will be seen that a density of just over 1·2 gives the minimum specific resistance, and therefore, the minimum internal resistance of the cell.

As stated before, the capacity of a cell is reckoned by the number of ampere-hours on discharge at the normal discharge rate. If the discharge rate is less than normal the capacity will be greater, but with an excessive discharge rate the capacity will be very much reduced, as shown by Fig. 157. For a pasted positive plate the average capacity is about $\frac{1}{2}$ ampere-hour for each square inch of positive surface (reckoning one side); the normal discharge rate, about $\frac{1}{8}$ amp. per square inch of positive surface. Temperature has a great effect on the action of a cell, rise of temperature increasing the E.M.F. slightly, and the capacity, very considerably. Thus, a rise of temperature of 30° C. will increase the capacity at normal discharge rate by 30 per cent.

The efficiency of a cell can be reckoned in two ways—

$$\text{Quantity efficiency} = \frac{\text{ampere-hours of discharge}}{\text{ampere-hours of charge}}$$

$$\text{Energy efficiency} = \frac{\text{watt-hours of discharge}}{\text{watt-hours of charge}}$$

If E is the terminal P.D. per cell, E_1 the E.M.F. per cell on charge, and E_2 the E.M.F. on discharge, then

$$E = E_1 + RI$$
$$E = E_2 - RI$$

The average terminal P.D. during charge is therefore greater than during discharge, the difference being twice the internal drop RI, plus the quantity $(E_1 - E_2)$. The E.M.F. during charging is greater than during discharging, because of the presence of the gases evolved. It therefore follows that the energy efficiency is less than the quantity efficiency. Increase in discharge rate, as we have seen, reduces the capacity and therefore the quantity efficiency to a certain extent, but it decreases the energy efficiency still further, since the greater the discharge rate the smaller the terminal P.D.

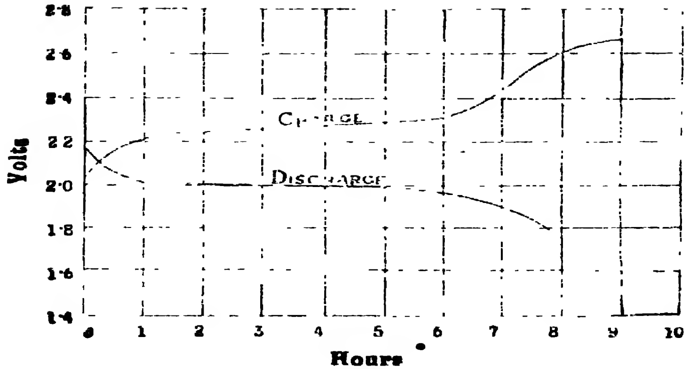


Fig. 155

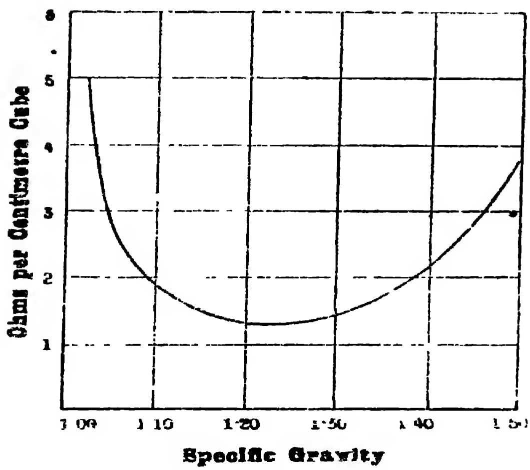


Fig. 156

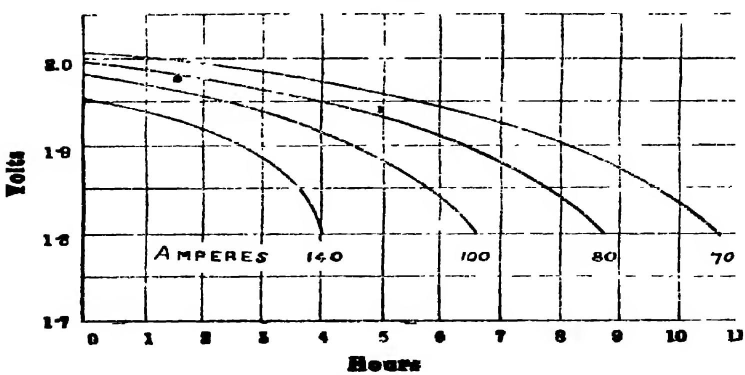


Fig. 157

Accumulator Characteristics

of the cell. Average values are 90 per cent for the quantity efficiency and 75 per cent for the energy efficiency.

Cells of the lead-acid type weigh about $\frac{1}{2}$ lb. per ampere-hour for ordinary construction, $\frac{1}{3}$ lb. per ampere-hour for high discharge rate cells such as are used in power stations, and about $\frac{1}{5}$ lb. per ampere-hour for cells used in electric vehicles.

6. **Testing.** In order to carry out on a cell a test of any real value, it is necessary to make observations during charging and discharging, and to ensure that the electrical and chemical states of the cell at the end of the test are the same as at the beginning. This is done by plotting the E.M.F.-time curves and Sp. Gr.-time curves for successive charges and discharges until the curves are exactly repeated, the charge and discharge being performed at constant current. The cell is then said to be in the "cyclic state." Then

$$\text{Quantity efficiency} = \frac{\text{discharge current} \times \text{time}}{\text{charging current} \times \text{time}}$$

To determine the energy efficiency, readings of the P.D. have to be taken at frequent intervals and plotted against time. The area of the curve gives the volt-hours, which, when multiplied by the constant current of charge or discharge, gives the watt-hours.

$$\text{Energy efficiency} = \frac{\text{discharge current} \times \text{volt-hours of discharge}}{\text{charge current} \times \text{volt-hours of charge}}$$

If E_0 is the open-circuit E.M.F. of a cell, and E_1, the terminal P.D. when delivering a current I, then

$$E_0 - E_1 = \text{drop of volts in cell} = RI$$

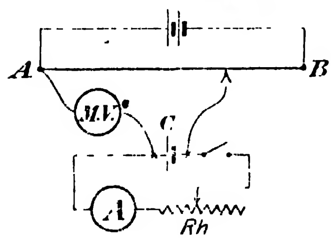


FIG. 158

MEASUREMENT OF INTERNAL RESISTANCE

$$\therefore \text{ Resistance of cell } R = \frac{E_0 - E_1}{I}$$

In this way the resistance of a cell can be determined, but the method is not very accurate because $(E_0 - E_1)$ is small. A better method is to balance the E.M.F. of the cell C, in Fig. 158, against the drop along a potentiometer AB, using a milli-voltmeter (M.V.) in place of a galvanometer. The cell is then discharged at, say, I amp. through an adjustable resistance Rh, the milli-voltmeter then giving directly the drop in the cell. Hence,

$$R = \frac{\text{reading of M.V.}}{\text{current}}$$

7. **The Alkaline Accumulator.** The active materials in this cell are nickel oxide on the positive, and iron oxide on the negative,

plate. The electrolyte is a 21% solution of caustic potash, KOH, with the addition of a small quantity of lithium hydrate. The action of the latter is not yet understood, but it very materially increases the capacity of the cell. The exact formula of the nickel oxide is not yet established, but the action of the cell can be followed by assuming the peroxide NiO_2, or its hydrated form $Ni(OH)_4$ On discharge the OH ions of the KOH travel to the negative, the iron therefore becoming oxidized. The K ions travel to the positive and reduce the $Ni(OH)_4$ to $Ni(OH)_2$. During charge the converse action takes place, so that the action can be represented by the reversible equation

$$Ni(OH)_4 + KOH + Fe \rightleftarrows Ni(OH)_2 + KOH + Fe(OH)_2$$

It will be noticed that the electrolyte acts merely as a vehicle for the transfer of OH from one plate to another, and it does not take part in any chemical change. As a result, the density does not change to the same extent as in the ordinary lead-acid cell.

The positive plate consists of a number of tubes made of perforated steel ribbon, wound spirally, and held together by steel rings. The tubes are very heavily nickel plated, packed with alternate layers of nickel hydroxide and flake nickel, and then clamped in a steel frame, which is also nickel plated. The flake nickel is added because the hydroxide is rather a poor conductor. The negative plate is made from finely perforated nickelled steel strip stamped into pockets, the pockets being filled with iron oxide. Here also the conductivity is not very good, and it is improved by adding a little mercury. In order to render the iron oxide susceptible to chemical changes in the cell, it is prepared by alternate reduction and oxidation of iron sulphide under KOH. The plates are separated from one another by hard rubber strips, and are held in a nickel-plated sheet steel container with welded seams.

The normal density of the electrolyte is 1·22, and this falls to about 1·16 during the first year's use. A fresh electrolyte of density 1·25 is then added, this higher density being to compensate for the old weak solution left in the plates.

The E.M.F. per cell when fully charged is 1·4 volts, and it is usually allowed to fall to 1·1 before recharging. Actually, the cell can be completely discharged, or even short-circuited, without damage, but it is not economical to allow the E.M.F. to fall below about 1 volt. The efficiencies are lower than for lead-acid cells, average values being 80% for the quantity efficiency and 60% for the energy efficiency. The weight of the cell is about 1 lb. per 10 ampere-hours.

The effect of an increase in temperature is to lower the E.M.F. slightly but to increase the capacity, and there is a critical temperature of 53° F. below which the capacity falls off rapidly.

8. **Primary Cells.** The simplest cell consists of a vessel of dilute sulphuric acid in which electrodes of copper and zinc are placed. The copper becomes positive relatively to the zinc, and if the two are joined by a wire a current flows. The sulphuric acid is thereby decomposed, and the hydrogen, which travels in the direction of the current, adheres to the copper electrode in the form of small bubbles. This gas can be regarded as constituting the positive of a new cell consisting of the elements hydrogen, sulphuric acid, and zinc, the E.M.F. of this cell opposing that of the original cell. The E.M.F. of the latter therefore dies away, the phenomenon being that of polarization. Since any practical form of cell must have a reasonably constant E.M.F., polarization must be prevented. The problem is thus the prevention of the formation of free hydrogen. There are two methods: a metal can be deposited on the cathode instead of hydrogen, or the hydrogen can be combined with some other substance the moment it is evolved.

The first method is used in the Daniell cell, in which two liquids, copper sulphate and dilute sulphuric acid, separated by a porous pot, are used. The positive plate, of copper, is immersed in the copper sulphate, and the negative plate, zinc, in the dilute acid. When current flows, the direction inside the cell is from zinc to copper, the hydrogen thus travelling to the copper sulphate solution. Here it throws down copper and forms sulphuric acid. The SO_4 ion attacks the zinc and forms zinc sulphate. This change is an oxidation and it supplies an amount of energy equal to the electrical energy supplied by the cell, plus the energy required to precipitate the copper. The electrodes are not changed, and therefore, there is no polarization, the E.M.F. remaining constant at 1·07 volts.

The second method is made use of in the Leclanché cell. The electrolyte is in this case a solution of ammonium chloride, NH_4Cl, in which a zinc rod (the negative electrode) is dipped. When the cell delivers current, zinc chloride, $ZnCl$, is formed, and the NH_4 group acts as the carrier of positive electricity. The positive electrode is a carbon plate placed in a porous pot and packed round with a mixture of manganese dioxide and carbon. This mixture becomes permeated with the electrolyte, and the NH_4 ions are able to travel to the carbon plate. They are there split up into ammonia and hydrogen gas, the latter being oxidized to water by the manganese dioxide. The removal of hydrogen in this way is necessarily slow, and therefore the cell is only suitable for intermittent use. The energy in this cell is supplied by the formation of zinc chloride.

PART TWO
ALTERNATING CURRENT

CHAPTER XIII

ALTERNATING CURRENT CIRCUITS

1. **An Alternating Quantity** is one which acts in alternate directions, and whose magnitude undergoes a definite cycle of changes in definite intervals of time. The graph of such a quantity is shown in Fig. 159. It will be seen that the graph repeats after regular time intervals, and one repeat is called a complete "cycle." The time T of one cycle is called the "periodic time," and the number of cycles per second, the "frequency," f. Hence,

$$T = \frac{1}{f} \text{ and } f = \frac{1}{T}$$

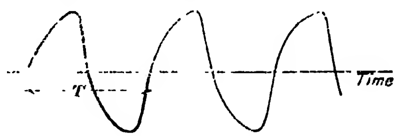


FIG. 159
GRAPH OF AN ALTERNATING QUANTITY

2. **The Simple Alternator.** Consider a coil rotating with angular velocity ω radians per sec., as shown in Fig. 160, in a uniform magnetic field, and let time be reckoned from the instant the plane of the coil includes the OX axis. In this position the flux Φ linking with the coil has its maximum value Φ_{max}. Hence, if the linkage is expressed as a function of the time t,

$$\Phi N = \Phi_{max} N \cos \omega t,$$

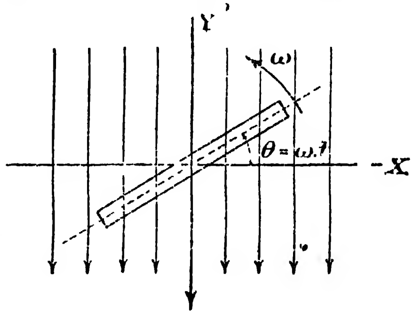


FIG. 160
SIMPLE ALTERNATOR

N being the number of turns on the coil. The induced E.M.F. in the coil is equal to minus the rate of change of linkage. Denoting the instantaneous E.M.F. by e,

$$e = -\frac{d}{dt}(\Phi_{max} N \cos \omega t) \times 10^{-8} \text{ volts}$$

$$= (\Phi_{max} N \omega \times 10^{-8}) \sin \omega t.$$

Obviously the quantity in the brackets is the maximum value of the E.M.F., E_{max}, and we can therefore write

$$e = E_{max} \sin \omega t.$$

The induced voltage is therefore sinusoidal and, unless otherwise stated, alternating current calculations are always made on the assumption that the voltage and current are sinusoidal.

It is obvious that in the case of the rotating coil, one complete cycle is gone through in one revolution, i.e. 2π radians. Hence, the equation for the instantaneous voltage can also be written in the form

$$e = E_{max} \sin 2\pi ft.$$

Example. An alternating current of frequency 50 cycles per sec. has a maximum value of 100 amp. Reckoning time from the instant the current is zero and is becoming positive, calculate (*a*) the instantaneous value after 1/300th sec.; (*b*) the time taken for the current to reach 80 amp. for the first time.

(*a*) $$i = I_{max} \sin 2\pi ft.$$

$$= 100 \sin\left(2\pi \times 50 \times \frac{1}{300}\right) \ldots \text{angle in radians}$$

$$= 100 \sin\left(360 \times 50 \times \frac{1}{300}\right) \ldots \text{angle in degrees}$$

$$= 100 \sin 60° = 86{\cdot}6 \text{ amperes}$$

(*b*) $$80 = 100 \sin (2\pi \times 50t) \ldots \text{angle in radians}$$

$$= 100 \sin (360 \times 50t) \ldots \text{angle in degrees}$$

$$\therefore 360 \times 50 \times t = \sin^{-1} 0{\cdot}8 = 53° \text{ approx.} \quad \therefore t = \frac{53}{360 \times 50}$$

$$= {\cdot}00295 \text{ sec.}$$

3. Effective Value. The effective value of an alternating E.M.F. or current is given by that direct E.M.F. or current which, when applied to a given circuit for a given time, produces the same expenditure of energy as when the alternating E.M.F. or current is applied to the same circuit for the same time.

Consider an alternating current of any wave form (Fig. 161). Divide the base into a large number, n, equal intervals, each of $\frac{T}{n}$ sec., and let the mid ordinates be i_1, i_2, i_3, etc. Let this current be flowing through a resistance of R ohms.

Then energy expended in 1st interval $= i_1^2R \times \frac{T}{n}$ joules

„ „ 2nd „ $= i_2^2R \times \frac{T}{n}$ „

„ „ 3rd „ $= i_3^2R \times \frac{T}{n}$ „

etc.

Hence, total energy expended in time T

$$= T \times R\left(\frac{i_1^2 + i_2^2 + i_3^2 + \ldots}{n}\right)$$

Now let I be the effective current; then the energy expended in time T will be I^2RT joules. By definition these two expressions for the energy are equal

$$\therefore \quad I^2RT = TR\left(\frac{i_1^2 + i_2^2 + i_3^2 + \ldots}{n}\right)$$

$$\therefore \quad I = \sqrt{\frac{i_1^2 + i_2^2 + i_3^2 + \ldots}{n}}$$

Hence, the effective value is equal to the square root of the mean of the squares of successive ordinates. It is, therefore, sometimes

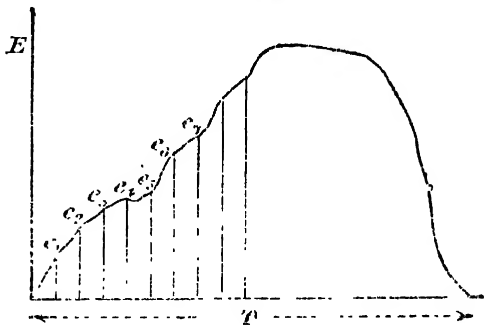


FIG. 161

called the "root mean square" (R.M.S.) value. It is also called the "virtual" value.

4. **Average and Effective Values of a Sinusoidal Wave.** We have

$$e = E_{max} \sin \theta$$

where $\theta = \omega t$. A half wave is completed when θ varies from 0 to π radians.

Hence, average value

$$E_{av} = \frac{1}{\pi}\int_0^{\pi} E_{max} \sin \theta d\theta$$

$$= -\frac{E_{max}}{\pi}\Big[\cos \theta\Big]_0^{\pi}$$

$$= \frac{2E_{max}}{\pi} \text{ or } \frac{E_{max}}{\pi/2}$$

Again,

$$E^2_{eff} = \frac{1}{\pi}\int_0^{\pi} E^2_{max} \sin^2 \theta d\theta$$

$$= \frac{E^2_{max}}{2\pi}\int_0^{\pi} (1 - \cos 2\theta)\, d\theta$$

$$= \frac{E^2_{max}}{2\pi}\Big[\theta - \tfrac{1}{2}\sin 2\theta\Big]_0^{\pi}$$

$$= \tfrac{1}{2} E^2_{max}$$

$$\therefore \quad E_{eff} = \frac{E_{max}}{\sqrt{2}}$$

The ratio E_{eff}/E_{av} is called the " Form Factor " of a wave. For a sinusoidal wave we therefore have

$$\text{Form Factor} = \frac{E_{max}}{\sqrt{2}} \div \frac{E_{max}}{\pi/2}$$

$$= 1{\cdot}11$$

5. Graphical Representation of Alternating Quantities. Consider a vector OP rotating with angular velocity ω, where $\omega = 2\pi f$ (Fig 162). Let the length of OP be E_{max}. Then at any instant the intercept on the Y axis will be

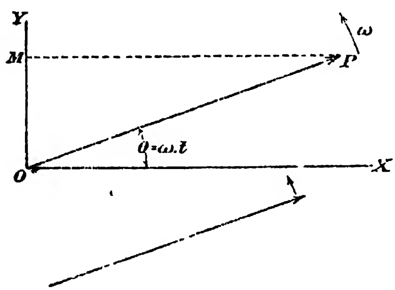


FIG. 162

$$OM = OP \sin \omega t$$
$$= E_m \, . \sin \omega t$$

Hence, $OM = e$, the instantaneous voltage.

Now the axes OX and OY are fixed in space, and it is not necessary to include them in the diagram ; we are thus left simply with the vector OP. Finally, since the effective value E bears a definite relationship to the maximum value, namely, $E_{max}/\sqrt{2}$, the length of the vector can be made equal to the effective value if desired.

Hence, an alternating voltage or current can be represented by a simple vector. It is to be remembered that this vector is rotating, so that its position on a diagram gives the conditions at a certain instant of time. By drawing the OY axis and projecting the vector on to it, the instantaneous value at the particular instant defined by the position of the vector is obtained.

6. Phase Difference. Suppose two alternating quantities are represented at a given instant by two vectors, OP and OQ (Fig. 163). Then the angle φ between them is called the "phase" angle. With reference to the direction of rotation of the vectors, OP is in front of OQ, and OP is said to "lead" OQ, while OQ is said to "lag" behind OP. If the two quantities are also represented by wave diagrams as shown in Fig. 163, the phase difference φ is given by the relative displacement of the curves along the angle, or time, axis. With reference to such a diagram the leading quantity is that which goes through its zero or crest value the first. The lag or lead obviously depends upon the assumed direction of rotation of the vectors, and in all the vector diagrams following the direction is counter-clockwise.

If the vectors are voltage vectors and the leading vector is represented by

$$e_1 = OP \sin \omega t$$

then the lagging vector will be represented by

$$e_2 = OQ (\sin \omega t - \varphi)$$

If a number of alternating voltages act at the same time in a given circuit their resultant is given by their vector sum, just as the resultant of a number of forces acting at a point is given by the vector sum. The resultant voltage *at any instant* is given by the algebraic sum of the instantaneous values. This algebraic sum is obviously the same as the algebraic sum of the projections of all the vectors on the Y axis.

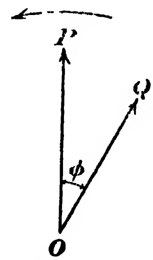


FIG. 163
PHASE DIFFERENCE

7. The Addition of Alternating E.M.F.s, or Alternating Currents. In the great majority of alternating current circuits it is necessary to consider the combined action of several E.M.F.s, all acting in the circuit at the same time, or in the case of a branched circuit, the action of several currents. One method of determining the resultant of a number of alternating E.M.F.s or currents is to regard their vectors as forces and to make the calculations exactly as in the case of a similar problem dealing with the effect of a number of forces acting at a point. For example, if there are two E.M.F.s acting at the same time, their equations being

$$e_1 = E_1 \sin \omega t$$

and

$$e_2 = E_2 \sin (\omega t + \phi)$$

the vectors are of lengths E_1 and E_2 (maximum values) and their phase displacement ϕ, with the result that their resultant is given by

$$E^2_{max} = E_1^2 + E_2^2 + 2E_1E_2 \cos \phi$$

$$\therefore \quad E_{eff} = \cdot 707 \times \sqrt{E_1^2 + E_2^2 + 2E_1E_2 \cos \phi}$$

Alternatively, we have, dealing in instantaneous values,

$$e = e_1 + e_2 = E_1 \sin \omega t + E_2 \sin (\omega t + \phi)$$
$$= E_1 \sin \omega t + E_2 (\sin \omega t \cos \phi + \cos \omega t \sin \phi)$$
$$= (E_1 + E_2 \cos \phi) \sin \omega t + E_2 \sin \phi \cdot \cos \omega t$$

This shows how we can resolve the total E.M.F.s into two sinusoidal components in quadrature with one another, the magnitudes of these two components (maximum values) being $(E_1 + E_2 \cos \phi)$ and $E_2 \sin \phi$ respectively. Denoting these by X and Y respectively, we see that they will form the base and perpendicular of a right-angled triangle (Fig. 164). Let the base angle of this triangle be θ, then

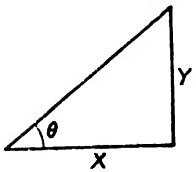


FIG. 164

$$\sin \theta = \frac{Y}{(X^2 + Y^2)^{\frac{1}{2}}} \text{ and } \cos \theta = \frac{X}{(X^2 + Y^2)^{\frac{1}{2}}}$$

$$\therefore\ e = X \sin \omega t + Y \cos \omega t$$
$$= (X^2 + Y^2)^{\frac{1}{2}} \left\{ \frac{X}{(X^2 + Y^2)^{\frac{1}{2}}} \cdot \sin \omega t + \frac{Y}{(X^2 + Y^2)^{\frac{1}{2}}} \cdot \cos \omega t \right\}$$
$$= (X^2 + Y^2)^{\frac{1}{2}} (\sin \omega t \cdot \cos \theta + \cos \omega t \cdot \sin \theta)$$
$$= (X^2 + Y^2)^{\frac{1}{2}} \cdot \sin (\omega t + \theta)$$

Thus the resultant E.M.F. is a sinusoidal E.M.F. of maximum value $(X^2 + Y^2)^{\frac{1}{2}}$, leading the E.M.F. E_1 by θ where $\theta = \text{arc tan } \dfrac{Y}{X}$

For the effective value of the resultant, we have

$$E_{eff} = \cdot 707\ (X^2 + Y^2)^{\frac{1}{2}}$$
$$= \cdot 707\ \{(E_1 + E_2 \cos \phi)^2 + E_2^2 \sin^2 \phi\}^{\frac{1}{2}}$$
$$= \cdot 707 \times \sqrt{E_1^2 + E_2^2 + 2E_1E_2 \cos \phi}, \text{ as before.}$$

The special case in which E_1 and E_2 are of equal magnitude is of importance. Denoting each by E_1 we then have

$$e = e_1 + e_2 = E_1 \sin \omega t + E_1 \sin (\omega t + \phi)$$
$$= E_1 \{\sin \omega t + \sin (\omega t + \phi)\}$$
$$= 2E_1 \cos \frac{\phi}{2} \cdot \sin \left(\omega t + \frac{\phi}{2}\right)$$

showing that the resultant is of maximum value $2E_1 \cos \phi/2$, and therefore of effective $1{\cdot}414E_1 \cos \phi/2$, and that it leads the component $E_1 \sin \omega t$ by $\phi/2$.

8. Pure Resistance Circuit. A pure resistance circuit is one possessing neither inductance nor capacity. Hence, if a current passes through the circuit no back E.M.F. will be set up by any change in current. The applied voltage has therefore to overcome

the ohmic drop only as in a direct current circuit. We thus have, using effective values,

$$I = \frac{E}{R} \text{ in phase with } E \text{ (Fig. 165).}$$

The power, w, in the circuit at any instant is the product of the instantaneous voltage and instantaneous current. Hence, if the

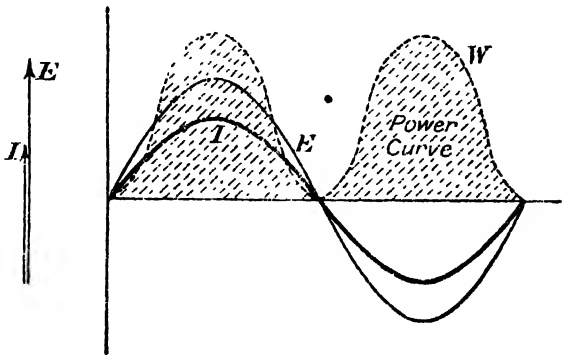


FIG. 165

wave diagrams are drawn and the products of the ordinates at given instants plotted, the resulting curve will be the curve of power.

We have $w = e \times i$

Let $e = E_{max} \sin \omega t$

Then $i = I_{max} \sin \omega t$

and $w = E_{max} I_{max} \sin^2 \omega t$

$= \frac{1}{2} E_{max} I_{max} (1 - \cos 2\omega t)$

since $\sin^2 \omega t = \frac{1}{2}(1 - \cos 2\omega t)$.

It is clear that the power, w, has a constant part $\frac{1}{2} E_{max} I_{max}$ and a fluctuating part $-\frac{1}{2} E_{max} I_{max} \cos 2\omega t$, the latter averaging zero over a complete cycle. Now the power in a circuit, as measured by a wattmeter for example, is the *average* of the instantaneous power, w, because power is a scalar and not a vector quantity. Hence average power is

$$\begin{aligned} W &= \text{average of } w \\ &= \text{steady part of } w \\ &= \tfrac{1}{2} E_{max} I_{max} \\ &= \frac{E_{max}}{\sqrt{2}} \times \frac{I_{max}}{\sqrt{2}} = EI \end{aligned}$$

Thus the power in a pure resistance circuit is given by the product of the effective voltage and current. The fluctuating power has twice the frequency of the voltage or current. This is illustrated by the curve of power on the wave diagram.

9. Purely Inductive Circuit. A pure inductive circuit is one which possesses inductance only, but no resistance or capacity. The nearest approach to such a circuit is obtained by winding a coil of heavy section copper wire on a laminated iron core. Such a coil is called a "choking coil." The magnetic field set up by the alternating current will be alternating; hence, its magnitude will be changing at every instant. Now a self-induced E.M.F. is set up whenever the magnetic flux, linking with a circuit, changes, and since there is no ohmic resistance, the applied E.M.F. has to oppose

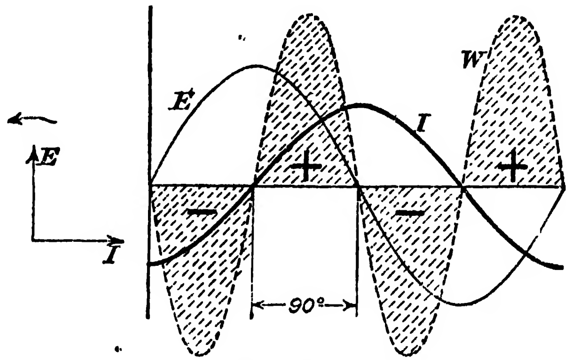


FIG. 166

the self-induced E.M.F. only. Hence, the applied E.M.F. is equal and opposite to the self-induced E.M.F. at every instant.

Let $i = I_{max} \sin \omega t$

Then self-induced E.M.F. at any instant

$$e' = -L\frac{di}{dt}$$

$$= -L\omega I_{max} \cos \omega t$$

Hence, applied voltage at the same instant,

$$e = -e' = +L\omega I_{max} \cos \omega t$$

The applied voltage is therefore represented by a cosine function, and consequently leads the current (which is represented by a sine function) by 90° (Fig. 166). Again

$$E_{max} = \text{max. of } (L\omega I_{max} \cos \omega t)$$

$$= L\omega I_{max}$$

$$\therefore \quad E = L\omega I \text{ or } I = \frac{E}{L\omega}$$

The quantity $L\omega$ is called the "Reactance." If L is in henrys and ω in radians per sec., the reactance is expressed in ohms.

From the above we see that the current in a purely inductive circuit lags 90° behind the voltage applied.

For the instantaneous power we have, as before,

$$w = e \times i$$

If we put $e = E_{max} \sin \omega t$

$$\text{Then } i = I_{max} \sin\left(\omega t - \frac{\pi}{2}\right) = -I_{max} \cos \omega t$$

$$\therefore\ w = -E_{max} I_{max} \sin \omega t \cos \omega t$$

$$= -\tfrac{1}{2} E_{max} I_{max} \sin 2\omega t$$

and average power

$$W = -\tfrac{1}{2} E_{max} I_{max} \times \text{average of } \{\sin 2\omega t\}$$

$$= 0$$

We therefore see that the total power is zero, a result which at first sight is surprising when it is realized that both E and I are finite. The power is, in fact, a pure sine wave of double frequency and maximum value $\frac{1}{2}E_{max}I_{max} = EI$.

The power is alternately positive and negative, the alternate lobes of the curve being of equal magnitude as shown in the wave diagram. This is explained physically as follows. When the current is zero, there is zero magnetic field. As the current increases, the magnetic field increases, and since work has to be done to create a magnetic field, the circuit has to supply positive power. This goes on until the current is a maximum. The current, and therefore, the magnetic field, now begin to decrease. Since the field is decreasing, its potential energy also decreases; it is, in fact, returned to the circuit. This means that the coil is supplying power to the circuit, or in other words the power is negative. When the magnetic field has become zero again, the whole of its energy will have been returned to the circuit, and the total work done during the creation and destruction of the field will have been zero. Consequently, the *average* demand for power will have been zero.

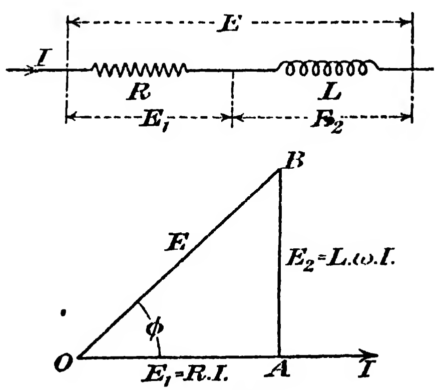


FIG. 167

10. Resistance and Inductance in Series.

Let E = total applied voltage

E_1 = drop along R, and E_2 = drop along L (Fig. 167).

Then $E_1 = IR$, in phase with I

and $E_2 = L\omega I$, leading I by 90°

Draw OI to represent the current I in phase and mark off $OA = E_1$. Then OA represents E_1 in magnitude and phase. Draw AB perpendicular to OI and equal to E_2. Then AB represents E_2 in magnitude and phase. Hence, OB, the vector sum of E_1 and E_2, is the applied voltage E. We therefore have

$$E = \sqrt{OA^2 + AB^2}$$

$$= I \times \sqrt{R^2 + (L\omega)^2}$$

or
$$I = \frac{E}{\sqrt{R^2 + (L\omega)^2}}$$

The quantity $\sqrt{R^2 + (L\omega)^2}$ is called the "Impedance," and it is measured in ohms. It is represented by the symbol Z.

Again, the vector OB represents the applied voltage in magnitude and phase, whereas OI represents the current in phase. Hence, the current lags behind the voltage by an angle

$$\varphi = \widehat{BOA} = \tan^{-1}\left(\frac{L\omega}{R}\right) \text{ or } \cos^{-1}\left(\frac{R}{Z}\right)$$

If each side of the triangle OAB is divided by I the new triangle so formed will have a base equal to R, perpendicular equal to $L\omega$, and hypotenuse equal to Z. We therefore have the relation

$$(\text{Impedance})^2 = (\text{Resistance})^2 + (\text{Reactance})^2$$

and this holds for any alternating current circuit.

Now put $e = E_{max} \sin \omega t$

$$\therefore\ i = I_{max} \sin (\omega t - \varphi)$$

$$\therefore\ w = e \times i = E_{max} I_{max} \sin \omega t \,.\, \sin (\omega t - \varphi)$$

$$\therefore\ W = E_{max} I_{max} \times \text{av of } \{\sin \omega t \,.\, \sin (\omega t - \varphi)\}$$

$$= E_{max} I_{max} \times \text{av. of } \tfrac{1}{2} \{\cos \varphi - \cos (2\omega t - \varphi)\}$$

$$= \frac{E_{max}}{\sqrt{2}} \times \frac{I_{max}}{\sqrt{2}} \cos \varphi$$

$$= EI \cos \varphi$$

The cosine of the angle of lag (or lead) of the current is called the "Power Factor," because it is the factor by which the apparent power EI must be multiplied in order to obtain the true power. The presence of the term $\cos (2\omega t - \varphi)$ indicates, as before, that the power has a periodic component of double frequency. In this

case the positive lobes of the power curve (Fig. 168) are greater than the negative lobes, because positive power has always to be drawn from the supply on account of the I^2R loss due to ohmic resistance.

It is useful to regard the above problem in another way. The current I lags behind E by the angle φ, as shown in Fig. 169. Resolve I into two components, one in phase with E and the other lagging 90°, or in " quadrature " with E. Then

Component in phase with E, OA $= I \cos \varphi$
Power contributed by it $= E \times OA = EI \cos \varphi$
Component in quadrature with E, $OB = I \sin \varphi$
Power contributed by it $= O$ (from par. 9)
$\therefore$ Total power $= EI \cos \varphi$

The component OA is called the working, or "wattful," component; while OB is called the idle, or " wattless," component.

Example. An arc lamp (which may be regarded as being non-inductive) takes 10 amp. at 50 volts. Calculate the impedance of a choker of 1 ohm resistance to be placed in series with it in order that it may be worked off a 200 volt 50 cycle supply. Find also the total power used and the power factor.

The circuit and vector diagram are given in Fig. 170. Resistance of arc $= \frac{50}{10} = 5$ ohms. Hence, total resistance $R = 6$. Total drop across this resistance $IR = 60$ volts in phase with I. The total drop along the whole circuit is 200, and the drop, E_L, along the inductance L is in quadrature with I.

Hence, $E_L = \sqrt{(200)^2 - (60)^2} = 190{\cdot}8$ volts.

In the vector diagram OA is the drop of 50 volts across the arc, AB the drop of 10 volts across the resistance in the choker, and BC the drop of 190 volts across the inductance of the choker. The vector sum OC is the total applied voltage of 200, and AC is the total drop across the choker

$$= \sqrt{190^2 + 10^2} = 190{\cdot}3 \text{ volts.}$$

The angle φ is the angle of lag of I behind E.

$$\therefore \text{ Power factor} = \cos \varphi = \frac{60}{200} = 0{\cdot}3.$$

Power in circuit $EI \cos \varphi = 200 \times 10 \times 0{\cdot}3 = 600$ watts

Or, $= I^2R$, where R is the total resistance

$= 10^2 \times 6 = 600$ watts

11. **Circuit Containing Capacity Only.**

Let $C =$ condenser capacity in farads.

If the voltage at the terminals has at any moment a value e, then the quantity of electricity in the condenser at that instant is

$$q = Ce \text{ coulombs.}$$

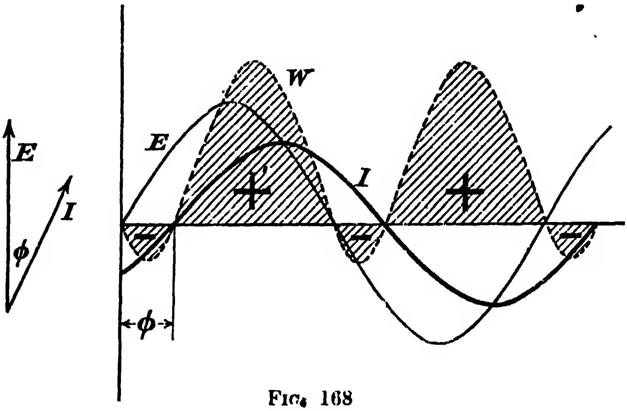
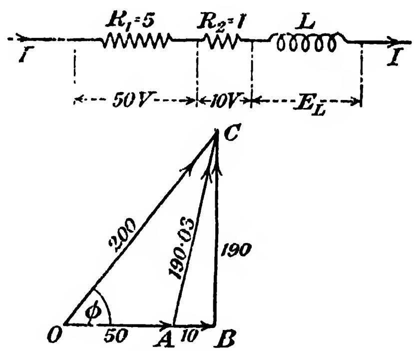


FIG. 168



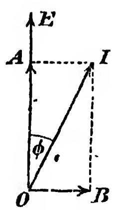


FIG 169

FIG. 170

Now quantity is the product of current and time, and if the current is varying, then

$$q = \int i \,.\, dt$$

or $$dq = i \,.\, dt$$

$$\therefore \quad i = \frac{dq}{dt} = C \,.\, \frac{de}{dt}$$

Put $$e = E_{max} \sin \omega t$$

$$\therefore \quad i = CE_{max}\, \omega \,.\, \cos \omega t$$

$$\therefore\ I_{max} = CE_{max}\,\omega$$

$$I = CE\,\omega \text{ or } \frac{E}{1/C\omega}$$

Hence, the reactance of a condenser is given by $1/C\omega$.

Again, since the voltage is represented by a sine function and the current by a cosine function, we see that the current *leads* the applied voltage by 90° (Fig. 171).

Again, if $e = E_{max} \sin \omega t$

we can write $i = I_{max} \sin\left(\omega t + \frac{\pi}{2}\right) = I_{max} \cos \omega t$

$$\therefore\ w = E_{max} I_{max} \sin \omega t \cos \omega t$$

$$= \tfrac{1}{2} E_{max} I_{max} \sin 2\omega t$$

$$\therefore\ W = \text{average of } w = 0.$$

The total power in this case also is zero, and as before, it is periodic with twice the supply frequency.

Since a condenser acts as an open circuit when connected in series with a direct current circuit, there is often difficulty in realizing how it is that the condenser can carry a current when placed in an alternating current circuit. When a continuous P.D. is applied to the plates of a condenser there is a momentary flow of electricity in the external circuit, which carries a sufficient quantity of electricity to the plates to make their P.D. equal to the applied P.D., after which all flow ceases. Similarly, if the plates of the charged condenser are joined by a wire, the quantity of electricity originally given to the condenser will flow in the opposite direction, thus producing a momentary current in the opposite direction, which ceases as soon as the plates are again at the same potential. This shows that by alternately charging and discharging a condenser it is possible for an alternating current in the condenser circuit to flow. But this current is confined to the external circuit and does not in any way flow through the dielectric, and, since the path of the current is broken at the plates, it follows that there must be alternate storing up and giving up of electrons by the plates.

The simple analogous hydraulic circuit of Fig. 172 will make this clear. The circuit consists of a tube in the form of a ring provided at one side with a piston, and at the opposite side with an expansion across which an elastic membrane is stretched. The whole tube is filled with water. If a steady pressure is applied to the piston in the direction indicated by the arrow there will be a displacement of water round the ring in a counter-clockwise direction, with consequent storing of water in the right-hand compartment and giving

up of water by the left-hand compartment. Equilibrium will be established, and all flow of water then cease, when the reaction of the stressed membrane is equal to the applied pressure. The establishment of the stress in the membrane is thus accompanied by a momentary displacement of water, in exactly the same way that the establishment of an electrostatic stress in the dielectric of a condenser is accompanied by a momentary flow, or displacement, of electricity. If the piston is released the membrane takes up its

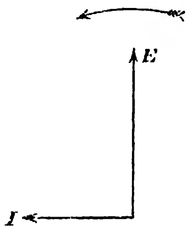


FIG. 171

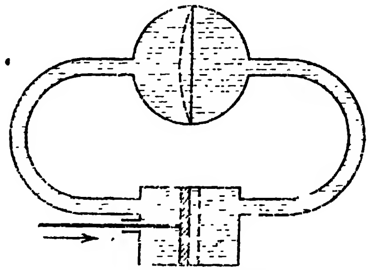


FIG. 172. HYDRAULIC ANALOGY

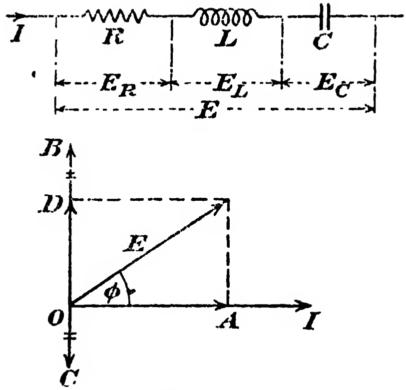


FIG. 173

unstressed position, accompanied this time by a displacement of water in the opposite direction. Similarly when a condenser is discharged there is a displacement of electricity in the reversed direction, this displacement ceasing when the P.D., and therefore the electrostatic stress, has been reduced to zero. This has been considered from a mathematical point of view in paragraph 18, Chapter IV.

Now, suppose that the piston is given a simple harmonic displacement, then there will be an alternating flow of water in the circuit, but it is obvious that this flow will be bounded by the elastic membrane, the current in no sense flowing through the membrane. In spite of this the flow of water in the rest of the circuit is exactly

the same as though the water did flow through the membrane, and similarly in the circuit of a condenser to which an alternating P.D. is applied. From this point of view we thus see that it is justifiable to speak of the current as the condenser current.

12: **General Series Circuit.** Fig. 173 represents a circuit with resistance, inductance, and capacity all in series. We have

$$E_R = RI \text{ in phase with } I$$

$$E_L = L\omega I \text{ in quadrature (leading) with } I$$

$$E_C = \frac{I}{C\omega} \text{ in quadrature (lagging) with } I$$

Draw OI to represent the current in phase, and represent the above voltage drops by OA, OB, and OC respectively. The resultant of OB and OC is OD, where

$$OD = L\omega I - \frac{I}{C\omega}$$

Hence, for the total applied voltage we have

$$E = \sqrt{OA^2 + OD^2} = I\sqrt{R^2 + \left(L\omega - \frac{1}{C\omega}\right)^2}$$

$$I = \frac{E}{\sqrt{R^2 + \left(L\omega - \frac{1}{C\omega}\right)^2}}$$

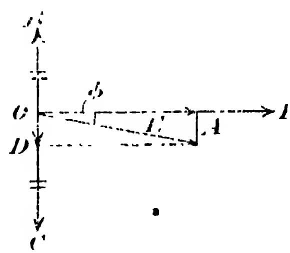


FIG. 174

The total impedance is now

$$Z = \sqrt{R^2 + \left(L\omega - \frac{1}{C\omega}\right)^2}$$

and the reactance

$$X = \left(L\omega - \frac{1}{C\omega}\right)$$

The condenser reactance is reckoned negative, and therefore, if it is greater than the inductive reactance $L\omega$, the total reactance will be negative, as in Fig. 174. If the total reactance is positive the current lags behind the applied voltage; if it is negative, the current leads.

For the angle of lag we have

$$\tan\varphi = \frac{OD}{OA} = \frac{I\left(L\omega - \frac{1}{C\omega}\right)}{RI}$$

$$\therefore \varphi = \tan^{-1}\frac{L\omega - \frac{1}{C\omega}}{R} = \tan^{-1}\frac{\text{reactance}}{\text{resistance}}$$

$$\text{Power factor} = \cos\varphi = \frac{R}{\sqrt{R^2 + \left(L\omega - \frac{1}{C\omega}\right)^2}} = \frac{\text{resistance}}{\text{impedance}}$$

13. Electrical Resonance. If the inductive and capacity reactances, $L\omega$ and $1/C\omega$, are equal to one another in the general series circuit, then the total reactance is zero, and the current is given by the Ohm's Law value E/R, and is in phase with E. In order to understand what is taking place in the circuit when this condition is fulfilled, imagine that the resistance R is zero, that a constant voltage E is applied, and that the frequency is varied from zero to a very high value.

Then the inductive reactance $X_1 = L\omega = 2\pi Lf$, and is represented graphically by a straight line through the origin (Fig. 175). The capacity reactance

$$X_2 = -\frac{1}{C\omega} = -\frac{1}{2\pi Cf}$$

It is therefore represented by a rectangular hyperbola in the fourth quadrant. The total reactance $X = X_1 + X_2$, and the graph of X is also a hyperbola which crosses the frequency axis at some point A.

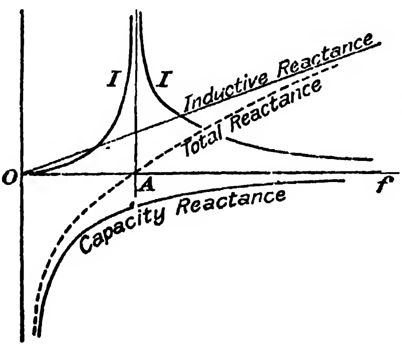


FIG. 175

At this point the total reactance X is zero, and since the resistance is zero, the total impedance is zero. The current I is therefore theoretically infinite at this point, as shown. For frequencies greater than OA, the inductive reactance predominates and the current lags. For frequencies less than OA, the capacity reactance predominates, and therefore, the current leads.

The condition for zero reactance is

$$L\omega = \frac{1}{C\omega}$$

$$\omega = \sqrt{\frac{1}{LC}} \quad \text{or } f = \frac{1}{2\pi}\sqrt{\frac{1}{LC}}$$

This condition is said to be the condition for "electrical resonance," for the following reason. When a condenser is discharged

through an inductive external circuit of small resistance, the discharge is oscillatory, the frequency of the oscillations being

$$f = \frac{1}{2\pi}\sqrt{\frac{1}{LC}}$$

When the condenser is in an alternating current circuit it is periodically charged and discharged, and if the frequency of the applied voltage is such that the total reactance is zero, then it coincides with the natural frequency of electrical oscillation of the circuit. In an actual circuit, the resistance is, of course, never zero, and there are also iron losses taking place in the choker, and dielectric losses in the condenser. The current therefore cannot become infinite, but when R is small it may reach a very high value. The voltages across the choker and condenser which then occur, namely,

$$E_2 = L\omega I, \text{ and } E_3 = \frac{I}{C\omega}$$

may be many times greater than the applied voltage. There will thus be the danger of break-down of the apparatus.

It will be seen that resonance is the result of the coincidence of the applied frequency with the natural frequency of the circuit. It is useful to consider a mechanical analogy. Consider a mass M suspended by a spring S. If M is pulled downwards and released, the system will have a definite frequency of oscillation. If M is pushed downwards by a series of timed impulses, then oscillation will take place, but no large amplitude will be set up if these impulses are not so timed as to occur each instant that the mass M is passing its equilibrium position in a downward direction. If the impulses are so timed, then a very large amplitude of oscillation will be set up, the amplitude being much greater than the displacement obtained if a steady downward force were applied.

Example. A choking coil of resistance 5 ohms and inductance 0·6 henry is in series with a capacity of 10 micro-farads. If a voltage of 200 is applied and the frequency is adjusted to resonance, find the current, and the voltages across the inductance and condenser.

$$\text{Frequency for resonance } f = \frac{1}{2\pi}\sqrt{\frac{1}{LC}}$$

$$= \frac{1}{2\pi}\sqrt{\frac{10^6}{0{\cdot}6 \times 10}}$$

$$= 65$$

Current at resonance $I = \dfrac{E}{R} = \dfrac{200}{5} = 40$ amp. in phase with E.

Voltage across condenser $\dfrac{I}{C\omega} = \dfrac{40 \times 10^6}{10 \times 2\pi \times 65}$

$= 9{,}800$ volts.

Voltage across choker $= I\sqrt{R^2 + (L\omega)^2}$

$= 40 \times \sqrt{25 + (\cdot 6 \times 2\pi \times 65)^2}$

$= 9{,}800$ volts.

14. **Circuits in Parallel.** In a direct current circuit we have for the total resistance of a combination of resistances, R_1, R_2, R_3, etc.

$$R = R_1 + R_2 + R_3 + \ldots \text{ if they are in series}$$

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \ldots \text{ if they are in parallel}$$

or $\quad G = G_1 + G_2 + G_3 + \ldots$ where G is the conductance, the reciprocal of the resistance.

For a combination of impedances in an alternating current circuit, we have

$$Z = Z_1 + Z_2 + Z_3 + \ldots \text{ (vector sum) if they are in series}$$

$$\frac{1}{Z} = \frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3} + \ldots \text{ (vector sum) if they are in parallel}$$

The reciprocal of the impedance is called the "Admittance," Y, so that

$$Y = Y_1 + Y_2 + Y_3 + \ldots \text{ (vector sum)}$$

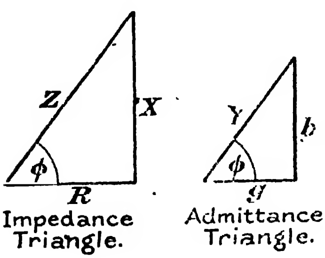


FIG. 176

Just as the impedance has two components, resistance R and reactance X, so also has the admittance, namely, the "conductance," g, and "susceptance," b. The impedance and admittance triangles (Fig. 176) are similar.

$$\therefore b/g = \tan\varphi = X/R$$

also $\quad g^2 + b^2 = Y^2 = \dfrac{1}{Z^2}$

$$= \frac{1}{X^2 + R^2}$$

The solution of these equations for g and b is

$$g = \frac{R}{Z^2} \text{ mhos}$$

$$b = \frac{X}{Z^2} \text{ mhos}$$

If the admittances of a number of circuits in parallel are represented on a vector diagram, as in Fig. 177, then

X component of total admittance

$$Y \cos \varphi = Y_1 \cos \varphi_1 + Y_2 \cos \varphi_2 + Y_3 \cos \varphi_3 + \ldots$$

$$\therefore g = g_1 + g_2 + g_3 + \ldots$$

Similarly, taking Y components,

$$b = b_1 + b_2 + b_3 + \ldots$$

Total admittance

$$Y = \sqrt{g^2 + b^2}$$

Total current

$$I = E \times Y$$

where E is the voltage applied to the branched network, and φ the phase angle of the total current with respect to E

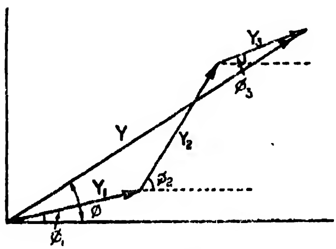


FIG. 177

$$\varphi = \tan^{-1}\frac{b}{g}$$

lagging if b is positive, and leading if b is negative

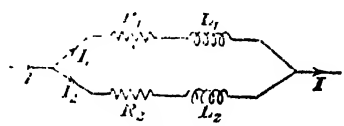


FIG. 178

Example. Two coils, one of resistance 2 ohms and self-induction 0·015 henry, the other having a resistance of 1 ohm and a self-induction of 0·08 henry, are arranged in parallel on a 100 volt, 50 frequency circuit. Find the magnitudes and phases of the currents flowing in each circuit, and of the resultant current flowing through the whole system. (C. and G.)

This problem can be solved by considering each branch separately, calculating the current in each, and then determining their vector sum. In order to illustrate the method, the admittance method is used in the following calculation. Fig. 178 shows the circuit.

$\omega = 2\pi f$; $R_1 = 2$; $L_1 = 0{\cdot}015$; reactance $X_1 = L_1\omega = 4{\cdot}7$ ohms.

Impedance $Z_1 = \sqrt{2^2 + 4{\cdot}7^2} = 5{\cdot}1$; current $I_1 = \dfrac{100}{5{\cdot}1} = 19{\cdot}6$ amp.

Power factor of branch 1 $= R_1/Z_1 = 2/5{\cdot}1 = 0{\cdot}39$.

Conductance $g_1 = \dfrac{R_1}{Z_1^2} = 0{\cdot}0765$ mho; susceptance $b_1 = \dfrac{X^1}{Z_1^2}$

$= 0{\cdot}177$ mho.

Again,

$R_2 = 1$; $L_2 = 0{\cdot}08$; $X_2 = L_2\omega = 25$ ohms; $Z_2 = \sqrt{25^2 + 1^2} = 25$, very approximately.

$\therefore I_2 = \frac{100}{25} = 4$ amp. ; power factor of branch 2 $= \frac{R_2}{Z_2} = \frac{1}{25} = 0{\cdot}04$

Conductance $g_2 = \frac{R_2}{Z_2^2} = 0{\cdot}0016$ mho ; susceptance $b_2 = \frac{X_2}{Z_2^2}$
$= 0{\cdot}04$ mho.

Total conductance $g = g_1 + g_2 = 0{\cdot}078$ mho.

Total susceptance $b = b_1 + b_2 = 0{\cdot}217$ mho (both susceptances are positive in this case).

Total admittance $Y = \sqrt{g^2 + b^2} = 0{\cdot}232$

Total current $I = 100 \times 0{\cdot}232 = 23{\cdot}2$ amp.

Power factor of whole circuit $= \frac{g}{Y} = 0{\cdot}34.$

15. **Graphical Solution of a Branched Circuit.** Since the impedance triangle is right-angled, the locus of its apex is a circle on the base Z as diameter. Draw a base AB (Fig. 179) to represent to scale the applied voltage E, and on it describe a semicircle. Draw a chord AC inclined at an angle φ_1 to AB, where $\varphi_1 = \tan^{-1} X_1/R_1$. Join CB. Then AC and CB measured on the voltage scale give the resistance and reactance drops respectively in branch 1.

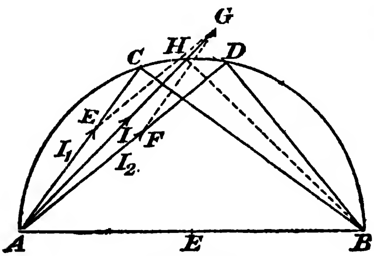


FIG. 179

Now the current in any branch is equal to the resistance drop divided by the resistance, and it is also in phase with this drop. Hence, to obtain I_1, divide AC (in volts) by R_1, and mark off AE to represent I_1 thus found.

Similarly obtain AF, the current I_2 in the second branch, the triangle ADB being the voltage drop triangle for this branch. Then AG, the resultant of AE and AF, gives the total current I. If there are more than two branches, then the individual branch currents are all found separately as above, the resultant of all of them being the total current.

Now triangle AHB is the drop of volts triangle for the whole branched circuit, so that the side AH in phase with I is the resistance drop, and HB, the reactance drop. Hence, if AH and HB are each measured on the voltage scale and divided by I, the equivalent single resistance R and reactance X are obtained.

16. **Resonance in a Branched Circuit.** Consider a circuit consisting of R, L, and C in parallel, as shown in Fig. 180.

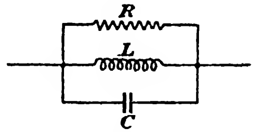


FIG. 180

Impedance of branch 1, $Z_1 = R$

Conductance of branch 1, $g_1 = \frac{R}{Z_1^2} = \frac{1}{R}$

Susceptance of branch 1, $b_1 = 0$

Impedance of branch 2, $Z_2 = L\omega$

Conductance of branch 2, $g_2 = 0$

Susceptance of branch 2, $b_2 = \frac{L\omega}{Z_2^2} = \frac{1}{L\omega}$

Impedance of branch 3, $Z_3 = \frac{1}{C\omega}$

Conductance of branch 3, $g_3 = 0$

Susceptance of branch 3, $b_3 = -\frac{\frac{1}{C\omega}}{Z_3^2} = -C\omega$

Total conductance $g = g_1 + g_2 + g_3$

$= \frac{1}{R}$

Total susceptance $b = b_1 + b_2 + b_3$ (algebraic sum)

$= \frac{1}{L\omega} - C\omega$

$\therefore$ Total admittance $Y = \sqrt{g^2 + b^2} = \sqrt{\frac{1}{R^2} + \left(\frac{1}{L\omega} - C\omega\right)^2}$ mhos.

$\therefore$ Total current $I = EY.$

Phase angle of total current with respect to applied voltage

$$\varphi = \tan^{-1}\frac{b}{g} = \tan^{-1} R\left(\frac{1}{L\omega} - C\omega\right)$$

From these equations we see that the total current is a minimum when $1/L\omega$ is equal to $C\omega$, and the current is then in phase with the applied voltage.

If the circuit contains L and C only in parallel, but no resistance branch, then total current

$$I = E\left(\frac{1}{L\omega} - C\omega\right)$$

so that, if $1/L\omega$ is equal to $C\omega$, I will be zero. But the branch currents will be finite, namely $E/L\omega$ and $EC\omega$ respectively. This apparent anomaly is explained by the fact that the two branch currents are exactly equal and are opposite in phase, their vector sum, which gives the total current, being therefore zero.

This phenomenon is called "current resonance" to distinguish it from the voltage resonance which takes place in series circuits. The condition for current resonance is the equality of inductive and capacity susceptances, namely,

$$1/L\omega = C\omega \text{ or } \omega = \sqrt{\frac{1}{LC}}$$

This is also the condition for voltage resonance in a series circuit. Current resonance does not produce a dangerous rise in pressure.

A useful hydraulic analogy of current resonance is shown in Fig. 181. A branched pipe has an elastic membrane across one branch and a column of mercury in the other. The elastic membrane possesses elasticity only, but no inertia,

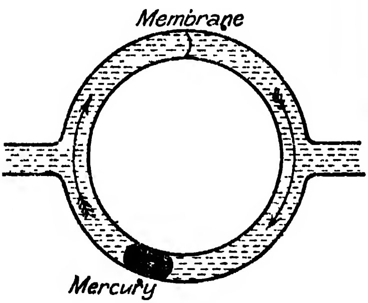


FIG. 181

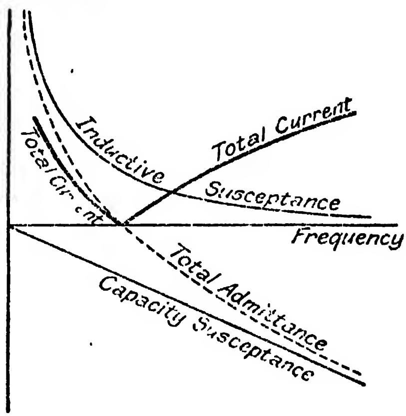


FIG. 182

and it therefore corresponds to the condenser branch. The mercury possesses inertia only and corresponds to the inductive branch. It is possible for an alternating current of water to exist in the circular tube as a whole since this closed system will have a natural period of oscillation of its own. This alternating current can exist when the water in the main pipe is stationary, a condition which corresponds with the branched circuit when current resonance takes place.

In Fig. 182 the characteristics of the circuit containing L and C in parallel are plotted against frequency, the applied voltage being assumed constant.

Inductive susceptance $= \dfrac{1}{L\omega} = \dfrac{1}{2\pi Lf}$, and is represented by a rectangular hyperbola in the first quadrant.

Capacity susceptance $= -C\omega = -2\pi Cf$, and is represented by a straight line through the origin.

The total susceptance is the sum of these two, and the curve is a hyperbola which crosses the frequency axis. The total admittance is equal to the susceptance since there is zero conductance. We see that for one particular frequency the admittance is zero, and therefore, the total current at this frequency is zero.

Since the applied voltage is constant, the curves of inductive susceptance, capacity susceptance, and admittance represent to scale the inductive, capacity, and total currents respectively.

For frequencies below that which makes the admittance zero, the inductive susceptance predominates and the total current is lagging. For greater frequencies the capacity susceptance predominates, so that the total current leads the applied voltage.

17. **Circuit with Mutual Inductance.** Since the magnetic field set up by any appliance may extend throughout a considerable space, it is inevitable that there must be many cases where the flux produced by one part of a circuit links with the windings in another part of the circuit. Thus changes in current in one part will be associated with induced E.M.F.s in the other part. If M is the coefficient of mutual induction in henrys and I_1 is the current in coil 1, then the mutually induced E.M.F. in coil 2 is given by

$$e_2 = -M\frac{di_1}{dt}$$

As a simple numerical example, take the following: a coil carrying a current of 5 amp. (R.M.S.) at frequency 50 is adjacent to a second coil of resistance 20 ohms, and inductance 1 henry. If the mutual inductance is 0·5 henry, calculate the current in the second coil if its circuit is closed.

$$i_1 = (I_1)_{max}\sin\omega t$$

$$\therefore\ e_2 = -\frac{d}{dt}\left\{(I_1)_{max}\sin\omega t\right\}\times M$$

$$= (I_1)_{max}\,\omega\cos\omega t\times M$$

$$\therefore\ (E_2)_{max} = M\omega(I_1)_{max}$$

$$\therefore\ E_2 = M\omega I_1$$

$$= \cdot 5\times 2\pi\times 50\times 5$$

$$= 78\cdot 5 \text{ volts}$$

$$\therefore\ I_2 = \frac{E_2}{\sqrt{R_2^2 + X_2^2}}$$

$$= \frac{78\cdot 5}{\sqrt{20^2 + (1\times 314)^2}}$$

$$= 0\cdot 25 \text{ amp.}$$

The following additional example is taken from a London University examination paper.

Example. Two coils, with terminals $T_1 T_2$ and $T_3 T_4$ respectively, are placed side by side. Measured separately, the inductance of the first coil is 1,200 microhenrys, and that of the second coil is 800 microhenrys. When T_2 is joined to T_3 the inductance between T_1 and T_4 is 2,500 microhenrys. What is the mutual inductance between the two coils, and what would be the inductance between T_1 and T_3 with T_2 joined to T_4?

Let L_1 and L_2 be the two self-inductances and M the mutual inductance, Fig. 183. Then, when the combination carries a current i, we have

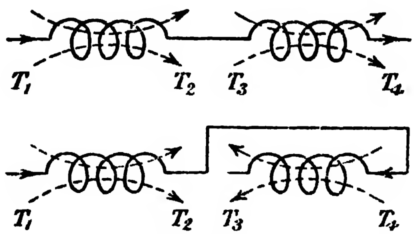


FIG. 183

Self induced E.M.F. in coil 1 $\quad = -L_1 \dfrac{di}{dt}$

Mutually induced E.M.F. in coil 1 due to change of current in coil 2 $\quad = +M\dfrac{di}{dt}$

Self induced E.M.F. in coil 2 $\quad = -L_2 \dfrac{di}{dt}$

Mutually induced E.M.F. in coil 2 due to change of current in coil 1 $\quad = +M\dfrac{di}{dt}$

The $-$ sign is given to $M\dfrac{di}{dt}$ when self and mutual fluxes are in the same direction, and the $+$ sign when they are in opposition.

$\therefore$ Total induced E.M.F. in the circuit

$$= -(L_1 \pm 2M + L_2)\frac{di}{dt}$$

$\therefore$ Equivalent self-induction of the whole circuit

$$= L_1 \pm 2M + L_2$$

the $+$ sign now being taken when self and mutual fluxes are in the same direction, and vice versa.

Since the total of 2,500 > (1,200 + 800) we have

$$1,200 + 2M + 800 = 2,500$$

$$\therefore\ M = 250 \text{ microhenrys.}$$

Hence, when the connections of coil 2 are reversed, total equivalent self-induction

$$= 1,200 - 2 \times 250 + 800$$

$$= 1,500 \text{ microhenrys.}$$

18. **The Symbolic Method.** When a number of forces acting at a point have to be combined into a single resultant, one method of determining this resultant is to resolve all the forces into two directions at right angles. The resultant is then the square root of the sum of the squares of the total OX and OY components. We have already seen that alternating current problems can be solved by this manner. Thus impedance has two components, resistance and reactance, and if there are a number of impedances in series the total impedance is given by

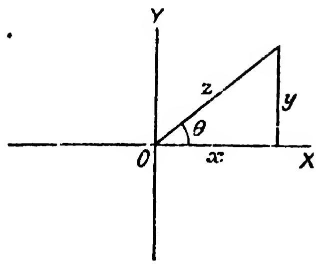


FIG. 184

$$Z = [\{\Sigma(R)\}^2 + \{\Sigma(X)\}^2]^{\frac{1}{2}}$$

Similarly, if there are a number of admittances in parallel the total admittance is given by

$$Y = [\{\Sigma(g)\}^2 + \{\Sigma(b)\}^2]^{\frac{1}{2}}$$

A more generally applicable method is to express a vector in terms of two components at right angles in the following manner. In Fig. 184, a vector of length z is resolved into two components x and y at right angles. In the symbolic notation the length of the vector is expressed in the form

$$z = x + jy$$

where $$j = \sqrt{-1}.$$

The quantity j, although an imaginary quantity, is here used to indicate a real operation, namely, that of rotating a vector in a counter-clockwise direction through 90°, and the appearance of the operator j in front of the component y means that this component is leading the x component by 90°. The operator j^2 in front of a vector, e.g. j^2a, means that the a vector is 90° ahead of y, and therefore 180° ahead of x. Similarly if the operator is j^3, e.g. j^3b, the vector b will be 270° ahead of x. If the operator is j^4 the vector operated on will be in phase with x.

Hence we have, for any vector z,

$$j^2z = -z$$
$$j^3z = -jz$$
$$j^4z = z$$

Again, from the diagram

$$x = z\cos\theta, \text{ and } y = z\sin\theta$$

$$\therefore \qquad \dot{z} = z\cos\theta + jz\sin\theta$$

This is commonly written in an abbreviated form

$$\dot{z} = z\,\underline{/\theta}$$

It can also be written in the form

$$\dot{z} = z\varepsilon^{j\theta}$$

the proof being as follows

$$\varepsilon^{j\theta} = 1 + j\theta + \frac{(j\theta)^2}{1\,.\,2} + \frac{(j\theta)^3}{1\,.\,2\,.\,3} + \cdots \qquad (1)$$

$$= 1 + j\theta - \frac{\theta^2}{1\,.\,2} - \frac{j\theta^3}{1\,.\,2\,.\,3} + \cdots$$

and

$$\varepsilon^{-j\theta} = 1 - j\theta - \frac{\theta^2}{1\,.\,2} + \frac{j\theta^3}{1\,.\,2\,.\,3} + \cdots \qquad (2)$$

Adding (1) and (2) and dividing by 2

$$\frac{\varepsilon^{j\theta} + \varepsilon^{-j\theta}}{2} = 1 - \frac{\theta^2}{1\,.\,2} + \frac{\theta^4}{1\,.\,2\,.\,3\,.\,4} - \cdots$$

$$= \cos\theta \qquad (3)$$

Subtracting (1) and (2) and dividing by $2j$

$$\frac{\varepsilon^{j\theta} - \varepsilon^{-j\theta}}{2j} = \theta - \frac{\theta^3}{1\,.\,2\,.\,3} + \frac{\theta^5}{1\,.\,2\,.\,3\,.\,4\,.\,5} - \cdots$$

$$= \sin\theta \qquad (4)$$

$$\therefore \qquad \frac{\varepsilon^{j\theta} - \varepsilon^{-j\theta}}{2} = j\sin\theta \qquad (5)$$

Adding (3) and (5) we have

$$\varepsilon^{j\theta} = \cos\theta + j\sin\theta$$

whence

$$\dot{z} = z\,/\theta = z\cos\theta + jz\sin\theta = z\,.\,\varepsilon^{j\theta}$$

In the case of an impedance Z made up of ohmic resistance R in series with reactance X, we have

$$\dot{Z} = R + jX = Z\varepsilon^{j\theta}$$
$$= Z \underline{/\theta}$$

where $\theta = \text{arc tan } \frac{X}{R}$

The admittance can be expressed symbolically in the following manner—

$$Y = \frac{1}{Z}$$

$$\therefore \quad \dot{Y} = \frac{1}{\dot{Z}} = \frac{1}{R + jX}$$
$$= \frac{R - jX}{(R + jX)(R - jX)}$$
$$= \frac{R}{R^2 + X^2} - j \cdot \frac{X}{R^2 + X^2}$$
$$= g - j \cdot b$$

$$\therefore \quad \dot{Y} = g - jb = Y\varepsilon^{-j\theta} = Y \underline{/-\theta}, \text{ or } Y \setminus\theta$$

We will now consider several applications of the method to ordinary alternating current calculations.

(*a*) The Resultant of a Number of Alternating E.M.F.s all Acting at the Same Time in a Given Circuit. Suppose there are two E.M.F.s represented by

$$e_1 = E_1 \sin \omega t, \text{ and } e_2 = E_2 \sin (\omega t + \varphi)$$

Then taking E_1 as the reference vector, we have

$$\dot{E}_1 = E_1 + j \cdot 0$$

and $\dot{E}_2 = E_2 \cos \varphi + j \cdot E_2 \sin \varphi$

$\therefore$ $\dot{E} = \dot{E}_1 + \dot{E}_2 = (E_1 + E_2 \cos \varphi) + j \cdot E_2 \sin \varphi$

Writing $\dot{E} = \overline{X} + j \cdot \overline{Y}$, we have

$\overline{X} = (E_1 + E_2 \cos \varphi)$

and $\overline{Y} = E_2 \sin \varphi$, thus giving

$$E = \{(E_1 + E_2 \cos \varphi)^2 + E_2^2 \sin^2 \varphi\}^{\frac{1}{2}}$$

and $$\tan \theta = \frac{\overline{Y}}{\overline{X}} = \frac{E_2 \sin \varphi}{E_1 + E_2 \cos \varphi}$$

(*b*) Simple Series Circuit. Fig. 185 shows a circuit consisting of two inductive resistances in series with one another, the resistances

and reactances 5 and 6, 3 and 7 respectively. A sinusoidal P.D. of effective value 200 is applied to the combination. Calculate the current and power factor, and the voltage drops along the two coils.

The expression for impedance in symbolic notation is

$$Z = R + jX$$

$$\therefore \quad Z = (5 + 3) + j(6 + 7) = 8 + j13$$

Put $\quad E = 200 + j \,.\, 0$

$$I = \frac{200}{8 + j13}$$

$$\frac{200\,(8 - j13)}{(8 + j13)\,(8 - j13)} \qquad \frac{200\,(8 - j13)}{64 + 169} = 6{\cdot}88 - j11{\cdot}15$$

$$I = \sqrt{(6{\cdot}88)^2 + (11{\cdot}15)^2} = 13{\cdot}2 \text{ amp.}$$

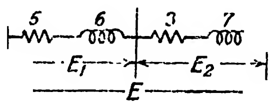


FIG. 185

This is the effective value since we used the effective value of 200 for the applied P.D. Again, the minus sign in the expression for I shows that the current lags behind E, the angle, φ, being

$$\varphi = \text{arc tan}\,\frac{11{\cdot}15}{6{\cdot}88} = 58^\circ\ 16'$$

and $\quad \text{P.F.} = \cos\varphi = {\cdot}524$

Again, for the voltage drops, we have

$$E_1 = IZ_1 = (6{\cdot}88 - j\,.\,11{\cdot}15)\,(5 + j\,.\,6)$$
$$= (101{\cdot}3 - j\,.\,14{\cdot}47)$$

$$\therefore \quad E_1 = \sqrt{(101{\cdot}3)^2 + (14{\cdot}47)^2} = 102{\cdot}5 \text{ volts}$$

its phase angle with respect to E being

$$\varphi_1 = \text{arc tan}\,\frac{-\,14{\cdot}47}{101{\cdot}3} = \text{arc tan} - {\cdot}1429 = -\,8^\circ\ 13'$$

Similarly

$$E_2 = IZ_2 = (6{\cdot}88 - j\,.\,11{\cdot}15)\,(3 + j\,.\,7)$$
$$= 98{\cdot}7 + j\,.\,14{\cdot}5$$

$$\therefore \quad E_2 = \sqrt{(98{\cdot}7)^2 + (14{\cdot}5)^2} = 99{\cdot}7 \text{ volts}$$

its phase angle with respect to E being

$$\varphi_2 = \text{arc tan}\,\frac{14{\cdot}5}{98{\cdot}7} = \text{arc tan}\ {\cdot}1471 = +\,8^\circ\ 14'$$

We have also as a check

$$\dot{E} = \dot{E}_1 + \dot{E}_2$$
$$= (101{\cdot}3 - j\,.\,14{\cdot}47) + (98{\cdot}7 + j\,.\,14{\cdot}5)$$
$$= 200 + j \cdot 0$$

(*c*) A DIVIDED CIRCUIT. When making calculations on a divided circuit it is often convenient to reduce the circuit to the equivalent series circuit containing resistance and reactance. Thus let the symbolic impedances of the individual branches be $(R_1 + jX_1)$,

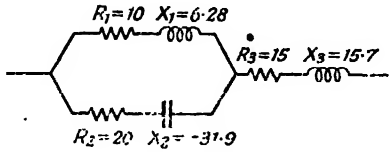


FIG. 186

$(R_2 + jX_2)$, $(R_3 + jX_3)$, etc., and the impedance of the whole branched circuit be $(R + jX)$. Then, since

$$\dot{I} = \dot{I}_1 + \dot{I}_2 + \dot{I}_3 + \ldots$$

$$\frac{\dot{E}}{R+jX} = \frac{\dot{E}}{R_1+jX_1} + \frac{\dot{E}}{R_2+jX_2} + \frac{\dot{E}}{R_3+jX_3} + \ldots$$

Dividing throughout by $\dot{E}$ and rationalizing, we have

$$\frac{R-jX}{R^2+X^2} = \frac{R_1-jX_1}{R_1^2+X_1^2} + \frac{R_2-jX_2}{R_2^2+X_2^2} + \frac{R_3-jX_3}{R_3^2+X_3^2} + \ldots$$

Equating real and imaginary parts, we have

$$\frac{R}{R^2+X^2} = \frac{R_1}{R_1^2+X_1^2} + \frac{R_2}{R_2^2+X_2^2} + \frac{R_3}{R_3^2+X_3^2} + \ldots$$

and

$$\frac{X}{R^2+X^2} = \frac{X_1}{R_1^2+X_1^2} + \frac{X_2}{R_2^2+X_2^2} + \frac{X_3}{R_3^2+X_3^2} + \ldots$$

In other words

$$g = g_1 + g_2 + g_3 + \ldots$$

and $$b = b_1 + b_2 + b_3 + \ldots$$

the results thus being identical with those obtained by the admittance method.

As a numerical example consider the following circuit. A resistance of 10 ohms is in series with an inductive reactance of 6·28, and the combination is parallel with a circuit consisting of a resistance of 20 ohms in series with a capacity of 100 m.f. The above branched circuit is in series with a resistance of 15 ohms and an inductive reactance 15·7 ohms, Fig. 186. Calculate the equivalent resistance, reactance, and impedance of the whole circuit.

The reactance of the condenser, assuming a 50-cycle supply, is

$$x_2 = -\frac{1}{314C} = -\frac{10^6}{314 \times 100} = -31{\cdot}9 \text{ ohms}$$

The circuit is, therefore, as shown in Fig. 181.

For branch 1

$$Y_1 = \frac{10}{10^2 + 6{\cdot}28^2} - j \cdot \frac{6{\cdot}28}{10^2 + 6{\cdot}28^2} = {\cdot}0718 - j \times {\cdot}045$$

For branch 2

$$Y_2 = \frac{20}{20^2 + 31{\cdot}9^2} - j \cdot \frac{-31{\cdot}9}{20^2 + 31{\cdot}9^2} = {\cdot}0141 + j \times {\cdot}0225$$

For the branched portion, we thus have

$$\begin{aligned} Y_{1\cdot2} &= ({\cdot}0718 + {\cdot}0141) - j({\cdot}045 - {\cdot}0225) \\ &= {\cdot}0859 - j \times {\cdot}0225 \end{aligned}$$

$$\therefore \quad R_{1\cdot2} = \frac{{\cdot}0859}{{\cdot}0859^2 + {\cdot}0225^2} = 10{\cdot}87$$

and $\quad X_{1\cdot2} = \dfrac{{\cdot}0225}{{\cdot}0859^2 + {\cdot}0225^2} = 2{\cdot}86$ and is positive

Hence, for the whole circuit, we have

$$\begin{aligned} R &= R_{1\cdot2} + R_3 = 10{\cdot}87 + 15 &&= 25{\cdot}87 \text{ ohms} \\ X &= X_{1\cdot2} + X_3 = 2{\cdot}86 + 15{\cdot}7 &&= 18{\cdot}56 \quad \text{,,} \\ Z &= (25{\cdot}87^2 + 18{\cdot}56^2)^{\frac{1}{2}} &&= 31{\cdot}8 \quad \text{,,} \end{aligned}$$

In order to determine the manner in which a current divides between two parallel branches, we have the two equations

and
$$\left.\begin{aligned} I &= I_1 + I_2 \\ \frac{I_1}{I_2} &= \frac{Z_2}{Z_1} \end{aligned}\right\}$$

These give

$$\begin{aligned} \frac{I_1}{I} &= \frac{Z_2}{Z_1 + Z_2} \\ &= \frac{R_2 + jX_2}{(R_1 + R_2) + j(X_1 + X_2)} \end{aligned}$$

$$\therefore \qquad \frac{I_1}{I} = \left\{\frac{R_2^2 + X_2^2}{(R_1 + R_2)^2 + (X_1 + X_2)^2}\right\}^{\frac{1}{2}}$$

and similarly

$$\frac{I_2}{I} = \left\{\frac{R_1^2 + X_2^2}{(R_1 + R_2)^2 + (X_1 + X_2)^2}\right\}^{\frac{1}{2}}$$

(*d*) Alternating Current Bridge Networks. Consider first of

all a simple network such as that shown in Fig. 187. The condition for a balance with direct current is

$$\frac{R_1}{R_2} = \frac{R_3}{R_4}$$

The condition for a balance with alternating current is

$$\frac{Z_1}{Z_2} = \frac{Z_3}{Z_4}$$

$$\therefore \qquad \frac{R_1 + jX_1}{R_2 + jX_2} = \frac{R_3 + j\,.\,0}{R_4 + j\,.\,0}$$

$$\therefore \qquad R_1R_4 + jX_1R_4 = R_2R_3 + jX_2R_3$$

Equating real and imaginary parts, we have

$$\frac{R_1}{R_2} = \frac{R_3}{R_4}$$

as for direct current

and $$\frac{X_1}{X_2} = \frac{R_3}{R_4}$$

Hence $$\frac{R_1}{R_2} = \frac{R_3}{R_4} = \frac{X_1}{X_2}$$

As a more complicated bridge network take the case of the Wien bridge shown in Fig. 188*. This consists of two non-inductive arms

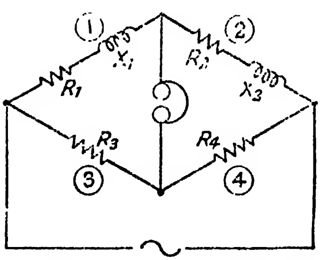


FIG. 187. SIMPLE A.C. BRIDGE NETWORK

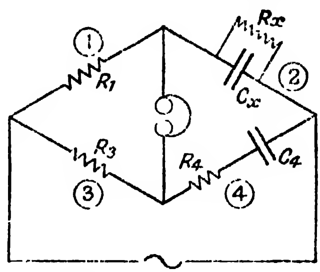


FIG. 188. WIEN A.C. BRIDGE

R_1 and R_3, a shunted condenser for arm 2, and a condenser in series, with a non-inductive resistance for arm 4. First of all reduce arm 2 to an equivalent series circuit of resistance R_2 and reactance x_2; we then have

$$R_2 = \frac{R_x}{1 + \omega^2 C_x^2 R_x^2} \text{ and } x_2 = -\frac{1}{\omega C_x\left(1 + \dfrac{1}{\omega^2 C_x^2 R_x^2}\right)}$$

and for arm 4 $$x_4 = -\frac{1}{\omega C_4}$$

* See Hague, *A.C. Bridge Measurements* (Pitman).

The condition for a balance is thus

$$\frac{R_1}{R_3} = \frac{\dfrac{R_x}{1 + \omega^2 C_x^2 R_x^2} + j \cdot \dfrac{1}{\omega C_x\left(1 + \dfrac{1}{\omega^2 C_x^2 R_x^2}\right)}}{R_4 + j \cdot \dfrac{1}{\omega C_4}}$$

Equating real and imaginary parts, as before, this gives

$$\left.\begin{aligned} R_1 R_4 &= \frac{R_3 R_x}{1 + \omega^2 C_x^2 R_x^2} \\ \text{and} \qquad \frac{R_1}{\omega C_4} &= \frac{\omega C_x R_3 R_x^2}{1 + \omega^2 C_x^2 R_x^2} \end{aligned}\right\}$$

This particular network is used for the determination of R_x, without the necessity for the determination of φ_x, hence if φ_x is eliminated in the usual manner, we have

$$R_x = R_1 \times \frac{1 + \omega^2 C_4^2 R_4^2}{\omega^2 R_3 R_4 C_4^2}$$

(*e*) CALCULATION OF POWER. We have

$$\underset{\cdot}{E} = E(\cos a + j \sin a)$$

and

$$\underset{\cdot}{I} = I(\cos \beta + j \sin \beta)$$

$(a - \beta)$ being the phase difference between E and I.

Multiplying these, we have

$$\begin{aligned} \underset{\cdot}{E}\underset{\cdot}{I} &= EI\{(\cos a \cos \beta - \sin a \sin \beta) + j(\sin a \cos \beta + \cos a \sin \beta)\} \\ &= EI\{\cos (a + \beta) + j \sin (a + \beta)\} \end{aligned}$$

The real part of this is $EI \cos (a + \beta)$, and this obviously is not the power in the circuit, since this should be $EI \cos (a - \beta)$. In addition, the imaginary part is $EI \sin (a + \beta)$, and this also is not the correct expression for the reactive volt-amperes. Hence, when calculating power by the symbolic method, it is necessary to reverse the real part of either E or I. Thus, if we leave $\underset{\cdot}{E}$ as written above, but write

$$\underset{\cdot}{I} = I(\cos \beta - j \sin \beta)$$

then

$$\underset{\cdot}{E}\underset{\cdot}{I} = EI\{\cos (a - \beta) + j \sin (a - \beta)\}$$

The real term is the *mean* power, and the imaginary term gives the reactive volt-amperes.

Alternatively we can write the voltage and current in the form

$$\underset{\cdot}{E} = E\varepsilon^{ja};\ \underset{\cdot}{I} = I\varepsilon^{j\beta}$$

Reversing the sign of β, we have

$$\underset{\cdot}{I}_1 = I\varepsilon^{-j\beta}$$

$$\begin{aligned} \therefore \qquad \underset{\cdot}{E}\underset{\cdot}{I}_1 &= EI\varepsilon^{j(a-\beta)} \\ &= EI\{\cos (a - \beta) + j \sin (a - \beta)\} \end{aligned}$$

as before. The vector obtained by reversing the real part of an original vector is sometimes referred to as the conjugate of the original vector.

As an example let

$$\underset{\bullet}{E} = 100 + j\,200$$
$$\underset{\bullet}{I} = 25 + j\ 10$$

Then $E = \sqrt{100^2 + 200^2} = 223{\cdot}6$

$I = \sqrt{25^2 + 10^2} = 26{\cdot}93$

$\alpha = \text{arc tan } \frac{200}{100} = 63° \ 26'$

$\beta = \text{arc tan } \frac{10}{25} = 21° \ 48'$

$\therefore W = EI \cos (\alpha - \beta)$

$= 223{\cdot}6 \times 26{\cdot}93 \times {\cdot}7474$

$= 4500$ watts.

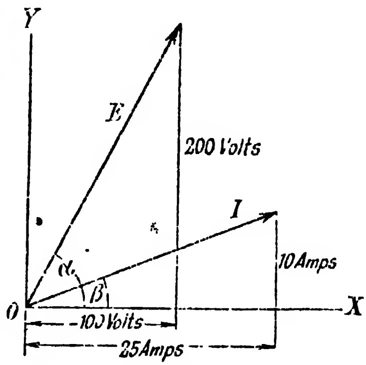


Fig. 189

Alternatively, since the imaginary voltage component of 200 is in phase with the current component of 10, while the real voltage component of 100 is in phase with the current component of 25, we have

$$W = 200 \times 10 + 100 \times 25$$
$$= 4500 \text{ watts.}$$

Examples on Chapter XIV.

(1) An alternating current of frequency 50 has a maximum value of 100 amp. Calculate (*a*) its value 1/600th second after the instant the current is zero; (*b*) in how many seconds after the zero value will the current attain the value of 86·6 amp.?

Ans.—(*a*) 50 amp.; (*b*) $\frac{1}{300}$th sec.

(2) Two currents represented by $i_1 = 50 \sin \omega t$ and $i_2 = 100 \sin \left(\omega t + \frac{\pi}{4}\right)$ are led into a common conductor. Find an expression for the total current in the form $i = I_{max} \sin (\omega t + \varphi)$. If the conductor has a resistance of 10 ohms, calculate the number of calories produced in 20 min.

Ans.—28×10^6 calories.

(3) Three voltages, represented by $e_1 = 20 \sin \omega t$, $e_2 = 30 \sin \left(\omega t - \frac{\pi}{4}\right)$, and $e_3 = 40 \cos \left(\omega t - \frac{\pi}{6}\right)$, act together in a circuit. Find an expression for the resulting voltage.

Ans.—$E_{max} = 62{\cdot}6$.

(4) A wooden ring of mean circumference 50 cm. and a circular section of 3 cm. diameter is uniformly wound with 400 turns of 1 mm. copper wire. Calculate its resistance, inductance, and impedance at frequency 100. Take the specific resistance of copper to be $1{\cdot}7 \times 10^{-6}$ ohms per cm. cube.

Ans.—(*a*) 0·82; (*b*) $2{\cdot}85 \times 10^{-4}$; (*c*) 0·83.

(5) The air gap under the pole of a series motor is 0·06 in. in length, and the effective area is 80 sq. in. There are 5 turns of cable round the pole, having a resistance of 0·005 ohm. Assuming that the reluctance of the iron parts of the magnetic circuit is one fifth of the reluctance of the air gap, calculate the voltage drop in a series coil when 150 amp. R.M.S. at 25 cycles per sec. are passing through it. (C. and G.)

Ans.—21 volts.

(6) A condenser and an inductive conductor are placed in series across alternate current mains, whose voltage is kept constant while the frequency is varied. Show by means of a curve how the current varies with the frequency. What determines the maximum value reached by the current ? If this maximum occurs at a frequency of 20, and if the condenser has a capacity of 50 microfarads, what is the value of the inductance ? If the voltage of the mains contains a pronounced third harmonic, how will the curve connecting current with frequency be affected ? (London Univ., 1916.)

Ans.—1·27 henrys.

(7) An alternate current circuit includes two sections AB and BC in series. The section AB consists of two branches in parallel. The first of these is formed of a non-inductive resistance of 80 ohms in series with a condenser of 50 microfarads, while the second consists of a resistance of 60 ohms having an inductance of 250 millihenrys. The section BC consists of a resistance of 100 ohms having an inductance of 300 millihenrys. The frequency of the current is 50 cycles per sec. The voltage across the section AB is 500. What is the voltage across the section BC ? (London Univ., 1914.)

Ans.—960.

(8) An air-cored choking coil is subjected to an alternating voltage of 100. The current taken is 0·1 amp., and the power factor 0·2, when the frequency of the current is 50. Find the capacity of a condenser which, if placed in parallel with the coil, will cause the main current to be a minimum. What will be the impedance of this parallel combination (*a*) for currents of frequency 50, and (*b*) for currents of frequency 40 ? (London Univ., 1914.)

Ans.—3·1 microfarads. (*a*) 5,000 ohms ; (*b*) 1,940 ohms.

(9) A choke coil of negligible resistance connected across a 500 volt 50 cycle circuit takes 1 amp. at 0·8 power factor. What capacity must be placed in parallel with it in order to make the power factor of the combination equal to unity ? (London Univ., 1921.)

Ans.—3·82 m.f.s.

(10) A coil having a resistance of 5 ohms and an inductance of ·02 henry is arranged in parallel with another coil with a resistance of 1 ohm, and an inductance of ·08 henry. Calculate the current flowing through each coil when a pressure of 100 volts at 50 cycles is applied to them. Find the total current passing, and estimate the resistance and inductance of a single coil which will take the same current at the same power factor. (London Univ., 1922.)

Ans.—12·5, 3·97, and 15·9 amps. ; 3·2 ohms and ·0175 henry.

(11) A transmission line 40 miles long consists of two wires each ·46 in diameter, spaced 5 ft. If the conductors are short-circuited at the far end what voltage at frequency 25 must be applied to produce a current of 200 amp. in the loop so formed ?

Ans.—6,140 volts.

CHAPTER XIV

POLYPHASE CURRENTS.

1. **General Principles.** Up to the present the alternating current systems we have considered have possessed only one electrical circuit, as in the case of two-wire direct current circuits. In alternating current working it is possible to use two, three, or more individual circuits in the same apparatus or machine. The voltages and frequencies in the individual circuits are the same, but they have definite phase differences, the amounts of these differences depending on the number of circuits or "phases." The complication involved is only small, but the advantages of polyphase over single phase working are enormous They may be summed up as follows—

(*a*) For a given size frame a polyphase generator or motor has a bigger output than a single phase.

(*b*) To transmit a given amount of power at a given voltage over a given distance, a polyphase transmission line requires less copper than a single phase line.

(*c*) Polyphase motors have an absolutely uniform torque, whereas single phase motors (except commutator motors) have a pulsating torque.

(*d*) Single phase motors (except commutator motors) are not self-starting. Polyphase motors are self-starting.

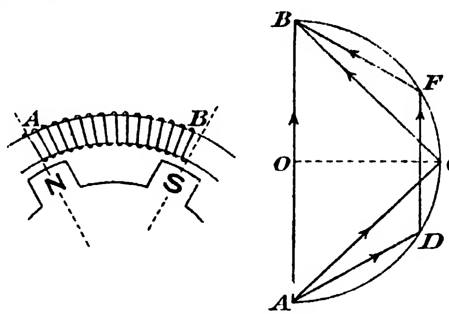


Fig. 190

Relation of Output to Number of Phases

(*e*) The pulsating nature of the armature reaction in single phase alternators causes difficulty with parallel running unless the poles are fitted with exceedingly heavy dampers. Polyphase generators work in parallel without difficulty.

2. **Relation of Output to Number of Phases.** Imagine an armature with a uniformly distributed winding, and consider that portion of the winding lying in one pole pitch (Fig. 190). The points A and B are 180 electrical degrees apart, and therefore, the phase difference in the E.M.F.s induced in the conductors at A and B is 180°. Also, the change of phase from A to B is uniform, and

the vector polygon for a large number of conductors is a semicircle, as shown. For a given winding the maximum allowable current I is fixed.

(*a*) SINGLE PHASE. All the winding is utilized and therefore the terminal voltage is the closing side of the whole polygon, that is, the diameter. Let this be E_1; then output $W_1 = E_1 I$.

(*b*) TWO PHASE. Each phase contains half the winding, and the closing sides for the two phases are the vectors AC and CB. Let these be E_2.

$$\therefore \text{ Total output, } W_2 = 2E_2 I = 2\frac{E_1}{\sqrt{2}} I$$

$$= \sqrt{2}\, W_1.$$

Thus the output of the same frame when used two phase is 41·4 per cent greater than the single-phase output.

(*c*) THREE PHASE. The closing sides for the three phases are AD, DF, and FB. Calling these E_3, we have $E_3 = \frac{1}{2}E_1$. Hence, total output

$$W_3 = 3E_3 I = 1{\cdot}5E_1 I = 1{\cdot}5W_1.$$

The increase in output in this case is therefore 50 per cent.

(*d*) " m " PHASE, where m is a large number.

If the individual phase voltage is E_m, then

$$m \,.\, E_m = \text{circumference of semicircle} = \frac{\pi}{2} E$$

$$\therefore \text{ Output } W_m = mE_m I$$

$$= \frac{\pi}{2} E_1 I = 1{\cdot}57 W_1$$

The increase in output is therefore 57 per cent, and this is the maximum possible increase. Since it is only 7 per cent greater than the three-phase output the increase does not justify the extra complication, and it is only in exceptional cases that more than three phases are used.

3. **Two-phase Working.** The armature has two distinct windings arranged so that there is a phase difference of 90° in the E.M.F.s induced in them. The alternator can be connected to the receiving apparatus by either four or three conductors, as in Fig. 191, the circuit with four conductors being the same as two independent single-phase circuits.

Let E and I refer to line voltage and current, and E_p and I_p

refer to phase voltage and current. Then obviously in the four-wire case,

$$E = E_p \text{ and } I = I_p.$$

In the three-wire case the outers carry current I, $= I_p$, and the voltages between outers and middle wire are $E = E_p$. The middle wire carries the resultant of the currents I in the two outers, and since these are 90° apart, the current in the middle wire is $\sqrt{2}\,I$. Similarly the voltage between the two outers is $\sqrt{2}\,E$. In both four- and three-wire systems, the power

$$W = 2EI \cos \varphi$$

where φ is the phase of I with respect to E.

4. **Three-phase Working.** The individual phases can be connected either in star ($\curlywedge$), or in mesh or delta (Δ), as shown in

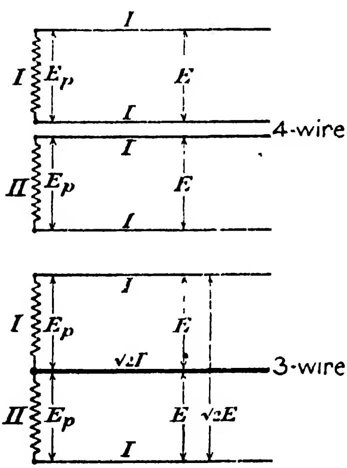


FIG. 191
TWO-PHASE WORKING

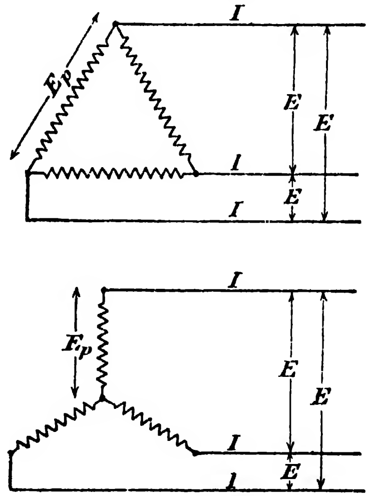


FIG. 192
THREE-PHASE WORKING

Fig. 192. The E.M.F.s in the individual phases have a mutual phase difference of 120°.

Consider first the mesh connection. The line voltages E are obviously equal to the phase voltages E_p. The line currents are the vector differences of the two-phase currents fed into a given line. This is easily understood by replacing the ordinary three-wire arrangement by a six-wire, the three phases being thus independent. This is shown in Fig. 193. Since each line wire in the actual circuit carries the currents flowing in a pair of wires in the six-wire arrangement, it follows that to obtain the line current the difference and not the sum of the phase currents must be taken. The phase

difference of $_1I_p$ and $_2I_p$ reversed is 60°. Hence, in the case of a current balance, that is,

$$_1I_p = {_2I_p} = {_3I_p} = I_p, \text{ say, we have}$$
$$I = \sqrt{3}\, I_p.$$
$$\text{Power } W = 3E_pI_p \cos\varphi$$
$$= \sqrt{3}\, EI \cos\varphi$$

In the star-connected arrangement the line currents are obviously equal to the phase currents. The voltage E between any pair of lines is the vector difference of the voltages in the two phases supplying that pair. The phase difference between the voltage in one phase and the reversed voltage in the next phase is 60°, so that $E = \sqrt{3}\, E_p$.

Hence, $W = 3E_pI_p \cos\varphi = \sqrt{3}\, EI \cos\varphi$

as in the case of the mesh connection.

If a middle wire is connected to the junction of the three phases, called the "star" point, the current in this wire will be the vector *sum* of the three line currents, as can be seen by considering the three phases separated, as in Fig. 194. If the system is balanced,

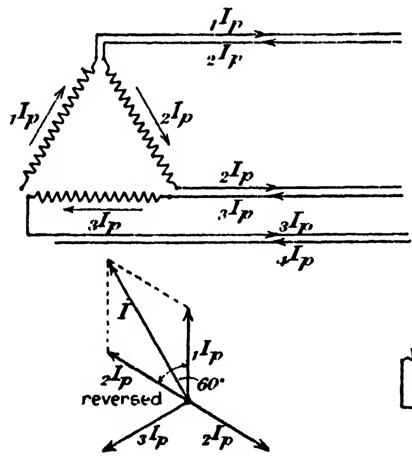


FIG. 193
EQUIVALENT MESH CONNECTION

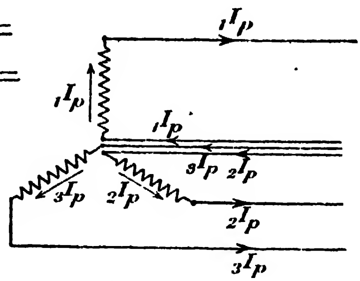


FIG. 194
EQUIVALENT STAR CONNECTION

i.e. the three line currents equal and having the same phase angle with their respective phase voltages, then the line current vectors will be equal and will have a mutual phase difference of 120°. Hence, their vector sum is zero. The current in the neutral wire of a balanced star-connected system is therefore zero, for which reason the neutral wire is generally omitted.

Example. A 100 h.p. three-phase star-connected motor works on a supply whose line voltage is 3,000. The motor efficiency is 92 per cent, and power factor 90 per cent. Calculate the line current.

$$\text{Motor intake } W = \frac{\text{output}}{\text{efficiency}} = \frac{746 \times 100}{\cdot 92}$$
$$= 81{,}000 \text{ watts}$$

$\therefore$ From $W = \sqrt{3}\, EI \cos \varphi$

$$I = \frac{W}{\sqrt{3}\, E \cos \varphi} = \frac{81{,}000}{\sqrt{3} \times 3{,}000 \times \cdot 9}$$
$$= 17 \cdot 3 \text{ amp.}$$

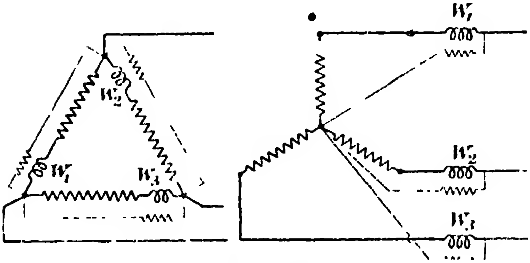


FIG. 195
MEASUREMENT OF POWER

5. **Measurement of Power in Three-phase Circuits.** (*a*) THREE-WATTMETER METHOD. The wattmeters are arranged one in each phase as in Fig. 195, the total power being the sum of the three readings. This method is only used in special cases. If the star point is not available when the alternator is star-connected, an "artificial star" can be made by connecting three high resistances in star to the three line conductors.

(*b*) TWO-WATTMETER METHOD. Consider the instantaneous values of the currents and voltages (Fig. 196). Since all three currents meet at the star point, the sum of their *instantaneous* values is zero, whether the system is balanced or not.

$$\therefore \quad i_1 + i_2 + i_3 = 0.$$

The instantaneous total power is the sum of the instantaneous powers in the three phases

$$w = e_1 i_1 + e_2 i_2 + e_3 i_3.$$

Now the current i_2 does not flow through either of the wattmeter current coils, hence we can eliminate it.

We have

$$i_2 = -(i_1 + i_3).$$

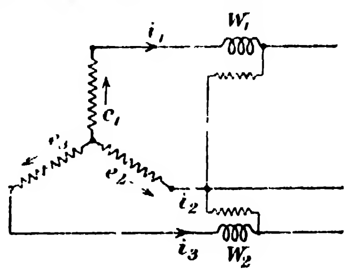


FIG. 196
TWO-WATTMETER METHOD OF MEASURING POWER

And substituting this in the equation for w, we have

$$w = e_1 i_1 + e_3 i_3 - e_2 (i_1 + i_3) = i_1 (e_1 - e_2) + i_3 (e_3 - e_2)$$

Now $(e_1 - e_2)$ is the instantaneous voltage across the pressure coil of wattmeter W_1, and $(e_3 - e_2)$ is the instantaneous voltage across the pressure coil of W_3.

$\therefore$ w = (instantaneous power through W_1) + (instantaneous power through W_3)

$\therefore$ Total average power

W = av. power through W_1 + av. power through W_3

= *algebraic* sum of the wattmeter readings.

This is the most important method of measuring power in a three-phase circuit, since only two wattmeters are required, and the method applies whether the system is balanced or not

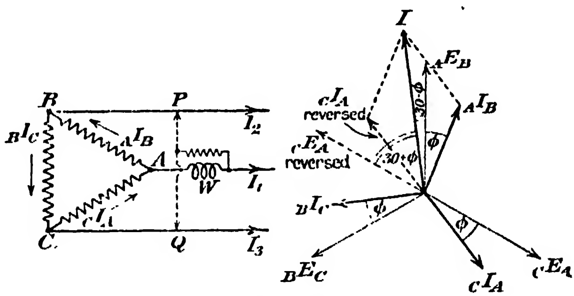


FIG. 197

MEASUREMENT OF POWER IN A BALANCED SYSTEM

(c) ONE-WATTMETER METHOD. This method can only be used when the system is balanced. The current coil is connected in one line, as in Fig. 197, the pressure coil being connected alternately between this and the other two lines, and the readings taken. In each case the wattmeter is connected exactly like one of the wattmeters in the two-wattmeter method, and therefore, the analysis following applies equally well to the two-wattmeter method when used on a balanced circuit. The current I through the wattmeter current coil is the vector difference of $_AI_B$ and $_CI_A$, i.e. the vector sum of $_AI_B$ and $_CI_A$ reversed, as shown in the vector diagram. In this diagram the phase currents are shown lagging behind the phase voltages at an angle φ, cos φ therefore being the power factor of the load.

$\therefore$ $I = \sqrt{3}\, I_\varphi$

where I_φ is the numerical value of each of the phase currents. When the wattmeter is connected to the point P, the pressure

across it is ${}_AE_B$, the phase difference between this pressure and I being $(30° - \varphi)$. Hence, if W_1 is the reading,

$$W_1 = {}_AE_BI \cos(30 - \varphi)$$
$$= \sqrt{3}\, EI_p \cos(30 - \varphi)$$

where E is the numerical value of each of the line or phase voltages.

When the pressure coil is disconnected from P and connected to Q the phase relation between the current and pressure in it is that between I and ${}_CE_A$ *reversed*, because to preserve a true cyclic rotation the pressure coil should have the end previously at P connected to A, and the end at A connected to Q, the reverse of what is actually done. The phase angle between I and ${}_CE_A$ reversed is $(30° + \varphi)$. Hence, if W_2 is the new reading,

$$W_2 = {}_CE_AI \cos(30 + \varphi)$$

Hence,

$$= \sqrt{3}\, EI_p \cos(30 + \varphi)$$
$$W_1 + W_2 = 3EI_p \cos\varphi = \text{total power.}$$

Again,

$$W_1 - W_2 = \sqrt{3}\, EI_p \sin\varphi$$
$$\therefore \frac{W_1 - W_2}{W_1 + W_2} = \frac{1}{\sqrt{3}} \tan\varphi$$

or

$$\tan\varphi = \sqrt{3} \times \frac{W_1 - W_2}{W_1 + W_2}$$

In this way the phase angle of the load current, and therefore, the power factor, can be calculated.

For the second reading the phase angle in the wattmeter is $(30 + \varphi)$; hence

(*a*) If $\varphi = 0$ both readings will be the same.

(*b*) If $\varphi < 60°$ the second reading will be positive and the total power will be the sum of the readings.

(*c*) If $\varphi = 60°$ the second reading will be zero.

(*d*) If $\varphi > 60°$ the angle $(30 + \varphi)$ will be $> 90°$ and the second reading will be negative. In this case the total power is the difference of the two readings.

It must be noted that the one-wattmeter method is only a special case of the two-wattmeter method, and the relationships deduced for it also apply to the two-wattmeter method when the system is balanced.

6. **Corrections to be Applied to Wattmeter Readings.** Suppose that a wattmeter W is measuring the power in a circuit AB, Fig. 198. Then it is usual to connect the pressure coil across AB. If i is the instantaneous current in AB, and i_1 the instantaneous current in the pressure coil, $(i + i_1)$ is the instantaneous current in the current coil.

$\therefore$ Torque at any instant, which is proportional to the product of the instantaneous currents in the pressure and current coils, is proportional to $i_1(i + i_1)$.

Now the current in the pressure coil is equal at any instant to e/R, since we can apply the ordinary Ohm's Law equation when dealing with instantaneous values.

$$\therefore \text{ Instantaneous torque } \propto \left(\frac{ei}{R} + \frac{ei_1}{R}\right)$$
$$\propto (ei + ei_1)$$

$\therefore$ Reading, which $\propto$ average torque
$\propto$ average (of ei) + average (of ei_1)
$\propto$ average power in AB + average power in pressure coil

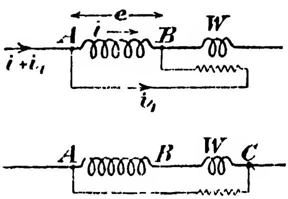


Fig. 198

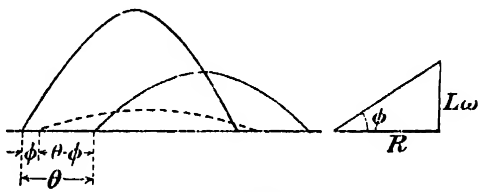


Fig. 199

WATTMETER CORRECTIONS

A correction for the power used in the pressure coil has therefore to be applied. In standard wattmeters this correction is applied automatically by means of a compensating coil. This is a small coil placed with its axis along the axis of the current coil, and having the same number of turns as the current coil, but connected in series with the pressure coil. It is arranged so that its ampere-turns neutralize the extra ampere-turns in the current coil due to the current i_1

When the current in the circuit AB is very small the pressure coil of the wattmeter is sometimes connected to the points A and C (Fig. 198). In this case the wattmeter reading

= power in AB + power in current coil.

In addition to the above corrections, it is necessary to apply a second correction when the wattmeter is working on a circuit whose power factor is less than unity.

Let θ = angle of lag in the main circuit

φ = angle of lag of current in the pressure coil behind the pressure across it

i = instantaneous value of main current

i_1 = instantaneous current in pressure coil.

Then from the phase relations shown in Fig. 199, in which the current i_1 is represented by the dotted wave, we see that if

$$i = I_{max} \sin \omega t \text{ ; then } i_1 = I_{1max} \sin (\omega t + \overline{\theta - \varphi})$$

$$\therefore \text{ Wattmeter reading } \propto \text{ average torque}$$
$$\propto \text{ average of } i i_1$$
$$\propto \text{ average of } I_{max} I_{1max} \times \tfrac{1}{2}[\cos (\theta - \varphi) - \cos \{2\omega t + (\theta - \varphi)\}]$$

Now the average of $\cos \{2\omega t + (\theta - \varphi)\}$ is zero, because it is a periodic quantity.

$$\therefore \text{ Reading } \propto \frac{I_{max}}{\sqrt{2}} \cdot \frac{I_{1max}}{\sqrt{2}} \cdot \cos (\theta - \varphi)$$
$$\propto I_{eff} \cdot I_{1eff} \cos (\theta - \varphi)$$
$$\propto I_{eff} \times \frac{E_{eff}}{\sqrt{R^2 + (L\omega)^2}} \cos (\theta - \varphi)$$

where R and L are the resistance and inductance of the pressure coil

$$\propto I_{eff} E_{eff} \times \frac{R}{\sqrt{R^2 + (L\omega)^2}} \times \frac{1}{R} \cos (\theta - \varphi)$$
$$\propto I_{eff} E_{eff} \cos \varphi \cos (\theta - \varphi)$$

omitting $\frac{1}{R}$ because it is constant.

But true reading should be proportional to $I_{eff} E_{eff} \cos \theta$.

$$\therefore \text{ Correcting factor } = \frac{\text{true reading}}{\text{actual reading}}$$
$$= \frac{\cos \theta}{\cos \varphi \cos (\theta - \varphi)}$$
$$\text{or } = \frac{1 + \tan^2 \varphi}{1 + \tan \theta \tan \varphi}$$

In a standard instrument, φ should not be greater than 5′ or 10′. In a switchboard instrument, especially if used with instrument transformers, φ may be as high as 5° or 10°.

Example. A wattmeter is measuring the power supplied to a circuit whose power factor is 0·7. The frequency is 50. The meter has a shunt winding whose self induction is 0·4 henry, and resistance 1,000 ohms. Calculate its percentage error.

Power factor of load $\cos \theta = 0{\cdot}7 \quad \therefore \tan \theta = 1{\cdot}03$ approx.

$$\tan \varphi = \frac{L\omega}{R} = \frac{0{\cdot}4 \times 314}{1{,}000} = 0{\cdot}126$$

$$\therefore \text{ Correcting factor } = \frac{1 + (0{\cdot}126)^2}{1 + 1{\cdot}03 \times 0{\cdot}126} = 0{\cdot}9 \text{ approx.}$$

$$\therefore \text{ Error } = 10\%$$

We will now show how the symbolic method can be applied to the calculation of the wattmeter correction factor. Let $\dot{E}$ be the voltage applied to the load and to the pressure coil. Then, taking the vector of E as the reference vector, we have

$$\dot{E} = E\varepsilon^{j.0}$$

Let I_1 be the load current, Z_1 the load impedance, and $\cos\theta$ its power factor. Then

$$\dot{I}_1 = I_1\varepsilon^{-j\theta}$$
$$= \frac{E}{Z_1} \cdot \varepsilon^{-j\theta}$$

Similarly for the shunt circuit of the wattmeter we can write

$$\dot{I}_2 = \frac{E}{Z_2} \cdot \varepsilon^{-j\varphi}$$

Now the torque on the moving coil is proportional to the product of the currents I_1 and I_2, but when using the symbolic method we have to use the complement of $\dot{I}_2$, namely, $\dot{I}_2'$, where

$$\dot{I}_2' = \frac{E}{Z_2} \cdot \varepsilon^{+j\varphi}$$

$\therefore$ Torque, $\propto$ real part of $\dot{I}_1\dot{I}_2'$

$$\propto \text{real part of } \frac{E^2}{Z_1Z_2}\,\varepsilon^{(\varphi-\theta)}$$

$$\propto \text{real part of } \frac{E^2}{Z_1Z_2}(\cos\overline{\varphi-\theta} + j\sin\overline{\varphi-\theta})$$

$$\propto \frac{E^2}{Z_1Z_2}\cos(\varphi-\theta).$$

Now, if the shunt circuit acted like a pure resistance of value R_2 φ would be zero. The above expression would then reduce to

$$\frac{E^2}{Z_1R_2}\cos(-\theta) = \frac{E^2}{Z_1R_2}\cos\theta.$$

Hence the correcting factor is

$$\left(\frac{E^2}{Z_1R_2}\cos\theta\right) \div \left(\frac{E^2}{Z_1Z_2}\cos\overline{\varphi-\theta}\right)$$

$$= \frac{Z_2\cos\theta}{R_2\cos(\varphi-\theta)} = \frac{\cos\theta}{\cos\varphi\,(\cos\varphi\cos\theta + \sin\varphi\sin\theta)}$$

$$= \frac{1}{\cos^2\varphi(1+\tan\varphi\tan\theta)} = \frac{\sec^2\varphi}{1+\tan\varphi\tan\theta}$$

$$= \frac{1+\tan^2\varphi}{1+\tan\varphi\tan\theta},$$

which is the correction factor obtained previously.

7. **Polyphase Vector Diagrams.** Consider a star-connected alternator connected to a star-connected consuming device. Draw three vectors OA, OB, and OC to represent the phase voltages (Fig. 200) Then, on joining the ends, we obtain vectors AB, BC, and CA, which give the line voltages at the alternator terminals. If the line and load are non-inductive, the line currents will be in phase with the alternator phase voltages, so that the directions of OA, OB, and OC will also give the directions of the line currents. The drop of volts in a non-inductive line will be in phase with the line currents, so that, if we deduct lengths AA_1, BB_1, and CC_1, where each length is equal to RI, R being the resistance of each line, then OA_1, OB_1, and OC_1 are the phase voltages of the load, and A_1B_1, B_1C_1, C_1A_1, the line voltages at the load.

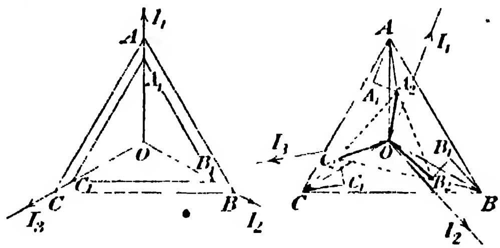


FIG. 200
THREE-PHASE VECTOR DIAGRAMS

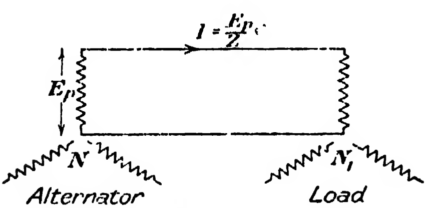


FIG. 201

Now let the line possess resistance R and reactance X per phase, and let the load be inductive. The voltage triangle ABC for the alternator end is drawn as before, but the line current vectors OI_1, OI_2, and OI_3 will lag by equal amounts behind the phase voltages OA, OB, and OC. Draw AA_1 parallel to OI_1 and equal to RI, where I is the line current. Then AA_1 is the resistance drop in line 1. Draw A_1A_2 perpendicular to OI_1 and equal to XI. Then A_1A_2 is the reactance drop in the line. The total line drop is therefore AA_2, and the phase voltage of the corresponding phase at the load end is OA_2. Similarly with the other phases. The line voltages at the load end are given by the sides A_2B_2, B_2C_2, and C_2A_2 of the triangle $A_2B_2C_2$.

In a symmetrical system, the alternator neutral N, and the load neutral N_1 (Fig. 201), will be at the same potential, so that they can be considered as joined by a cable of zero resistance and reactance. Hence, each phase can be considered separately. Let E_p be the phase voltage of the alternator, and Z the total impedance of one phase of the whole system, including alternator, line,

and load. Then line current $I = \frac{E_p}{Z}$, and power factor $\cos\varphi = \frac{R}{Z}$, R being the total resistance of one phase of the system.

8. **Calculation of Unbalanced Three-phase Circuits.** When making calculations on balanced three-phase circuits, as, for example, three-

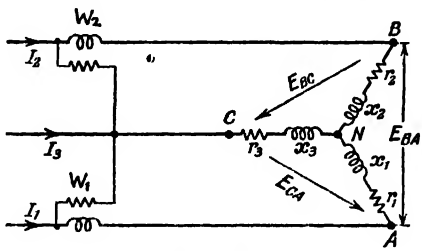


FIG. 202

phase machines working under normal conditions, it is usually the best to work in terms of phase voltages and phase currents, and then perform the calculation exactly as for a single-phase circuit. With unbalanced circuits this is not possible, and the laws of networks have to be applied. Consider the star-connected network shown in Fig. 202. Take the vector of E_{BA} as the reference vector, and let E be the numerical value of each line voltage, then we have

$$E_{BA} = E + j0$$

$$E_{BC} = -\frac{E}{2} + j{\cdot}866E$$

$$E_{CA} = -\frac{E}{2} - j{\cdot}866E$$

Hence, for the circuit ANB, we have

$$E + j0 = (r_1 + jx_1)\,I_1 - (r_2 + jx_2)\,I_2 \quad . \quad . \quad (1)$$

For the circuit BNC we have

$$-\frac{E}{2} + j{\cdot}866E = (r_2 + jx_2)\,I_2 - (r_3 + jx_3)\,I_3 \quad . \quad . \quad (2)$$

We now have to find a third equation which is independent of (1) and (2) above, and we therefore cannot use a similar equation for the circuit CNA, since the information it would give is already contained in (1) and (2). We, therefore, use

$$I_1 + I_2 + I_3 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The above equations can be written in the form

$$\begin{array}{lllll} (r_1 + jx_1)\,I_1 & - (r_2 + jx_2)\,I_2 & + 0I_3 & - E & = 0 \\ 0I_1 & + (r_2 + jx_2)\,I_2 & - (r_3+jx_3)\,I_3 & + \left(\dfrac{E}{2} - j{\cdot}866E\right) & = 0 \\ I_1 & + I_2 & + I_3 & + 0 & = 0 \end{array}$$

The solution can now be expressed in determinant form as follows

$$\frac{-I_1}{\begin{vmatrix} -(r_2 + jx_2) & 0 & -E \\ (r_2+jx_2) & -(r_3+jx_3) & \left(\dfrac{E}{2} - j{\cdot}866E\right) \\ 1 & 1 & 0 \end{vmatrix}} = \frac{I_2}{\begin{vmatrix} 0 & -E & (r_1 + jx_1) \\ -(r_3 + jx_3) & \left(\dfrac{E}{2} - j{\cdot}866E\right) & 0 \\ 1 & 0 & 1 \end{vmatrix}}$$

$$\frac{-I_3}{\begin{vmatrix} -E & (r_1 + jx_1) & -(r_2 + jx_2) \\ \left(\dfrac{E}{2} - j{\cdot}866E\right) & 0 & (r_2 + jx_2) \\ 0 & 1 & 1 \end{vmatrix}} = \frac{1}{\begin{vmatrix} (r_1 + jx_1) & -(r_2 + jx_2) & 0 \\ 0 & (r_2 + jx_2) & -(r_3 + jx_3) \\ 1 & 1 & 1 \end{vmatrix}}$$

Abbreviating this to the form

$$\frac{-I_1}{D_1} = \frac{I_2}{D_2} = -\frac{I_3}{D_3} = \frac{1}{D_4}$$

we will evaluate D_4, since this is necessary for the calculation of all three currents. On the other hand, it is preferable only to evaluate D_1, D_2, and D_3 when numerical values are given for the circuit constants. We then have

$$D_4 = (r_1 + jx_1)(r_2 + jx_2) + (r_2 + jx_2)(r_3 + jx_3) + (r_3 + jx_3)(r_1 + jx_1)$$

Suppose, for example, that

$$E = 100 \text{ volts R.M.S.}$$
$$r_1 = 1,\ r_2 = 2,\ r_3 = 3$$
$$x_1 = 2,\ x_2 = 3,\ x_3 = 4$$

Then $D_1 = \begin{vmatrix} -(2 + j3) & 0 & -100 \\ (2 + j3) & -(3 + j4) & (50 - j86{\cdot}6) \\ 1 & 1 & 0 \end{vmatrix} = -(140{\cdot}2 + j723{\cdot}2)$

$$D_2 = \begin{vmatrix} 0 & -100 & 1 + j2 \\ -(3 + j4) & (50 - j86{\cdot}6) & 0 \\ 1 & 0 & 1 \end{vmatrix} = -(523{\cdot}3 + j413{\cdot}4)$$

$$D_3 = \begin{vmatrix} -100 & 1 + j2 & -(2 + j3) \\ (50 - j86{\cdot}6) & 0 & (2 + j3) \\ 0 & 1 & 1 \end{vmatrix} = -(383 + j309{\cdot}8)$$

$$D_4 = (1 + j2)(2 + j3) + (2 + j3)(3 + j4) + (3 + j4)(1 + j2)$$
$$= (-15 + j33)$$
$$\therefore \quad I_1 = \frac{140 \cdot 2 + j723 \cdot 2}{-15 + j33}$$
$$= 16 \cdot 4 - j11 \cdot 3$$
$$I_2 = \frac{-(523 \cdot 3 + j413 \cdot 4)}{15 + j33}$$
$$= -4 \cdot 52 + j17 \cdot 4$$
$$I_3 = \frac{38 \cdot 3 + j309 \cdot 8}{15 + j33}$$
$$= -11 \cdot 85 - j6 \cdot 12$$
$$\therefore \quad I_1 = 19 \cdot 8, \; I_2 = 18 \cdot 0, \; I_3 = 13 \cdot 3 \text{ amp.}$$

This gives for the power intake of the circuit

$$W = I_1^2 r_1 + I_2^2 r_2 + I_3^2 r_3$$
$$= 392 + 648 + 530 = 1{,}570 \text{ watts}$$

But we can also calculate this power symbolically; for example, suppose that the power is being measured by the two wattmeter method, as indicated in the figure. Then, remembering that we have to use the complement of the current when calculating power, we have

$$W = W_1 + W_2$$
$$= \text{real part of } I_1 E_{AC} + \text{real part of } I_2 E_{BC}$$

with the necessary reversal of sign as explained previously

$$= \text{real part of } \{(50 + j86 \cdot 6)(16 \cdot 4 + j11 \cdot 3) + (-50 + j86 \cdot 6)(-4 \cdot 52 - j17 \cdot 4)\}$$
$$= 1{,}570 \text{ watts, as before}$$

In the case of a mesh-connected system the calculation is much simpler, as the voltage across each phase of the load and the phase currents can then be calculated independently of one another. For this reason it is often desirable to reduce an unbalanced star-connected system to the equivalent mesh-connected system, or alternatively, a mesh-connected system can be converted to the equivalent star. Let the systems indicated in Fig. 203 be electrically equivalent, then the measured impedances between the pairs of terminals must be the same for the two loads

$$\therefore \; Z_a + Z_b = \frac{Z_{ab}(Z_{bc} + Z_{ca})}{Z_{ab} + Z_{bc} + Z_{ca}} \quad . \quad . \quad . \quad . \quad (1)$$

$$Z_b + Z_c = \frac{Z_{bc}(Z_{ca} + Z_{ab})}{Z_{ab} + Z_{bc} + Z_{ca}} \quad . \quad . \quad . \quad . \quad (2)$$

$$Z_c + Z_a = \frac{Z_{ca}(Z_{ab} + Z_{bc})}{Z_{ab} + Z_{bc} + Z_{ca}} \quad . \quad . \quad . \quad . \quad (3)$$

These give

$$\dot{Z}_a = \frac{\dot{Z}_{ab} \cdot \dot{Z}_{ca}}{\dot{Z}_{ab} + \dot{Z}_{bc} + \dot{Z}_{ca}}$$

$$\dot{Z}_b = \frac{\dot{Z}_{bc} \cdot \dot{Z}_{ab}}{\dot{Z}_{ab} + \dot{Z}_{bc} + \dot{Z}_{ca}}$$

$$\dot{Z}_c = \frac{\dot{Z}_{ca} \cdot \dot{Z}_{bc}}{\dot{Z}_{ab} + \dot{Z}_{bc} + \dot{Z}_{ca}}$$

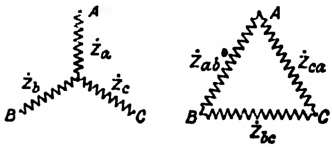


FIG. 203

Thus, suppose we re-connect the limbs of the previous star-connected system into a mesh-connected system. Then

$$\dot{Z}_{ab} = 1 + j2$$
$$\dot{Z}_{bc} = 2 + j3$$
$$\dot{Z}_{ca} = 3 + j4$$
$$\therefore\ \dot{Z}_{ab} + \dot{Z}_{bc} + \dot{Z}_{ca} = 6 + j9$$

$$\therefore\ \dot{Z}_a = \frac{(1 + j2)\,(3 + j4)}{6 + j9} = \cdot 513 + j\cdot 9$$

$$\dot{Z}_b = \frac{(2 + j3)\,(1 + j2)}{6 + j9} = \cdot 333 + j\cdot 666$$

$$\dot{Z}_c = \frac{(3 + j4)\,(2 + j3)}{6 + j9} = 1 + j1\cdot 33$$

9. The Drop of Volts in a Three-phase Line. We have seen that the inductance of the loop formed by a pair of parallel conductors is

$$L = 14\cdot 8 \times 10^{-4} \log_{10} \frac{d}{r} \text{ henrys per mile.}$$

The inductance of a single conductor in the case of three conductors placed at the corners of an equilateral triangle of side d is $\frac{1}{2}L$. For consider the individual phase current to be flowing in three separate loops of a six conductor line, as in Fig. 204. Let the currents in two of these loops be I_1 and I_2. Then the actual current, I_A, flowing in conductor A, is the vector sum of the currents I_1 and I_2. Again, the third loop is directly

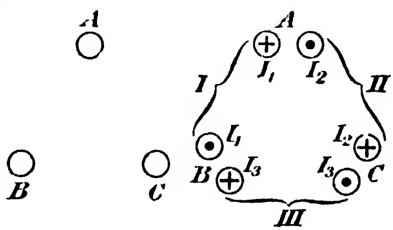


FIG. 204

opposite to the conductor A, so that the flux produced by it will not link with A, and therefore will not induce a voltage in A. It is thus not necessary to consider the third loop when calculating the inductance of A, and the E.M.F. induced in A is due to the current I_1 in B and I_2 in C. The E.M.F. induced in both conductors of loop 1 is $L\frac{di_1}{dt}$. Hence, E.M.F. induced in one conductor only of loop 1, say conductor A, is $\frac{1}{2}L\frac{di_1}{dt}$. Similarly, the E.M.F. induced in conductor A by the current I_2 in loop 2 is $\frac{1}{2}L\frac{di_2}{dt}$.

$\therefore$ Total E.M.F. induced in conductor A

$$= \frac{1}{2}L\left(\frac{di_1}{dt} + \frac{di_2}{dt}\right)$$

Now $\quad i_A = i_1 + i_2$

$$\therefore \quad \frac{di_1}{dt} + \frac{di_2}{dt} = \frac{di_A}{dt}$$

$\therefore$ Total E.M.F. induced in conductor A

$$= \frac{1}{2}L \quad dt$$

Hence, $\frac{1}{2}L$ is the equivalent self-induction of a single conductor.

In a single-phase line formed by two of the conductors and carrying current I_1, the drop, reckoning both conductors, is

$$\{(2I_1R)^2 + (I_1L\omega)^2\}^{\frac{1}{2}} = I_1 \times \{4R^2 + (L\omega)^2\}^{\frac{1}{2}}$$

The drop per conductor in a three-phase line carrying a line current of I_2 is

$$\{(I_2R)^2 + (\tfrac{1}{2}I_2L\omega)^2\}^{\frac{1}{2}} = I_2 \times \{R^2 + \tfrac{1}{4}(L\omega)^2\}^{\frac{1}{2}}$$

$\therefore$ Drop in potential difference between two conductors of the three-phase line carrying current I_2, i.e. the drop in line voltage

$$= \sqrt{3} \times \text{drop per conductor}$$

$$= \sqrt{3}I_2 \times \{R^2 + \tfrac{1}{4}(L\omega)^2\}^{\frac{1}{2}}$$

If the drops in the single-phase and three-phase lines are equal, then

$$I_1 \times \{4R^2 + (L\omega)^2\}^{\frac{1}{2}} = \sqrt{3}I_2 \times \{R^2 + \tfrac{1}{4}(L\omega)^2\}^{\frac{1}{2}}$$

$$\frac{I_1}{I_2} = \frac{\sqrt{3}}{2}$$

If the line pressures and the power factors are the same in the single and three-phase lines, say, E and $\cos\varphi$ respectively,

$$\text{Power conveyed by single-phase line} = EI_1 \cos\varphi$$

$$\text{Power conveyed by three-phase line} = \sqrt{3}\,EI_2 \cos\varphi$$

$$\therefore \text{ Ratio of powers} = \frac{I_1}{\sqrt{3}\,I_2} = \tfrac{1}{2}$$

Hence, we have the following rule: The total drop of volts in a balanced three-phase system with symmetrically arranged conductors is the same as that in a single-phase system having conductors of the same diameter and spacing, operating at the same

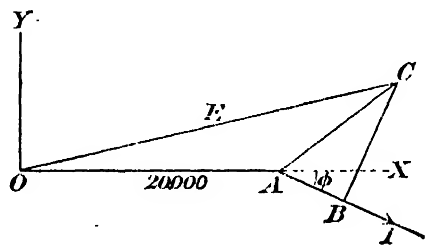


FIG. 205

line voltage and the same power factor, and transmitting only half the amount of power transmitted by the three-phase system.

Example. A three-phase transmission line consists of $\frac{1}{4}$ in. conductors placed at the corners of an equilateral triangle of side 4 ft. The line is 20 miles long and supplies a load of 5,000 kW at a voltage of 20,000 and a power factor of 0·8. The frequency is 50. Find the drop of volts in the line.

Resistance of one conductor $R = 5$ ohms.

$$d/r = 48/\tfrac{1}{4} = 192 \; : \; \log_{10}\frac{d}{r} = 2{\cdot}28$$

$\therefore$ Inductance of a loop formed by two conductors

$$L = 14{\cdot}8 \times 10^{-4} \times 2{\cdot}28 \times 20 = {\cdot}068 \text{ henry}$$

$$\text{Reactance of loop } X = L\omega = {\cdot}068 \times 314 = 21{\cdot}3 \text{ ohms}$$

The current in a single-phase line using two of the conductors and transmitting 2,500 kW would be

$$I = \frac{2{,}500{,}000}{20{,}000 \times 0{\cdot}8} = 156 \text{ amp}$$

$\therefore$ Resistance drop, AB in Fig. 205 $= 2 \times 5 \times 156 = 1{,}560$ volts.

Reactance drop, $BC = 21{\cdot}3 \times 156 = 3{,}330$ volts.

$\therefore$ Voltage at generating end E has for its X and Y components

$$(E)_x = 20{,}000 + 1{,}560 \times \cdot 8 + 3{,}330 \times \cdot 6$$
$$= 23{,}250$$
$$(E)_y = 3{,}330 \times \cdot 8 - 1{,}560 \times \cdot 6$$
$$= 1{,}730.$$
$$\therefore \quad E = \sqrt{(23{,}250)^2 + (1{,}730)^2} = 23{,}300 \text{ volts.}$$

$\therefore$ Voltage drop = 3,300 volts. This is also the drop in the actual three-phase system.

EXAMPLES ON CHAPTER XIV.

(1) A two-phase alternator having a voltage per phase of 1,000 and frequency 50, has an inductive resistance of 10 ohms and 0·1 henry connected to phase I, and a condenser of 50 mf. capacity connected to phase II. If the alternator is working on a three-wire circuit, find the current in the common return.

Ans.—38·2 amps.

(2) A single-phase and a two-phase three-wire system supply equal amount of power to transmission lines of equal length. If the maximum voltage between any two conductors is the same in the two cases, and the conductors are all worked at the same current density, compare the amount of copper used in the two cases.

Ans.—1 to 1·21.

(3) A 50 h.p. three-phase motor is supplied at a terminal voltage of 500. Its efficiency is 85 per cent, and power factor 0·8. Find (*a*) the line current; (*b*) the cost of running the motor at full load for 24 hours, the price of a B.O.T. unit being one penny.

Ans.—63·5 amps., £4 8*s.* 0*d.*

(4) Power in a balanced three-phase system is measured by the two-wattmeter method, and it is found that the ratios of the two readings are 2 to 1. What is the power factor of the system?

Ans.—0·866.

(5) In testing the power supplied to a three-phase induction motor by two wattmeters connected across the line wires, one of the wattmeters reads backwards at light loads, though its readings are forwards at heavy loads. Examine the reason for this and find what is the true power reading. (London Univ., 1908.)

(6) A single wattmeter is used to ascertain the power supplied from three-phase mains. It is only possible to insert the current coil in one of the mains, but each of the other mains can in turn be connected with the pressure coil. Two wattmeter readings W_1 and W_2 are so obtained, the current coil being connected with one or other of the pressure coil terminals. Explain carefully why it is necessary to alter the pressure coil connections for the second reading. Under what circumstances can the power and power factor be accurately obtained from the readings of W_1 and W_2? Prove in each case the formula to be used. (London Univ., 1915.)

(7) 500 kW at 11,000 volts are received from a three-phase transmission line, each wire of which has a resistance of 1·2 ohms and a reactance of 1 ohm. Calculate the supply pressure when the power factor of the load is (*a*) unity; (*b*) 0·5 leading. (C. and G., 1922.)

Ans.—(*a*) 11,055; (*b*) 10,985.

CHAPTER XV

THE TRANSFORMER

1. Construction. A transformer consists essentially of a magnetic circuit linking with two distinct windings, the primary and secondary. Fig. 200 shows the elementary scheme of a transformer. When the primary is connected to an A.C. supply an alternating flux will be set up in the core, and this flux linking with the secondary will induce an alternating E.M.F. in the secondary. The operation can thus be regarded as a case of mutual induction.

There are three main types of transformer, the type being decided by the disposition of the core. These are—

1. "Core Type" with single magnetic circuit.
2. "Shell Type" with double magnetic circuit.
3. "Berry Type" with distributed magnetic circuit.

These are shown in Fig. 206. The cores are built up of sheet iron, or alloyed steel, in order to keep down the eddy current loss to a minimum. Average thicknesses are 0·5 mm. for a frequency of 25 and 0·35 mm. for a frequency of 50. Fig. 207 shows the methods of arranging the core strips, and it will be seen that joints in alternate layers are staggered, so as to avoid the presence of narrow gaps right through the cross section of the core. Such joints are said to be "imbricated." The cross sections of small transformer cores are rectangular, but in large sizes it is common to adopt an approximately circular section since this section has the smallest

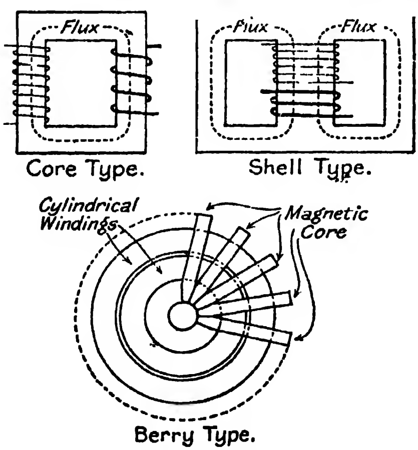


FIG. 206

TYPES OF TRANSFORMERS

FIG. 207

CONSTRUCTION OF JOINTS

perimeter for a given area, and therefore requires less copper than the rectangular section. In large sizes the strips are arranged in packets, as in armature cores, the ducts between them helping with the ventilation.

The windings can be either of the "cylindrical" or "sandwich" type, as illustrated in Fig. 208, the former type being used mainly on core transformers and the latter on shell. In elementary diagrams it is necessary to represent the primary and secondary windings as being placed on separate limbs, but in an actual transformer primary and secondary coils are arranged on each limb in order to keep down magnetic leakage. This is explained more fully in par. 5.

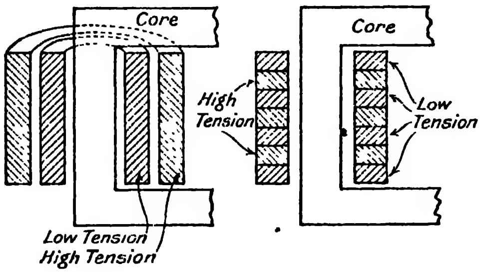


FIG. 208
TRANSFORMER WINDINGS

2. **Transformation Ratio.** The ratio

$$\frac{\text{Secondary voltage}}{\text{Primary voltage}} = \frac{E_2}{E_1}$$

is called the "Transformation" ratio.

Since the flux is alternating it can be written

$$\Phi = \Phi_{max} \sin \omega t.$$

$\therefore$ E.M.F *induced* in the primary, instantaneous value,

$$e_1' = -\frac{d\Phi}{dt} \times N_1 \times 10^{-8} \text{ volts}$$

where N_1 and N_2 are the numbers of turns on primary and secondary respectively.

$$\therefore e_1' = -\Phi_{max}\,\omega \,.\cos \omega t\,.\,N_1 \times 10^{-8}$$
$$= -2\pi\Phi_{max} N_1 f\,.\cos \omega t \times 10^{-8}$$
$$\therefore (E_1')_{max} = -2\pi\Phi_{max} N_1 f \times 10^{-8}$$

and the effective value

$$E_1' = -\frac{2\pi}{\sqrt{2}}\,.\,\Phi_{max}\,.\,N_1 f \times 10^{-8}$$
$$= -4{\cdot}44\Phi_{max} N_1 f \times 10^{-8}$$

Since the resistance of a transformer winding is very low, the

ohmic drop is small and the applied primary voltage, E_1, has therefore only to oppose the induced primary voltage E_1'. Hence

$$E_1 = -E_1' = 4{\cdot}44\Phi_{max} N_1 f \times 10^{-8}$$

The voltage induced in the secondary is

$$E_2' = 4{\cdot}44\Phi_{max} N_2 f \times 10^{-8}$$

by the same reasoning, and this is the voltage which appears at the secondary terminals on no load, i.e. on no load $E_2 = E_2'$.

Hence, transformation ratio $\dfrac{E_2}{E_1} = \dfrac{N_2}{N_1}$

Obviously, the induced voltages E_1' and E_2' are in phase with one another, and they are therefore opposite in phase to the applied primary voltage E_1. Hence E_1 and E_2 are opposite in phase.

3. **Primary No Load Current.** On no load, i.e. with no current in the secondary winding, the primary carries a small current I_o, which has two components—

1. A magnetizing component I_μ which lags 90° behind E_1. This is an idle component and its function is to produce the magnetic flux.
2. A working component I_ω which produces the necessary real power to supply the hysteresis and eddy current losses in the iron. This component is in phase with E_1.

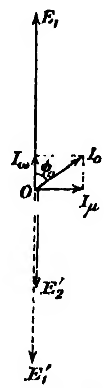


FIG. 209
NO LOAD VECTOR DIAGRAM

The no load vector diagram is therefore as shown in Fig. 209. For high efficiency and good regulation I_μ and I_ω must both be small. I_ω is kept small by using low loss iron and not working at too high a flux density. I_μ is kept small by having a closed magnetic circuit, i.e. one without an air gap. If an air gap were cut in the core, then the reluctance would be increased to such an extent that, in order to provide the necessary flux as given by the equation $E_1 = 4{\cdot}44\Phi_{max} N_1 f \times 10^{-8}$, the magnetizing component I_μ would have to be abnormally large.

The separate components I_μ and I_ω can be calculated as follows. The magnetizing force produced by I_μ is given by

$$H = \frac{4\pi I_\mu N_1}{10 l}$$

where l is the mean length in cm. of the magnetic path. Hence flux density

$$B = \mu H = \frac{4\pi I_\mu N}{10 l} \times \mu$$

$$\therefore\ B_{max} = \frac{4\pi(I_\mu)_{max} \times N_1\mu}{10l}$$

But $E_1 = \sqrt{2}\,.\,\pi\Phi_{max}N_1f \times 10^{-8}$

$$= \sqrt{2}\,.\,\pi B_{max}AN_1f \times 10^{-8}$$

where A is the magnetic cross section of the core

$$\therefore\ (E_1)_{max} = 2\pi B_{max}AN_1f \times 10^{-8}$$

$$\therefore\ B_{max} = \frac{(E_1)_{max} \times 10^8}{2\pi AN_1f}.$$

Equating these two expressions for B_{max} we have

$$(I_\mu)_{max} = (E_1)_{max} \times \frac{l \times 10^9}{8\pi^2AN_1^2\mu f}$$

$$\therefore\ I_\mu = \frac{E_1l \times 10^9}{8\pi^2AN_1^2\mu f} \text{ (effective values)}$$

This is only approximate, since it assumes a sinusoidal form of the magnetizing component and constant permeability of the iron. Neither assumption is correct, as will be shown in Chapter XXIII, but the equation gives a ready means of determining I_μ approximately.

In order to calculate the working component I_ω, we have

$$E_1I_\omega = W_h + W_e$$

where W_h = hysteresis loss and W_e = eddy current loss.

$$I_\omega = \frac{W_h + W_e}{E_1}$$

For the hysteresis loss, we have

$$W_h = \eta B_{max}^{1 \cdot 6}fV \times 10^{-7} \text{ watts}$$

where V is the volume of the iron in c.c. An average value for η is 0·002 for good quality transformer iron, and 0·00076 for alloyed steel containing 5 per cent silicon.

In order to calculate the eddy current loss imagine a portion of one core strip of thickness t cm., as shown in Fig. 210. Consider a rectangular prism of dimensions $t \times 1 \times 1$. The area of the front face is t sq. cm., and since the flux enters this face at right angles, the eddy currents will flow along paths parallel to the long sides. Consider two such paths each of width dx and distant x from the centre line.

Area of rectangle enclosed by paths $= 2x$ sq. cm.

$\therefore$ Maximum flux entering the rectangle

$$= 2B_{max}x \text{ lines.}$$

Regarding the rectangle as a single turn, the E.M.F. induced in it will be given by the transformer equation

$$\begin{aligned} E &= 4{\cdot}44\Phi_{max}Nf \times 10^{-8} \\ &= 4{\cdot}44 \times (2B_{max}x) \times 1 \times f \times 10^{-8} \\ &= 8{\cdot}88B_{max}fx \times 10^{-8} \end{aligned}$$

The current set up is confined to the two long sides, the total resistance of the path being therefore

$$\rho \cdot \frac{l}{a} = \frac{2\rho}{dx \times 1} = \frac{2\rho}{dx}$$

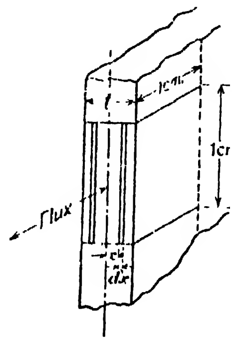


FIG. 210
CALCULATION OF EDDY CURRENT LOSS

Hence, eddy current loss in the two strips

$$\frac{E^2}{R} = \frac{(8{\cdot}88)^2B^2_{max}f^2x^2 \times 10^{-16}}{2\rho} \cdot dx$$

$\therefore$ Total eddy current loss in the rectangular prism considered

$$= \int_0^{\frac{t}{2}} \frac{(8{\cdot}88)^2B^2_{max}f^2 \times 10^{-16}}{2\rho} \times x^2dx$$

$$= \frac{(8{\cdot}88)^2B^2_{max}f^2 \times 10^{-16}}{48\rho} \times t^3$$

Now the volume of the prism is $t \times 1 \times 1$ c.c.s. Hence, eddy current loss per c.c.

$$= \frac{(8{\cdot}88)^2B^2_{max}f^2 \times 10^{-16}}{48\rho} \times t^2$$

Taking ρ as 12×10^{-6} ohm per cm. cube at a temperature of 50° C. the expression becomes

$$13{\cdot}6 \times 10^{-12}B^2_{max}f^2t^2 \text{ watts per c.c.}$$

$$\therefore W_e = 13{\cdot}6 \times 10^{-12}B^2_{max}f^2t^2V \text{ watts.}$$

Notice that the eddy current loss is proportional to the square of the thickness of the core strips; hence, the necessity for using thin strips. It is to be noticed in passing that whereas the flux in a transformer core is a purely alternating one, the flux in armatures is partly alternating and partly rotating. The above expression therefore requires correcting before it can be used to calculate eddy current loss in armature cores.

Example 1. **A transformer working on a 2,000 volt, 50 cycle circuit has 300 primary turns. The core has a mean magnetic path of 100 cm. and cross section 1,000 sq. cm., the iron having a permeability of 1,800. The iron loss is 400 watts. Calculate the primary no load current.**

The magnetizing current

$$I_\mu = \frac{E_1 l \times 10^9}{8\pi^2 N_1^2 A \mu f} = 0{\cdot}3 \text{ amp.}$$

The working component

$$I_\omega = \frac{\text{iron loss}}{\text{primary volts}} = \frac{400}{2{,}000} = 0{\cdot}2 \text{ amp.}$$

Hence, no load current

$$I_o = \sqrt{0{\cdot}3^2 + 0{\cdot}2^2} = 0{\cdot}36 \text{ amp.}$$

Example 2. When a transformer is connected to a 1,000 volt, 50 cycle supply, the core loss is 1,000 watts, of which 700 are hysteresis and 300 are eddy current loss. If the applied voltage is raised to 2,000 and the frequency to 100, find the new core loss.

The hysteresis loss W_h can be written in the form $PB_{max}^{1\cdot6} f$, and the eddy current loss W_e in the form $QB^2{}_{max}f^2$: P and Q being constants. Now from the E.M.F. equation

$$E = 4{\cdot}44 B_{max} A N_1 f \times 10^{-8}$$

we see that $B_{max} \propto \dfrac{E}{f}$

Hence, we can write for the hysteresis and eddy current losses

$$W_h = PE^{1\cdot6}f^{-0\cdot6}; \quad W_e = QE^2$$

where the constants P and Q now have different values.

From the data given, we have

$$700 = P \times (1{,}000)^{1\cdot6} \times (50)^{-0\cdot6}; \quad 300 = Q \times 1{,}000^2$$

$$\therefore \; P = 700 \times 1{,}000^{-1\cdot6} \times 50^{0\cdot6}; Q = 300 \times 1{,}000^{-2}$$

Hence, when the voltage is raised to 2,000 and the frequency to 100 we have

$$W_h = (700 \times 1{,}000^{-1\cdot6} \times 50^{0\cdot6}) \times 2{,}000^{1\cdot6} \times 100^{-0\cdot6} = 1{,}400$$

$$W_e = 300 \times 1{,}000^{-2} \times 2{,}000^2 = 1{,}200$$

$\therefore$ Total core loss under the new conditions = 2,600 watts.

4. **Transformer on Load.** Consider first of all a non-inductive load connected to the secondary terminals; then a secondary current I_2 will flow, and I_2 will be in phase with E_2. The secondary ampere-turns $N_2 I_2$ will tend to produce a secondary flux which, if allowed to exist, would disturb the flux conditions existing at no load. This would alter the primary induced voltage E_1' and the balance between E_1 and E_1' would no longer exist. The presence of the secondary M.M.F. therefore necessitates the production of a primary M.M.F. equal in magnitude but opposite in direction.

This is provided by a load current I_1' which flows from the supply through the primary. Since the M.M.F. produced by this is equal to the secondary M.M.F.,

$$1{\cdot}26 N_1 I_1' = 1{\cdot}26 N_2 I_2$$

$$\therefore\ I_1' = \frac{N_2}{N_1} \times I_2$$

The total primary current I_1 is therefore the vector sum of I_1' and I_0, and for all but light loads can be taken as equal to I_1'. We thus have

$$I_1 = \frac{N_2}{N^1} \times I_2 \text{ approx.}$$

or

$$\frac{I_2}{I_1} = \frac{N_1}{N_2}$$

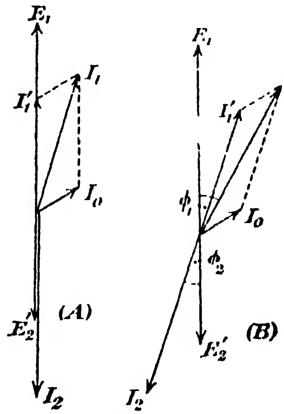


Fig. 211
Vector Diagrams for Loaded Transformer

The current transformation ratio is thus the inverse of the voltage transformation ratio. The vector diagram for the transformer on non-inductive load is shown in Fig. 211 (A). A similar action takes place if the secondary load has any phase angle φ. The *induced* primary current I_1' is always opposite in phase to the secondary current I_2, and since I_0 is small, the total primary current I_1 is almost exactly opposite in phase to I_2. (Fig. 211 (B).)

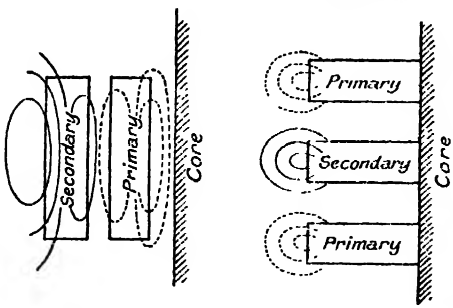


Fig. 212
Leakage in Transformers

Since the induced primary and the secondary ampere-turns always neutralize one another, the flux on load is the same as the flux on no load. Hence, the iron losses are constant and are independent of the load.

5. Effect of Resistance and Reactance in the Windings. So far we have assumed that the windings have no resistance and no reactance. In an actual transformer both are always present, the reactance being set up by the leakage fluxes. The paths of these fluxes for cylindrical and sandwich windings are as shown in Fig. 212. The primary leakage flux is defined as that flux which links with the primary, but not with the secondary.

Similarly, the secondary leakage flux is the flux which links with the secondary but not the primary winding. Since a leakage flux links with only one winding, it produces a self-induced back E.M.F. in that winding, and it is therefore equivalent to a small choker in series with that winding.

The effects of resistance and reactance are best understood by considering the case of an ideal transformer having no resistance and reactance of its own, but having separate resistance and reactance coils connected in series with each winding, as shown in Fig. 213.

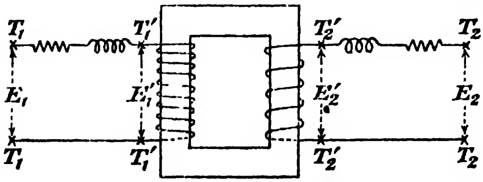


FIG. 213
EQUIVALENT CIRCUITS

Let R_1 = resistance of primary of actual transformer
R_2 = resistance of secondary of actual transformer
X_1 = reactance of primary of actual transformer
X_2 = reactance of secondary of actual transformer

Represent the applied primary voltage by E_1 (Fig. 214). Let OI_1 represent the primary current I_1 in phase, lagging any angle φ_1 behind E_1. Then primary resistance drop $= R_1I_1$ in phase with I_1, and reactance drop $= X_1I_1$ leading I_1 by 90°. Represent these drops by OB and BC respectively. Then OC, the vector sum of OB and BC, is the total primary drop. Deducting OC from OA, we obtain $CA = E_1'$, the voltage across the primary terminals $T_1'T_1'$ of the ideal transformer.

Drop the perpendicular CD on to OA. Then since in an actual transformer OC is small compared with OA we have

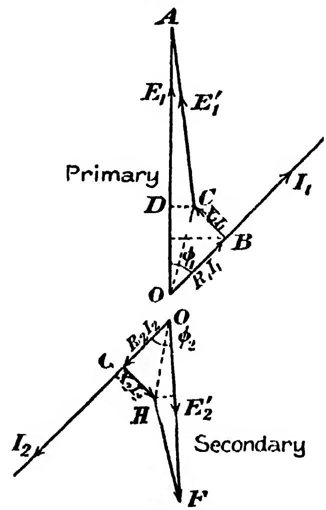


FIG. 214
COMPLETE VECTOR DIAGRAM

$$E_1' = E_1 - OD, \text{ approx.}$$
$$= E_1 - (R_1I_1 \cos\varphi_1 + X_1I_1 \sin\varphi_1)$$

Let K = voltage transformation ratio.

Then the induced voltage in the secondary, namely, the voltage across the terminals $T_2'T_2'$ of the ideal transformer is

$$E_2' = K \times E_1'$$
$$= KE_1 - K(R_1I_1 \cos\varphi_1 + X_1I_1 \sin\varphi_1)$$

Represent E_2' by the vector OF (in opposition to E_1') (Fig 214), and draw OI_2 to represent the secondary current I_2, lagging φ_2, behind E_2'. Then secondary resistance drop $= R_2 I_2$ in phase with I_2, and reactance drop $= X_2 I_2$ leading I_2 by $90°$.

Represent these drops by OG and GH respectively, then OH is the total secondary drop, and HF is the voltage E_2 which appears at the actual secondary terminals $T_2 T_2$. Hence, as before,

$$E_2 = E_2' - (R_2 I_2 \cos\varphi_2 + X_2 I_2 \sin\varphi_2)$$
$$= KE_1 - K(R_1 I_1 \cos\varphi_1 + X_1 I_1 \sin\varphi_1)$$
$$- (R_2 I_2 \cos\varphi_2 + X_2 I_2 \sin\varphi_2)$$

Now φ_1 and φ_2 are, as we have seen, approximately equal Replacing them by φ, we have

$$E_2 = KE_1 - \{(KR_1 I_1 + R_2 I_2)\cos\varphi + (KX_1 I_1 + X_2 I_2)\sin\varphi\}$$

Again, $\quad I_1 = KI_2$ approx.,

hence, the expression for the secondary terminal voltage, in terms of the applied primary voltage and the load current I_2, becomes

$$E_2 = KE_1 - \{(K^2R_1 + R_2)\cos\varphi + (K^2X_1 + X_2)\sin\varphi\}I_2$$

Now KE_1 is the secondary voltage on no load, and therefore, the expression

$$\{(K^2R_1 + R_2)\cos\varphi + (K^2X_1 + X_2)\sin\varphi\}I_2$$

is the drop of volts at the secondary terminals on load. Notice that the drop is proportional to the current I_2, and that it also depends upon the power factor, $\cos\varphi$, of the load.

If the load is non-inductive, $\varphi = 0$, $\cos\varphi = 1$, $\sin\varphi = 0$.

$\therefore \quad$ Drop $= I_2(K^2R_1 + R_2)$

It is therefore independent of the reactance.

If the load is purely inductive, $\varphi = 90°$, $\cos\varphi = 0$, $\sin\varphi = 1$

$\therefore \quad$ Drop $= I_2(K^2X_1 + X_2)$

In this case the drop is decided by the reactance and is independent of the resistance. The same applies when the load is a pure capacity load, except that in this case the drop is negative, i.e. it is a rise of voltage.

6. Equivalent Resistance and Reactance. The primary resistance drop $= R_1 I_1$, and this, when referred to the secondary, becomes $KR_1 I_1$, or $K^2 R_1 I_2$. The secondary resistance drop is $R_2 I_2$, and therefore, the total resistance drop in terms of I_2 is

$$(K^2R_1 + R_2)I_2$$
$$= \left\{\left(\frac{N_2}{N_1}\right)^2 R_1 + R_2\right\} I_2$$

The quantity in the large brackets is called the total resistance of the transformer *referred to the secondary*, because when multiplied by the secondary current it gives the total resistance drop. If multiplied by the square of the current it gives the total copper loss (I^2R loss) in the transformer.

The total reactance referred to the secondary is given by a similar expression. Hence, if $\mathbf{R}_2$ and $\mathbf{X}_2$ are the total resistance and reactance referred to the secondary, and $\mathbf{R}_1$ and $\mathbf{X}_1$ referred to the primary, we have

$$\mathbf{R}_1 = R_1 + \left(\frac{N_1}{N_2}\right)^2 R_2$$

$$\mathbf{X}_1 = X_1 + \left(\frac{N_1}{N_2}\right)^2 X_2$$

$$\mathbf{R}_2 = R_2 + \left(\frac{N_2}{N_1}\right)^2 R_1$$

$$\mathbf{X}_2 = X_2 + \left(\frac{N_2}{N_1}\right)^2 X_1$$

Also $\qquad \mathbf{Z}_1^2 = \mathbf{R}_1^2 + \mathbf{X}_1^2$ and $\mathbf{Z}_2^2 = \mathbf{R}_2^2 + \mathbf{X}_2^2$

Example. A transformer has a normal primary voltage of 1,000, and on open circuit it takes a current of 0·3 amp. at a power factor of 0·7. When the secondary is short-circuited and a reduced voltage of 50 is applied to the primary, it takes 12 amp. at a power factor of 0·3. Calculate the efficiency and percentage drop of volts when delivering the full output of 10 kVA at a power factor of 0·8.

Iron losses = open circuit intake = $1{,}000 \times 0{\cdot}3 \times 0{\cdot}7 = 210$ watts.

The *total* copper losses corresponding to a primary current of 12 amp. are equal to the intake of true power in the short circuit test, that is, to $50 \times 12 \times 0{\cdot}3 = 180$ watts. The full kVA capacity of the transformer is 10, and therefore, the primary current at this load (assuming 100 per cent efficiency) is $\frac{10 \times 1{,}000}{1{,}000} =$ 10 amp. The total copper losses at this load are therefore $180 \times \left(\frac{10}{12}\right)^2 = 125$ watts.

Output in true power = $10 \times 0{\cdot}8 = 8$ kw. = 8,000 watts.

$$\therefore \text{ Efficiency} = \frac{\text{output}}{\text{intake}} = \frac{8{,}000}{8{,}000 + 210 + 125} = 96\%.$$

Again, from the short circuit test,

$$\mathbf{Z}_1 = \frac{50}{12} = 4{\cdot}16 \text{ ohms}$$

$$\mathbf{R}_1 = \mathbf{Z}_1 \cos \varphi_s$$

where $\cos \varphi_s$ is power factor during the short circuit test (Fig. 215)

$$= 4{\cdot}16 \times 0{\cdot}3 = 1{\cdot}25 \text{ ohms}$$

$$\mathbf{X}_1 = \sqrt{(4{\cdot}16)^2 - (1{\cdot}25)^2} = 4{\cdot}0 \text{ ohms approx.}$$

Hence, since the primary current on ordinary load is 10 amp., the ohmic and reactive drops *referred to the primary* are $1{\cdot}25 \times 10 = 12{\cdot}5$, and $4 \times 10 = 40$ volts, respectively. If we deduct these drops vectorially from the applied primary voltage, the remainder

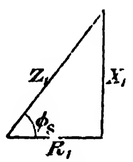


Fig. 215

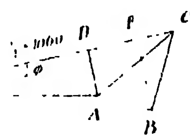


Fig. 216

will be the voltage available for transformation by the secondary winding. Also, it is not necessary to make any calculation for drop in the secondary since this is already taken into account by using the equivalent resistance and reactance referred to the primary. The vector diagram is given in Fig. 216. *OA* is the remainder of the voltage available for transformation; *AI* represents the phase of the current, this vector being therefore inclined at an angle of φ to *OC*, where $\cos \varphi = 0{\cdot}8$. *AB* is the resistance drop of 12·5 volts, in phase with *I*, and *BC*, the reactance drop of 40 volts in quadrature with *I*. Hence, *OC* is the total applied voltage of 1,000.

The voltage drop is $OC - OA$, which is approximately equal to *DC*. But

$$DC = DF + FC = AB \cos \varphi + BC \sin \varphi$$
$$= (12{\cdot}5 \times 0{\cdot}8) + (40 \times 0{\cdot}6) = 10 + 24$$
$$= 34 \text{ volts.}$$
$$\therefore \text{ Percentage drop} = 3{\cdot}4$$

7. **Kapp Regulation Diagram.** The "regulation" of a transformer, or an alternator, is the change of terminal voltage with load. Thus, if a transformer gives a secondary voltage of ${}_0E_2$ on no load and ${}_1E_2$ on full load, the regulation is $({}_0E_2 - {}_1E_2)$. The percentage regulation is

$$\frac{{}_0E_2 - {}_1E_2}{{}_0E_2} \times 100$$

This varies with the power factor of the load, as we have seen. Average values of the percentage regulation are 2 per cent at unity power factor, and 3·5 per cent at 0·8 power factor (lagging).

The graphical method of determining the voltage drop in a transformer is shown in Fig. 217. A vector OA is drawn to represent the secondary terminal voltage on load, and OX is drawn inclined at an angle φ to OA, where $\cos\varphi$ is the power factor of the load. Then OX represents the phase of the secondary current. Draw AB parallel to OX and equal to the resistance drop referred to the secondary, namely, $\mathbf{R}_2I_2$; and the perpendicular BC equal to the reactance drop referred to the secondary, namely, $\mathbf{X}_2I_2$. Then the triangle CAB is the drop triangle referred to the secondary, and CA is the total voltage drop. Hence OC is the secondary no load voltage ${}_0E_2$, and $(OC - OA)$, arithmetic difference, is the secondary drop from no load to full load. Now, if the triangle CAB is transferred to ODE, as shown, then $DA = OC = {}_0E_2$. Hence, for a given secondary current, the locus of C is a circle with centre O and radius ${}_0E_2$, while the locus of A is a circle with centre D and the same radius. In order to find the voltage drop on full load at any power factor $\cos\varphi$, a radius OQS is drawn inclined at an angle φ to OX. If the impedance triangle is drawn in the position PQR, then OP, $= OS$, is the no load secondary voltage, and OQ the voltage on load. Hence, the length QS is the voltage drop. It is obvious that the triangle PQR need not be drawn, but simply the radius OQS. It will be seen that for all phase angles φ included between $\angle\, XOF$ (which is leading) and $\angle\, GOX$ (which

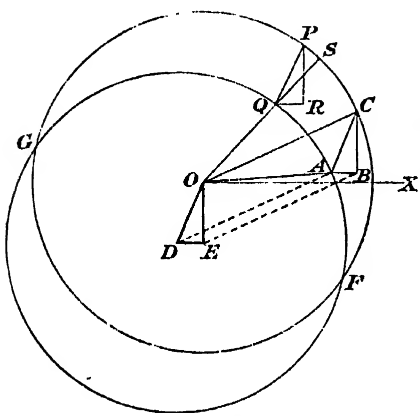


FIG. 217
KAPP REGULATION DIAGRAM

is lagging), there is a drop of volts on load. For phase angles equal to either of these two limits the voltage on load is equal to that on no load. For phase angles between $\angle\, GOX$ and $\angle\, XOF$ (reckoning in a counter-clockwise direction) there is actually a rise of voltage on load.

The Kapp diagram affords a very convenient means of determining the variation of regulation with power factor, but it has the disadvantage that since the lengths of the sides of the impedance triangle are small compared with the radius of the circles, the diagram has to be drawn to a very large scale if accurate results are to be obtained.

8. **Equivalent Circuit.** In Section 5 the primary and secondary reactances were treated as though they had independent existences, like the primary and secondary resistances. Actually this is not the

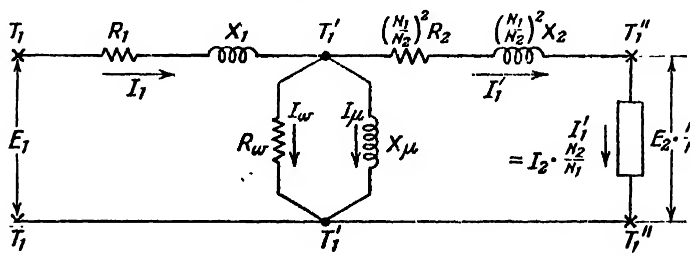


FIG. 218. EQUIVALENT CIRCUIT OF A STATIC TRANSFORMER

case, because the leakage fluxes responsible for these reactances only exist when the windings carry load currents, and as it is impossible to have a load current in, say, the secondary without the corresponding induced current in the primary, it is equally impossible for one of the leakage fluxes to be produced independently of the other. On the other hand, the voltage which is actually transformed in the transformer is the primary applied voltage less the primary drop. This is the voltage which just balances the primary induced voltage E_1'. In other words, it is the voltage which is responsible for the iron loss current I_w, and the magnetizing current I_μ, and we can therefore represent these two components of the no-load current by the current taken by a non-inductive resistance R_w, and a pure reactance X_μ having the voltage E_1' applied to them, as indicated in Fig. 218. The rest of the equivalent circuit consists of the secondary resistance and reactance referred to the primary, and the primary induced load current E_1' is taken off at the terminals $T_1''T_1''$. Now the calculation of the regulation from this exact circuit is somewhat cumbersome, and a simplification can be made by transferring the resistance R_w and reactance X_μ to the terminals T_1T_1. The error introduced by doing this is, for a transformer of normal performance, quite negligible, because the currents I_w and I_μ

are too small to affect the drop between T_1T_1 and $T_1'T_1'$. The simplified circuit is, therefore, as given in Fig. 219 (A). Now the resistance R_w and X_μ have no effect at all on the drop in the transformer when connected across T_1T_1, so that we can neglect them altogether, and we can make the further simplification of substituting the actual transformer current I_1 for the induced current I_1'. This

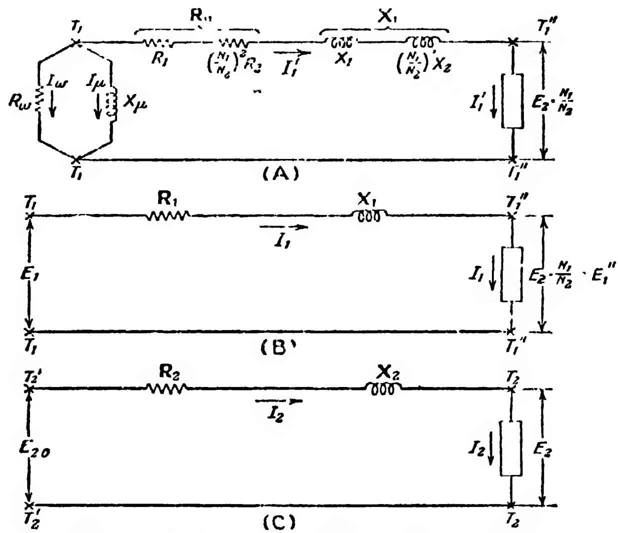


FIG. 219. SIMPLIFICATION OF THE EQUIVALENT CIRCUIT

is done in Fig. 219 (B). Alternatively, if the equivalent resistance and reactance are referred to the secondary side, we can use the equivalent circuit shown in Fig. 219 (C).

It is obvious that the circuits (B) and (C) lead to vector diagrams similar to that given in Fig. 216, but we will re-calculate the example on page 266, using the symbolic method. The data are as follows—

$$E_1 = 1{,}000 \text{ volts}, I_1 = 10 \text{ amp.}, \mathbf{R}_1 = 1{\cdot}25, \mathbf{X}_1 = 4{\cdot}0$$

Denote the voltage E_1'' (Fig. 207 (B)) by

$$x + j{\cdot}0$$

Since the current is at a power factor of 0·8 lagging, we have

$$\begin{aligned} I_1 &= 10 \times {\cdot}8 - j \times 10\sqrt{1-({\cdot}8)^2} \\ &= 8 - j6 \end{aligned}$$

Impedance referred to primary side

$$\mathbf{Z}_1 = 1{\cdot}25 + j4$$

$\therefore$ Drop of volts referred to primary side

$$I_1\mathbf{Z}_1 = (8 - j6)(1{\cdot}25 + j4)$$
$$= 34 + j24{\cdot}5$$
$$\therefore \quad E_1 = (x + j0) + (34 + j24{\cdot}5)$$
$$= (34 + x) + j24{\cdot}5$$

But $\quad E_1 = 1{,}000$

$\therefore \quad (34 + x)^2 + 24{\cdot}5^2 = 1{,}000^2$

$\therefore \quad x = 966$ volts, approx.

$\therefore$ Drop of volts referred to primary $= 1{,}000 - 966 = 34$

$\therefore \quad$ Percentage drop $= 3{\cdot}4$, as before.

9. **Percentage Resistance, Reactance, and Impedance.** It is usual to express the resistance, reactance, and impedance of a transformer as percentages of the normal voltage. This is done in the following manner—

$\dfrac{100\mathbf{R}I}{E_0}$ = percentage resistance, where E_0 is the normal voltage per phase of the winding on which the measurements or calculations are made

$$\frac{100\mathbf{X}I}{E_0} = \text{percentage reactance}$$

and $\dfrac{100\mathbf{Z}I}{E_0}$ = percentage impedance

We have seen previously that the percentage drop is given by the expression

$$\text{Percentage drop} = \frac{\mathbf{R}I\cos\varphi + \mathbf{X}I\sin\varphi}{E_0} \times 100$$

so that it can also be written

$$\frac{100\mathbf{R}I}{E_0}\cos\varphi + \frac{100\mathbf{X}I}{E_0}\sin\varphi$$

Denoting the percentage resistance and reactance by v_r and v_x respectively, we have

$$\text{Percentage drop} = v_r\cos\varphi + v_x\sin\varphi$$

The following numerical example will make the method clear. A 100 kVA single-phase transformer, ratio 10,000/200, 50 cycles, requires 300 volts at the H.T. winding to circulate full-load current with the L.T. winding short-circuited. The intake is then 1,000 watts. Calculate the percentage regulation and secondary terminal voltage on full load at 0·8 power factor lagging.

Full-load primary current

$$I_1 = \frac{1{,}000 \times \text{kVA}}{E_1} = \frac{1{,}000 \times 100}{10{,}000}$$
$$= 10 \text{ amp.}$$

$$\therefore \quad \mathbf{Z}_1 = \frac{300}{10} = 30 \text{ ohms}$$

$$\mathbf{R}_1 = \frac{1{,}000}{10^2} = 10 \text{ ohms}$$

$$\mathbf{X}_1 = \sqrt{30^2 - 10^2} = 28{\cdot}3 \text{ ohms}$$

$$\therefore \quad v_r = \frac{\mathbf{R}_1 I_1}{E_1} \times 100 = \frac{10 \times 10 \times 100}{10{,}000} = 1$$

$$v_x = \frac{\mathbf{X}_1 I_1}{E_1} \times 100 = \frac{28{\cdot}3 \times 10 \times 100}{10{,}000} = 2{\cdot}83$$

$\therefore$ Percentage drop

$$= v_r \cos \varphi + v_x \sin \varphi$$
$$= 1 \times 0{\cdot}8 + 2{\cdot}83 \times 0{\cdot}6$$
$$= 2{\cdot}5$$

$$\therefore \text{ Secondary drop } = \frac{2{\cdot}5}{100} \times 200 = 5 \text{ volts}$$

$$\therefore \ E_2, \text{ on load } = 200 - 5 = 195 \text{ volts.}$$

10. **"All-day" Efficiency.** The ordinary or commercial efficiency is defined as the ratio

$$\frac{\text{Output in watts}}{\text{Intake in watts}}$$

The all-day efficiency is the ratio

$$\frac{\text{Output in kilowatt hours}}{\text{Intake in kilowatt hours}}$$

This second efficiency is thus measured on an energy basis. The reason for the introduction of this efficiency is that transformers used for distribution have their primary winding connected to the line for 24 hours per day. The core losses are thus going on for the whole 24 hours, whereas the copper losses go on only when the transformer is on load. Hence, if the load is not on the transformer for the whole of the time it is connected to the line, the all-day efficiency is less than the commercial efficiency.

Example. A 20 kW lighting transformer of ordinary efficiency 95% is on full load for 6 hours per day. Find the all-day efficiency if the full load losses are equally divided between copper and iron.

Total losses on full load = 5% of 20 kW = 1 kW

$\therefore$ Iron losses = 0·5 kW, and full load copper losses = 0·5 kW

Output of 20 kW for 6 hours per day	=	120 kWh
Copper loss of 0·5 kW for 6 hours per day	=	3 kWh
Iron loss of 0·5 kW for 24 hours per day	=	12 kWh
Energy intake during 24 hours	=	135 kWh

$$\therefore \text{ All-day efficiency } = \frac{120}{135} = 89\%$$

11. Relation between Copper and Iron Loss.

$$\text{Output (watts)} = E_2 I_2 \cos\varphi$$

$$\text{Primary copper loss} = R_1 I_1^2$$

$$\text{Secondary copper loss} = R_2 I_2^2$$

$$\therefore \text{ Total copper loss} = (R_1 I_1^2 + R_2 I_2^2) = \mathbf{R}_2 I_2^2$$

$$\text{Iron loss} = W_i = a \text{ constant}$$

$$\therefore \text{ Efficiency} = \frac{\text{output}}{\text{intake}} = \frac{E_2 I_2 \cos\varphi}{E_2 I_2 \cos\varphi + \mathbf{R}_2 I_2^2 + W_i}$$

$$= \frac{E_2 \cos\varphi}{E_2 \cos\varphi + (\mathbf{R}_2 I_2 + W_i/I_2)}$$

For a given power factor the efficiency is a maximum when the expression in brackets is a minimum. Now the product of the two terms in the expression is a constant, and therefore, their sum is a minimum when they are equal. Hence, for maximum efficiency we have

$$\mathbf{R}_2 I_2 = W_i/I_2$$

$$\therefore \mathbf{R}_2 I_2^2 = W_i$$

The efficiency is thus a maximum at that load which makes the copper losses equal to the constant iron losses. If a transformer is intended to work most of the time on full load, it is designed to have the maximum efficiency at full load. If the load is variable it is usual to design it to have maximum efficiency at about three-quarter of full load.

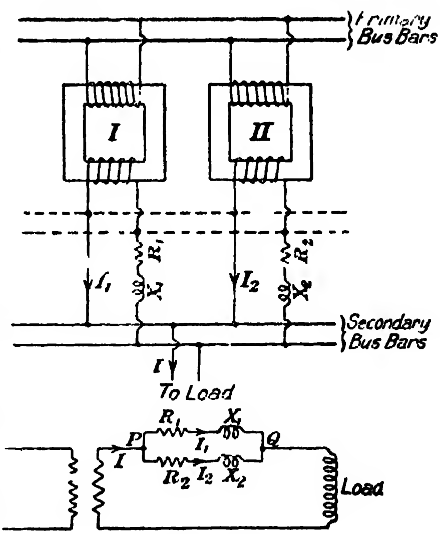


FIG. 220

12. Division of Load between Two Transformers in Parallel. In Fig. 220 two transformers are shown connected in parallel on a pair of bus-bars, and their resistances and reactances, *referred to the secondary*, are, for the purposes of calculation, shown external to the secondary windings. If the transformation ratios are the same, the voltages across the secondary terminals of the two ideal transformers will be the same, and we can therefore imagine these connected to a second pair of bus-bars, as shown dotted. The voltage across these bars will not vary with load, but will remain constant. The equivalent simple circuit is therefore as shown in the second diagram, and the load current I will divide itself between the two transformers in the ratio of the branch currents I_1 and I_2 in the equivalent circuit.

Let $\mathbf{Z}_1 = \sqrt{\mathbf{R}_1^2 + \mathbf{X}_1^2}$, and $\mathbf{Z}_2 = \sqrt{\mathbf{R}_2^2 + \mathbf{X}_2^2}$ be the equivalent impedances of the two transformers referred to the secondaries. If the ratios $\mathbf{R}_1/\mathbf{X}_1$ and $\mathbf{R}_2/\mathbf{X}_2$ are equal, then the currents I_1 and I_2 will be in phase, and we can write

$$\frac{I_1}{I_2} = \frac{\mathbf{Z}_2}{\mathbf{Z}_1};\ I_1 + I_2 = I$$

$$\therefore I_1 = \frac{\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2} \times I, \text{ and } I_2 = \frac{\mathbf{Z}_1}{\mathbf{Z}_1 + \mathbf{Z}_2} \times I.$$

If the ratios of resistance to reactance are not equal, then the graphical construction, as applied to branched circuits, can be used.

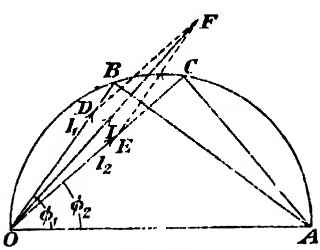


FIG. 221

Assume any drop OA (Fig. 221) between the points P and Q in Fig. 220, and on OA draw a semicircle. Draw chords OB and OC inclined at φ_1 and φ_2, where $\varphi_1 = \text{arc} \tan \mathbf{X}_1/\mathbf{R}_1$, and $\varphi_2 = \text{arc} \tan \mathbf{X}_2/\mathbf{R}_2$. Then these chords represent to scale the respective resistance drops, as before. Dividing them by the respective resistances $\mathbf{R}_1$ and $\mathbf{R}_2$, we obtain quotients proportional to the currents I_1 and I_2. Represent these quotients graphically by the lengths OD and OE, and find the resultant OF. Then OF represents the total current delivered to the load. But this current is I, which is known. Hence, we have

$$I_1 = \frac{OD}{OF} \times I \text{ and } I_2 = \frac{OE}{OF} \times I$$

The resistance drop represented by chord OB is then given numerically by $\mathbf{R}_1 I_1$. This gives the scale by which the total drop OA can be calculated.

We are now in a position to draw the complete vector diagram.

If the load is of power factor cos φ, the total current vector OF will be inclined φ to the direction of the secondary terminal voltage E_2, this direction therefore being given by the line $O'E_2$ in Fig. 222. With A as centre and $E_{2\cdot 0}$, the secondary no-load voltage, as radius, mark off AO' equal to $E_{2\cdot 0}$; then the length $O'O$ gives the secondary voltage on load, and the voltage regulation of the "bank" of transformer is given by

$$\frac{O'A - O'O}{O'A} \times 100$$

As an example consider the case of two transformers in parallel, their characteristics being (*a*) a resistance of 1 per cent, a reactance

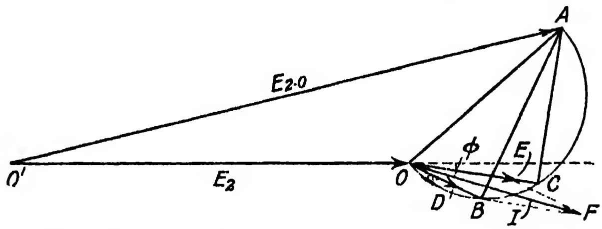


FIG. 222. COMPLETE VECTOR DIAGRAM FOR TWO TRANSFORMERS IN PARALLEL

of 7 per cent, and a capacity of 500 kVA; (*b*) a resistance of 2 per cent, a reactance of 5 per cent, and a capacity of 400 kVA. The load is one of 1,000 kVA at a power factor of 0·8 lagging.

Assume a secondary voltage of 100 giving full-load secondary currents of 5,000 and 4,000 amp. respectively. Hence, with this assumption, we have for the constants

$$\mathbf{R}_1 = \frac{1}{5{,}000} = 2 \times 10^{-4};\ \mathbf{X}_1 = \frac{7}{5{,}000} = 14 \times 10^{-4}$$

$$\mathbf{R}_2 = \frac{2}{4{,}000} = 5 \times 10^{-4};\ \mathbf{X}_2 = \frac{5}{4{,}000} = 12{\cdot}5 \times 10^{-4}$$

$$\varphi_1 = \text{arc tan } 14/2 \quad = 81^\circ\ 53'$$

$$\varphi_2 = \text{arc tan } 12{\cdot}5/2 = 68^\circ\ 12'$$

The drop triangles are drawn to scale in Fig. 223, and working backwards as explained above, we see that the length OA in volts is equal to 6·68 volts, with $E_{2\cdot 0}$ equal to 100, this gives 94·5 volts for E_2, so that the percentage regulation of the bank is 5·5.

We also see that the currents I_1 and I_2 are 4940 and 5150 respectively, and the phases of these currents with respect to E_2 are $\alpha_1 = 45^\circ$ and $\alpha_2 = 32^\circ$ respectively. Hence, for the output of the two transformers, we have

$$W_1 = E_2 I_1 \cos \alpha_1 \times 10^{-3} = 330 \text{ kW}$$

$$W_2 = E_2 I_2 \cos \alpha_2 \times 10^{-3} = 412 \text{ kW}$$

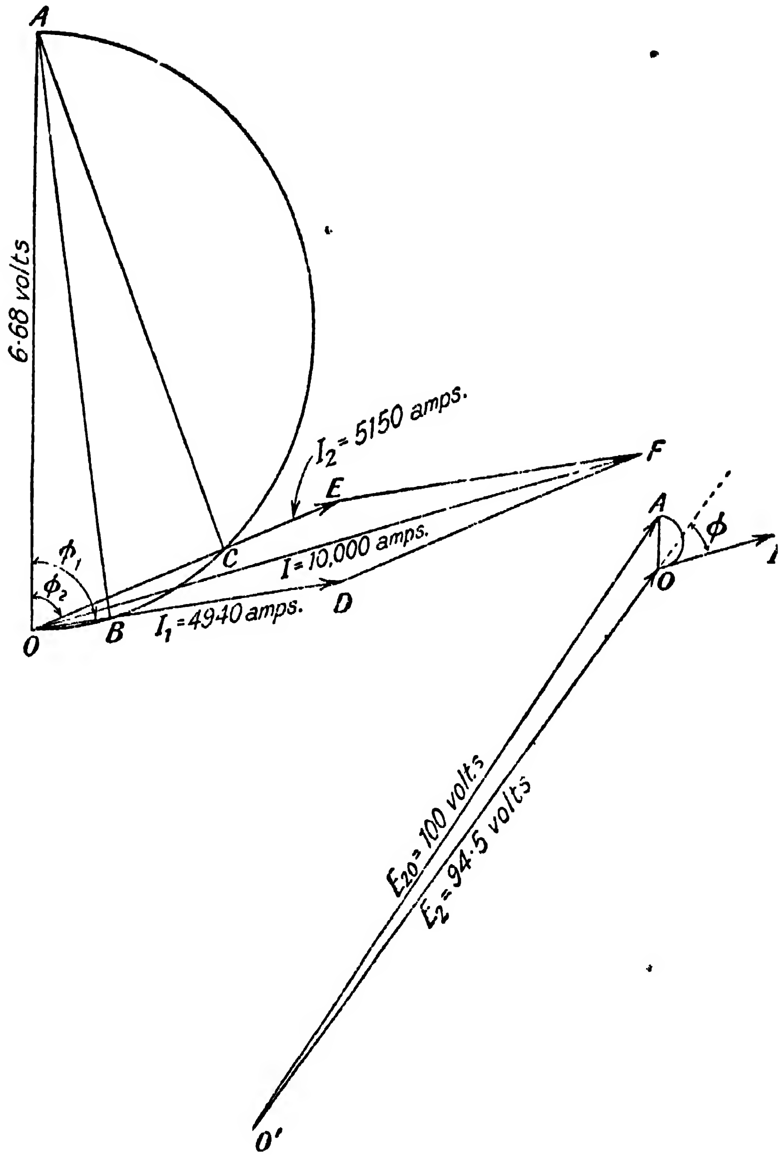

A
6·68 volts
I_2 = 5150 amps.
F
E
A
C
ϕ_1
I = 10,000 amps.
ϕ
ϕ_2
O
O
B
I_1 = 4940 amps.
D
E_{20} = 100 volts
E_2 = 94·5 volts
O'

FIG. 223

The sum of W_1 and W_2 is slightly less than the assumed load of 1,000 kvA, or 800 kW, because the currents are based on a terminal voltage of 100 per cent instead of the actual voltage of 94·5 per cent; the adjustment is easily made by multiplying each by the ratio $E_{2\cdot 0}/E$, and we then have for the corrected loads $W_1 = 350$ kW and $W_2 = 440$ kW. The sum is 790, and the slight error due to the setting off of the angles by means of a protractor.

We will now work out the symbolic method of treating this problem. Take the vector of secondary terminal voltage E_2 as the reference vector, so that

$$\dot{E}_2 = E_2 + j\,.\,0$$

The total load current can be written

$$\dot{I} = I \cos \varphi - jI \sin \varphi = a - jb, \text{ say,}$$

the minus sign being used because the current is assumed lagging with respect to E_2. The symbolic impedances of the two transformers are

$$\dot{z}_1 = \mathbf{R}_1 + j\mathbf{X}_1\,;\ \dot{z}_2 = \mathbf{R}_2 + j\mathbf{X}_2$$

and the currents I_1 and I_2 can be written

$$\dot{I}_1 = x_1 + jy_1\,;\ \dot{I}_2 = x_2 + jy_2$$

Since the drops are the same, we have

$$(\mathbf{R}_1 + j\mathbf{X}_1)(x_1 + jy_1) = (\mathbf{R}_2 + j\mathbf{X}_2)(x_2 + jy_2) \quad . \quad . \quad (1)$$

and since the total current is the resultant of the currents in the two transformers, we have

$$\dot{I}_1 + \dot{I}_2 = \dot{I} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

From (1) and (2) we have

$$\dot{I}_1 = \dot{I}(\mathbf{R}_2 + j\mathbf{X}_2) \div \{(\mathbf{R}_1 + \mathbf{R}_2) + j(\mathbf{X}_1 + \mathbf{X}_2)\}$$

and $$\dot{I}_2 = \dot{I}(\mathbf{R}_1 + j\mathbf{X}_1) \div \{(\mathbf{R}_1 + \mathbf{R}_2) + j(\mathbf{X}_1 + \mathbf{X}_2)\}$$

For the transformers used in the previous graphical construction, we have

$$\dot{\mathbf{Z}}_1 = 2 \times 10^{-4} + j\,14 \times 10^{-4}$$

$$\therefore\ \dot{\mathbf{Y}}_1 = \frac{2 \times 10^{-4} - j\,14 \times 10^{-4}}{(2 \times 10^{-4})^2 + (14 \times 10^{-4})^2} = \frac{2 \times 10^{-4}}{200 \times 10^{-8}} - j\frac{14 \times 10^{-4}}{200 \times 10^{-8}}$$

$$= 10^2 - j\,7 \times 10^2$$

and $$\dot{\mathbf{Y}}_2 = \frac{5 \times 10^{-4} - j\,12{\cdot}5 \times 10^{-4}}{5^2 \times 10^{-8} + 12{\cdot}5^2 \times 10^{-8}} = \frac{5 \times 10^{-4}}{180 \times 10^{-8}} - j\frac{2{\cdot}5 \times 10^{-4}}{180 \times 10^{-8}}$$

$$= 2{\cdot}76 \times 10^2 - j\,6{\cdot}9 \times 10^2$$

Hence, for the whole of the branched circuit

$$\dot{Y} = 10^2 - j\,7{\cdot}0 \times 10^2 + 2{\cdot}76 \times 10^2 - j\,6{\cdot}9 \times 10^2$$

$$= 3{\cdot}76 \times 10^2 - j\,13{\cdot}9 \times 10^2$$

$$\therefore \quad \dot{Z} = \frac{3{\cdot}76 \times 10^2 + j\,13{\cdot}9 \times 10^2}{3{\cdot}76^2 \times 10^4 + 13{\cdot}9^2 \times 10^4}$$

$$= 1{\cdot}81 \times 10^{-4} + j\,6{\cdot}69 \times 10^{-4}$$

Hence, voltage drop along the parallel path

$$\dot{Z}\dot{I} = (1{\cdot}81 \times 10^{-4} + j\,6{\cdot}69 \times 10^{-4})\,(10{,}000 \times {\cdot}8 - j\,10{,}000 \times {\cdot}6)$$

$$= 5{\cdot}47 + j\,4{\cdot}26$$

But $\dot{E}_2 = E_2 + j\,.\,0$

$$\therefore \quad \dot{E}_{2\cdot0} = E_2 + 5{\cdot}47 + j\,4{\cdot}26$$

$$E_{2\cdot0} = \{(E_2 + 5{\cdot}47)^2 + 4{\cdot}26^2\}^{\frac{1}{2}}$$

But $E_{2\cdot0} = 100$

$$\therefore \quad E_2 = 94{\cdot}43 \text{ volts}$$

For the calculation of the current carried by the two transformers, we have

$$\dot{I}_1 = \dot{Y}_1 \times \text{symbolic voltage drop}$$

$$= (10^2 - j\,7 \times 10^2)\,(5{\cdot}47 + j\,4{\cdot}26)$$

$$= 3{\cdot}53 \times 10^3 - j\,3{\cdot}43 \times 10^3$$

$$\therefore \quad I_1 = 10^3(3{\cdot}53^2 + 3{\cdot}43^2)^{\frac{1}{2}}$$

$$= 4{,}930 \text{ amp.}$$

$$\dot{I}_2 = \dot{Y}_2 \times \text{symbolic voltage drop}$$

$$= (2{\cdot}76 \times 10^2 - j\,6{\cdot}9 \times 10^2)(5{\cdot}47 + j\,4{\cdot}26)$$

$$= 4{\cdot}45 \times 10^3 - j\,2{\cdot}60 \times 10^3$$

$$\therefore \quad I_2 = 10^3(4{\cdot}45^2 + 2{\cdot}60^2)^{\frac{1}{2}}$$

$$\therefore \quad I_2 = 5{,}150 \text{ amp.}$$

As a check $\quad \dot{I} = 8 \times 10^3 - j\,6 \times 10^3$

$$\dot{I}_1 + \dot{I}_2 = 3{\cdot}53 \times 10^3 - j\,3{\cdot}43 \times 10^3 + 4{\cdot}45 \times 10^3 - j\,2{\cdot}60 \times 10^3$$

$$= 7{\cdot}97 \times 10^3 - j\,6{\cdot}03 \times 10^3$$

13. **The Transformer as a Mutually Inductive Circuit.** Since practically the whole of the flux produced by a transformer links with both primary and secondary windings, the transformer is an appliance with very "closely coupled" circuits. Each winding has its own separate resistance, R_1 and R_2, and each can be considered as having its own separate leakage inductance, L_1 and L_2, although these are not strictly separable, as has been pointed out before. The circuit is shown in Fig. 224, the load resistance and reactance

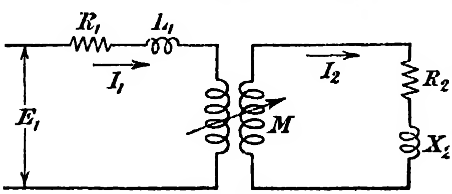


FIG. 224

being included in R_2 and X_2 on the secondary side. The applied primary E.M.F. has to do two things, viz. overcome the impedance drop in the primary and also overcome the mutually induced primary E.M.F. due to the current variations in the secondary. Hence, using the symbolic notation, we have

$$\dot{E}_1 = \dot{I}_1(R_1 + jL_1\omega) + jM\omega\dot{I}_2 \quad . \quad . \quad . \quad (1)$$

The secondary induced E.M.F. is

$$\dot{E}_2 = -jM\omega\dot{I}_1$$

and this has only to supply the total impedance drop in the secondary circuit

$$\therefore \ -jM\omega\dot{I}_1 = \dot{I}_2(R_2 + jL_2\omega) \ . \quad . \quad . \quad . \quad . \quad (2)$$

From equation (2) we have

$$\frac{\dot{I}_2}{\dot{I}_1} = \frac{-jM\omega}{R_2 + jL_2\omega}$$

$$\therefore \frac{I_2}{I_1} = \frac{M\omega}{\sqrt{R_2^2 + L_2^2\omega^2}}$$

Substituting in equation (1) the value of $\dot{I}_2$ obtained from (2), we have

$$\dot{E}_1 = \dot{I}_1(R_1 + jL_1\omega) + \dot{I}_1\frac{M^2\omega^2}{R_2 + jL_2\omega}$$

$$= \dot{I}_1\left\{(R_1 + jL_1\omega) + \frac{M^2\omega^2(R_2 - jL_2\omega)}{R_2^2 + L_2^2\omega^2}\right\}$$

$$= \dot{I}_1\left\{R_1 + R_2\frac{M^2\omega^2}{R_2^2 + L_2^2\omega^2} + j\omega\left(L_1 - L_2\frac{M^2\omega^2}{R_2^2 + L_2^2\omega^2}\right)\right\}$$

Thus the quantity

$$R_1 + R_2 \cdot \frac{M^2\omega^2}{R_2^2 + L_2^2\omega^2}\Bigg) + j\omega\left(L_1 - L_2 \cdot \frac{M^2\omega^2}{R_2^2 + L_2^2\omega^2}\right.$$

is the equivalent impedance of the primary circuit.

$$R_1 + R_2 \cdot \frac{M^2\omega^2}{R_2^2 + L_2^2\omega^2} \text{ is the equivalent resistance,}$$

and $\quad L_1 - L_2 \cdot \dfrac{M^2\omega^2}{R_2^2 + L_2^2\omega^2}$ is the equivalent inductance.

Denote these by r_1 and l_1 respectively, then the equivalent primary reactance is

$$\therefore \ x_1 = l_1\omega$$

$$\therefore \ \dot{I}_1 = \frac{\dot{E}_1}{r_1 + jx_1}$$

$$\therefore\ I_1 = \frac{E_1}{\sqrt{r_1^2 + x_1^2}}$$

But

$$I_2 = I_1 \times \frac{M\omega}{\sqrt{R_2^2 + L_2^2\omega^2}}$$

$$\therefore\ I_2 = \frac{E_1}{\sqrt{r_1^2 + x_1^2}} \times \frac{M\omega}{\sqrt{R_2^2 + L_2^2\omega^2}}$$

We see that $r_1 > R_1$ and $l_1 < L_1$, so that the effect of putting load on a transformer is to increase the apparent resistance and to decrease the apparent inductance.

14. Testing of Transformers. The various transformer tests are as follows—

(*a*) Core Loss Tests. The primary is connected to a supply of normal frequency and voltage, through a wattmeter W, as shown in Fig. 225, and the secondary is left on open circuit. Since the core loss is independent of the load, and since the primary no-load current I_0 is so small that the copper loss due to it can be neglected, we have

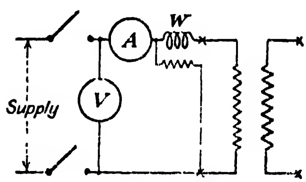


Fig. 225

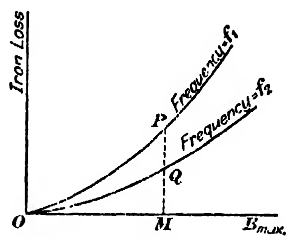


Fig. 226

Core loss = wattmeter reading = W_i, say

Power factor $\cos\varphi_0 = \dfrac{W_i}{EI_0}$

Working component $I_\omega = I_0 \cos\varphi_0$

Magnetizing compound $I_\mu = I_0 \sin\varphi_0$

Since the power factor $\cos\varphi_0$ will be low, it will be necessary to correct the wattmeter reading. (*See* Chap. XIV.)

If the dimensions of the core and the number of primary turns are known, the hysteresis and eddy current losses can be separated in the following way. We have, from the transformer E.M.F. equation,

$$B_{max} = \frac{E \times 10^8}{4{\cdot}44\, AN_1 f}$$

The applied voltage E is varied over as wide a range as possible, the

frequency being kept constant at, say, f_1. The iron loss, as measured by the wattmeter, is then plotted against the calculated values of B_{max} (Fig. 226). The test is then made at a second frequency f_2, thus giving two curves. For a given value of B_{max} the Hysteresis loss is proportional to the frequency, and the eddy current loss to the square of the frequency. Hence, if we take any ordinate PQM, we have

Total loss at frequency f_1

$$W_1 = Af_1 + Bf_1^2$$

While total loss at frequency f_2

$$W_2 = Af_2 + Bf_2^2, \text{ } A \text{ and } B \text{ being constants.}$$

From these equations A and B can be calculated. The separate losses at any frequency, *for the value of B_{max} equal to OM*, can then be calculated. By drawing a series of ordinates and making the above analysis for each, the losses at any value of B_{max}, as well as any value of the frequency, can be calculated.

It is usual to make the above test, not on the actual transformer itself, but on a test specimen built up of strips of transformer iron. In Epstein's apparatus the specimen is arranged in a square, the standard dimensions of a strip being 50 cm. × 3 cm. The total weight of strip used in building up the specimen is 10 kg. The arrangement of the specimen and the magnetizing coils is clearly shown in Fig. 227.

(*b*) IMPEDANCE AND COPPER LOSS TESTS. The secondary is short-circuited by a heavy conductor, as shown in Fig. 228, and a low voltage applied to the primary, the voltage being adjusted until the ammeter A indicates that the full load current is flowing. The losses now taking place will be the full load copper loss in both primary and secondary, and a small amount of iron loss. The voltage applied has to be so small that this iron

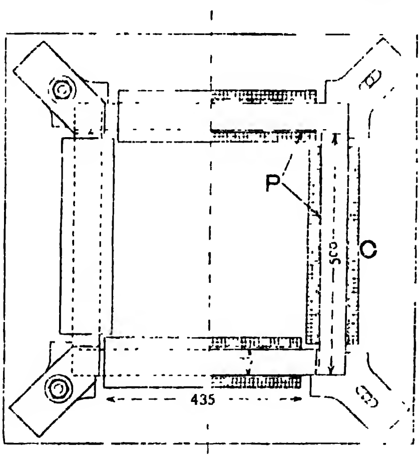


FIG. 227

EPSTEIN'S IRON TESTING APPARATUS

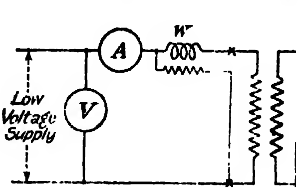


FIG. 228

loss is quite negligible. Hence, if W_s is the wattmeter reading, E_s the voltage applied, and I_s the current, then

$$\text{Full load copper loss} = W_s$$

Remembering that the copper loss is proportional to the square of the current, the copper loss at any load can be calculated. If it is required to determine the copper loss only, it is not absolutely essential that the supply should be of normal frequency because the copper loss (I^2R) is independent of frequency. If the windings are of very heavy section it is preferable that normal frequency should be used, because eddy currents are set up in such windings, and the additional loss due to them is a part of the total copper loss.

The impedance of the transformer can also be determined from the short circuit test, but in this case it is imperative that the voltage should be of normal frequency. If the voltage is applied to the primary,

$$\mathbf{Z}_1 = \frac{E_s}{I_s}$$

Also, from $\cos\varphi_s = \dfrac{W_s}{E_s I_s}$, we have

$$\mathbf{R}_1 = \mathbf{Z}_1 \cos\varphi_s\,; \quad \mathbf{X}_1 = \mathbf{Z}_1 \sin\varphi_s$$

Similarly, if the test is made with the primary short-circuited.

(c) LOAD TEST. The efficiency can be determined with considerable accuracy from the results of the open and short circuit tests, while full load tests are made to determine the temperature rise. A small transformer can be put on full load by means of an artificial load, say, a water resistance, but for large units a regenerative test is almost imperative, partly because of the saving in energy, but mainly because of the difficulty in arranging an artificial load to absorb very large amounts of power. The regenerative test on two transformers is called the Sumpner test, and sometimes the "back-to-back" test. Two similar transformers have their primaries connected to a supply of normal voltage and frequency as in Fig. 229. The wattmeter W_1 gives the total core losses. The two secondaries are connected together, care being taken to have

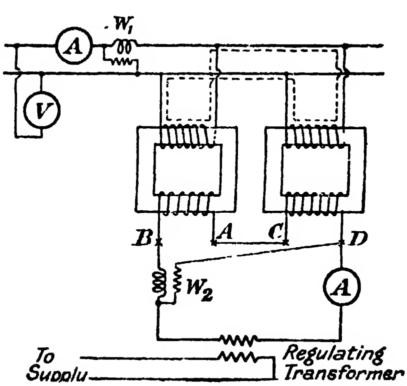


FIG. 229

their "polarities" in opposition. This is done as follows. Terminals A and C are connected together, and a voltmeter of range double the voltage of either transformer is connected across BD. If no reading is obtained, the voltages in the two secondaries are in opposition, and the terminals BD can be safely connected. If a reading is obtained, the voltages are acting in the same direction, so that it will be necessary to connect A to D. In the case of high voltage transformers, it is common practice to use a bank of incandescent lamps in series instead of a voltmeter, care being taken, of course, to see that all the lamps are sound. If the two

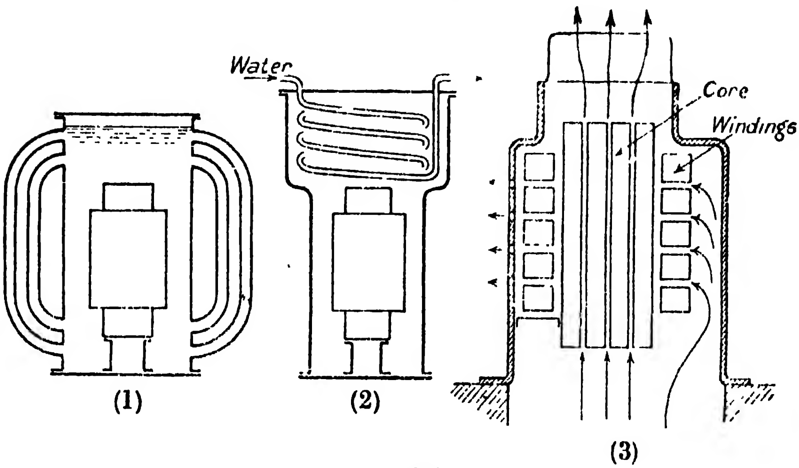


Fig. 230

Methods of Cooling Transformers

(1) Tank with external tubes (2) Water cooling (3) Air blast cooling

secondaries are connected in opposition no secondary current will flow, and to produce this current it is necessary to inject an E.M.F. into the secondary circuit by means of a regulating transformer This injected voltage is adjusted until full load secondary current is flowing. This will induce full load currents in the primaries, but the primary currents will circulate through the bus-bars, as shown by the dotted path, and the reading of W_1 will be unaffected. The wattmeter W_2 will give the total copper losses, and thus, knowing both iron and copper losses, the efficiencies can be calculated. The advantage of the test is that two large transformers can be put under full load conditions for several hours, so that the temperature rise can be measured, with an expenditure of energy equal to that required by the losses only.

The Sumpner test thus necessitates two transformers, and these are not always available. If a single transformer is to be tested,

then provided another transformer of the same voltage and the same, or greater, capacity is available, the test can be carried out. Failing this, the single transformer can be tested as follows. One side is short-circuited and a voltage applied to the other side of such value that the transformer intake is equal to the sum of the normal copper and iron losses. It is obvious that with this method the losses are almost entirely copper losses, with the result that the temperature rise of the copper will be greater than normal, and that of the iron less than normal. On the other hand, the temperature rise of the oil in which the transformer is immersed will be of normal value, the test thus being sufficient for the checking of a specification giving a figure for the temperature rise of the oil, as is usual.

Other tests which can only be applied to three-phase transformers are described in Chapter XIV.

15. **Methods of Cooling Transformers.** Since there are no rotating parts which induce a ventilating draught, transformers are more difficult to cool than rotating electrical machines. For small outputs, up to, say, 20 kW, the external surface is sufficient to enable the heat produced by the losses to be dissipated by radiation, but for larger sizes additional means of carrying away the heat have to be provided. The various methods are illustrated in Fig. 230. For outputs up to 500 kW it is sufficient to place the transformer in a tank of oil, and to provide as large a cooling surface as possible by (*a*) corrugating the sides of the tank, or (*b*) using a boiler plate tank provided with external tubes. This second method is being used very extensively at the present time. For still larger outputs the above method would necessitate a very large tank, and it is therefore common to use a small tank with a worm immersed in the oil, through which cooling water is passed. The disadvantage of this is that the water is under pressure, and if any flaws develop in the worm, water will find its way to the oil. It is therefore preferable to have the oil at higher pressure than the water, as in the case of the external oil cooler. The oil is pumped through a cooler very similar to a condenser, and is cooled by low pressure water. Where a cheap water supply is not available, as is often the case in substations, the transformer can be cooled by an air blast. In such a case it is necessary to filter the air, or dust particles will in time clog up the ventilating ducts. The main disadvantage in air-blast cooling is that the very valuable increase in insulation strength due to immersion in oil is lost.

16. **Heating-Time Curve.** With very large transformers a very long time is required for the temperature to attain its final value, and it is not convenient to leave the transformer on test for the whole of this time. If the temperature is taken at regular intervals for a few hours, then the final temperature can be calculated from the initial rate of rise so determined.

Let W = full load losses in copper and iron (watts)

H = amount of energy in joules required to raise the temperature of the whole transformer by 1° C.

K = watts dissipated by the transformer and its cooling appliances per degree rise in temperature

θ_m = final temperature rise attained

Then $W = K\theta_m$ by Newton's law of cooling.

Now at any temperature rise θ,

$$\text{(Rate of production of heat)} = \text{(Rate of storage of heat)} + \text{(Rate of dissipation of heat)}$$

$$\therefore\ W = H\frac{d\theta}{dt} + K\theta$$

the solution of which is

$$\theta = \frac{W}{K} + A\,.\,\varepsilon^{-\frac{K}{H}t}.$$

where A is a constant and t is the time in seconds.

Now when

$$\theta = 0,\ t = 0,\ \therefore\ A = -\frac{W}{K}$$

$$\therefore\ \theta = \frac{W}{K}\left(1 - \varepsilon^{-\frac{t}{B}}\right)$$

where $B = \dfrac{H}{K}$

B is called the "temperature-time" constant. Its components H and K can be calculated from the data of the transformer, and therefore, B is known. The equation for θ shows that the curve of θ against t is an exponential curve as shown in Fig. 231. The final temperature is theoretically attained when $t = \infty$. We therefore have

$$\theta_m = \frac{W}{K}$$

Differentiating the equation for θ with respect to t we have

$$\frac{d\theta}{dt} = \frac{W}{KB}\,.\,\varepsilon^{-\frac{t}{B}}$$

$\therefore$ Initial rate of rise of temperature

$$\left(\frac{d\theta}{dt}\right)_0 = \frac{W}{KB} \text{ when } t = 0$$

$\therefore$ From $\theta_m = \dfrac{W}{K}$

We have $\theta_m = B \times \left(\dfrac{d\theta}{dt}\right)_0$

$= $ (Temp.-time constant) $\times$ (Initial rate of rise of temperature)

If, after attaining its final temperature, the transformer is allowed to cool down, the curve of fall of temperature is complementary to the curve of rise of temperature. We then have

$$\theta = \frac{W}{K} \cdot \varepsilon^{-\frac{t}{B}}$$

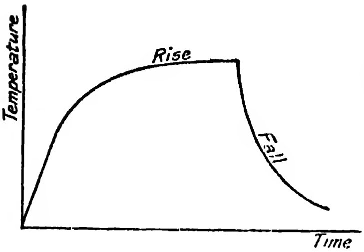


FIG. 231

Now, when carrying out a heat run on a small machine, the final temperature will be attained in a comparatively short time, say five or six hours, but in the case of a very large machine twenty-four hours, or even longer, may be required. In such a case it is desirable to be able to determine the final temperature from the results of a run of short duration. One method of doing this is as follows. We have seen that

$$W \quad H\frac{d\theta}{dt} + K\theta$$

$$\therefore \quad \frac{d\theta}{dt} = \frac{W}{H} - \frac{K}{H} \cdot \theta$$

$$= \frac{K}{H}\left(\frac{W}{K} - \theta\right)$$

$$= \frac{K}{H}(\theta_m - \theta)$$

$$\propto (\theta_m - \theta)$$

In other words, the slope of the curve at any instant is proportional to the difference between the actual temperature at that instant and the final temperature. Thus, in Fig. 232, the point P is taken at a temperature rise equal to half the final rise, and φ is taken at three-quarters of the final rise. The point O is at the commencement, for which there is no temperature rise. Hence, the slope of the tangent at O is proportional to the final rise θ_m, the slope of the tangent at P is proportional to $\frac{1}{2}\theta_m$, and that at Q is proportional

to $\frac{1}{4}\theta_m$. The procedure is therefore as follows. Observe the initial rate of rise from the results of the test of short duration, and determine the slope at successive periods until a point is reached at which

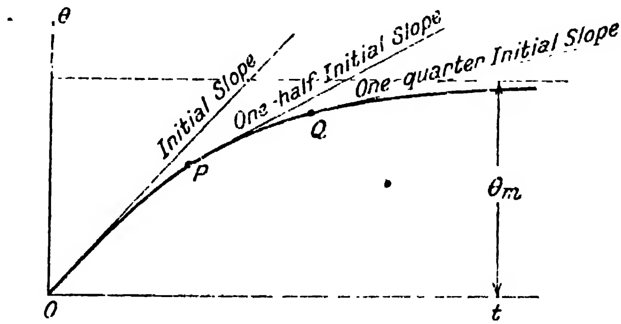


FIG. 232

the slope is half the initial slope. Then the final rise of temperature is twice the rise of temperature reached at that point. As a check on the accuracy of the determination of the slopes at various instants, the curve of $(\theta - d\theta/dt)$ can be plotted against time, and this should be a straight line.

Example. A 100 kVA transformer has a temperature-time constant of one hour. The ratio of iron loss to copper loss at full load current is 1 to 2, and the temperature rise on normal full load is 50° C. What overload will produce a temperature rise of 60° C. at the end of a run of 2 hours, starting from cold?

We have
$$\theta = \theta_m\left(1 - \varepsilon^{-\frac{t}{K}}\right)$$

Hence under overload conditions

$$60 = \theta_m\left(1 - \varepsilon^{-\frac{120}{60}}\right)$$
$$= \theta_m(1 - \varepsilon^{-2})$$
$$= \cdot 865\theta_m$$
$$\therefore\ \theta_m = \frac{60}{\cdot 865} = 69\cdot 4^\circ \text{ C.}$$

But
$$\theta_m = \frac{W}{K}$$

showing that θ_m is proportional to the total loss

$$\therefore\ \frac{\text{Total loss on overload}}{\text{Total loss on normal load}} = \frac{69\cdot 4}{50}$$
$$= 1\cdot 39$$

Denote the full load by unity and the total load at overload by x, also let W_i = iron loss.

Then $$2W_i = \text{copper loss on normal load}$$
and $$2W_i \times x^2 = \text{copper loss on overload}$$

$$\therefore \quad \frac{W_i + 2W_i}{W_i + 2W_i \times x^2} = \frac{1}{1{\cdot}39}.$$

$$\frac{3}{1 + 2x^2} = \frac{1}{1{\cdot}39}$$

which gives $$x = 1{\cdot}26.$$

Hence a 26 per cent overload will give a temperature rise of 60° C. in 2 hours.

17. **The Auto-Transformer.** An auto-transformer has one winding only as shown in Fig. 233, the secondary voltage being obtained by

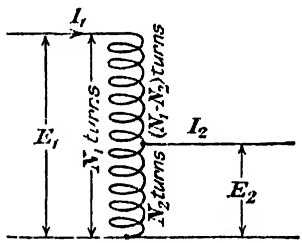


FIG. 233

tapping off at any convenient point. As in the ordinary transformer, the transformation ratio is equal to the turn ratio.

$$\frac{E_2}{E_1} = \frac{N_2}{N_1}$$

For the same capacity and voltage ratio the auto-transformer requires less copper than an ordinary transformer. The cross section of a conductor is proportional to the current carried, and the length of conductor in a winding is proportional to the number of turns. Hence, weight of copper in a winding is proportional to the product of the current and the number of turns.

(*a*) ORDINARY TRANSFORMER.

Weight of copper on primary $\propto N_1 I_1$

Weight of copper on secondary $\propto N_2 I_2$

Total weight of copper $\propto (N_1 I_1 + N_2 I_2)$

(*b*) AUTO-TRANSFORMER.

The top section of $(N_1 - N_2)$ turns carries current I_1

$\therefore$ Weight of copper in this section $\propto (N_1 - N_2) I_1$

The bottom section of N_2 turns carries a current of $(I_2 - I_1)$ because of the opposition of primary and secondary currents.

$\therefore$ Weight of copper in bottom section $\propto N_2(I_2 - I_1)$

$\therefore$ Total copper in auto-transformer $\propto$

$$(N_1 - N_2)I_1 + N_2(I_2 - I_1), \propto (N_1 - 2N_2)I_1 + N_2I_2$$

$$\therefore \frac{\text{Weight of copper on auto-transformer}}{\text{Weight of copper on ordinary transformer}} = \frac{(N_1 - 2N_2)I_1 + N_2I_2}{N_1I_1 + N_2I_2}$$

$$= \frac{N_1/N_2 - 2 + I_2/I_1}{N_1/N_2 + I_2/I_1}$$

Now $\quad N_1/N_2 = I_2/I_1 = m$, say, where $m = 1/K$

$$\therefore \quad \text{Ratio of weight of copper} = \frac{2m - 2}{2m} = \frac{m - 1}{m}$$

$$= 1 - K.$$

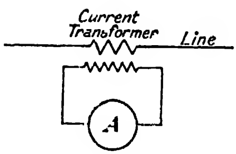


FIG. 234

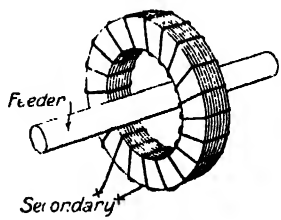


FIG. 235

Because of the saving of copper, auto-transformers are occasionally used where a transformation ratio not very different from unity is required; but where one side is at very high tension compared with the other, it is preferable to have the two sides electrically separate and to use ordinary transformers. Auto-transformers therefore have the biggest sphere of usefulness as regulating transformers. They are also sometimes used as boosters to raise the voltage in an alternating current feeder.

18. Use of Transformers with Measuring Instruments. (*a*) When measuring large currents in a direct current circuit it is usual to pass the current through a shunt, and to measure the P.D. across the shunt by means of a milli-voltmeter. This is not convenient with alternating current instruments, except hot-wire instruments, and it is therefore usual to pass the current to be measured through the primary of a "current" transformer, the secondary being connected to the ammeter (Fig. 234). Since ammeters are of very low resistance it will be seen that a current transformer normally

works short-circuited, and if for any reason the ammeter is taken from the circuit the secondary terminals must be short-circuited. If this is not done the primary winding will act like a choker connected in series with the line. The M.M.F. of the primary current will be unopposed and the core will carry an abnormally high flux, thus inducing high E.M.F.s in both primary and secondary windings. For the measurement of the very heavy currents carried by bus-bars or feeders the conductor itself constitutes the primary, as shown in Fig. 235.*

(*b*) When measuring very high pressures, the pressure is stepped down by means of a "pressure" transformer, the low tension secondary of which is connected to the voltmeter. For switchboard use a voltmeter works practically all the time on one part of the scale, and if the transformer gives the correct ratio at this

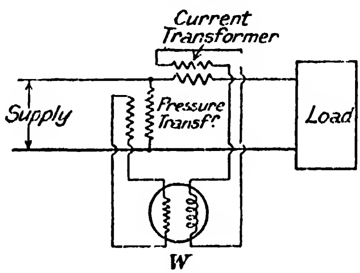


FIG. 236

point, a variation of the ratio at other points is not of much account. On the other hand an ammeter has to be correct at all points on the scale, and therefore, the current ratio of a current transformer must be the same at all loads

(*c*) When measuring power on a high pressure system both current and pressure transformers are used, the way in which they are connected in the circuit being illustrated in Fig. 236.

19. Transformation from Three to Two Phase. SCOTT'S METHOD. In some cases, mainly for electric furnace work, it is desirable to work with two phase current. At the present time bulk supplies are invariably three phase, and it is therefore necessary to convert from three to two phase. This can be done by means of two single phase transformers connected in "T," as in Fig. 237.

Let E_1 = voltage across each phase of the two phase side. Then, assuming unity transformation ratio in transformer *I*, we have $E_1/2$ across *CD* and *DA*. Hence, for the voltages on the three phase side, we have, if *K* is the transformation ratio in transformer II,

* The theory and use of instrument transformers are treated exhaustively in *Instrument Transformers*, by Hague (Pitman).

$$E_{AB} = \frac{E_1}{2} + K \cdot E_1, \text{ vector sum}$$

$$\cdot \; E_{BC} = -KE_1 + \frac{E_1}{2}, \text{ vector sum}$$

$$E_{CA} = -\frac{E_1}{2} - \frac{E_1}{2} = -E_1, \text{ vector sum}$$

Now on the two phase side the two voltages E_1 must have 90° phase difference. Hence, the complete vector diagram is as shown in Fig. 231. The three phase vectors must be 120° apart.

$$\therefore \angle POQ = 60°, \quad \therefore PQ = \sqrt{3} \times OQ = \frac{\sqrt{3}}{2} E_1$$

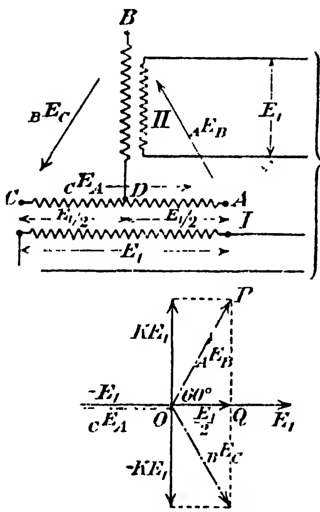


FIG. 237
SCOTT CONNECTIONS

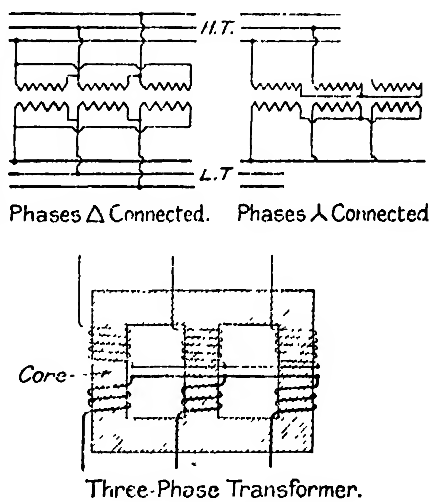


FIG. 238
THREE-PHASE TRANSFORMER

But PQ is equal to KE_1. Hence $K = \sqrt{3}/2$; and therefore, the coil BD of transformer II must have 0·866 as many turns as the coil CA in transformer I.

20. Three-phase Transformation. The ordinary transformation of three-phase current at one voltage to three-phase current at another voltage can be effected either by means of a single three-phase transformer, or by three separate single-phase transformers, as shown in Fig. 238. The advantage of using three separate single-phase transformers is that only one-third of the total output need be kept as spare, an equal single-phase transformer being

substituted for a faulty one, in the case of breakdown. With a three-phase transformer it is necessary to have another three-phase transformer of full output as a spare. If the three single-phase transformers are mesh connected, then, in the event of a fault developing in one of them, it can be removed from service, the two remaining transformers supplying the load in what is called "open delta" connection. This does not apply to the three single-phase transformers connected in star.

With one three-phase transformer, or a "bank" of three single-phase transformers, the primary may be star connected and the secondary mesh, or *vice versâ*. If a number of three-phase transformers, or a number of banks of single-phase transformers, are worked in parallel, then there are certain methods of connection which cannot be employed. A few arrangements are tabulated below—

1st Transformer.	Primary Δ,	Secondary ⅄	Impossible
2nd Transformer.	Primary Δ,	Secondary Δ	
1st Transformer.	Primary ⅄,	Secondary ⅄	Possible
2nd Transformer.	Primary Δ,	Secondary Δ	
1st Transformer.	Primary ⅄,	Secondary Δ	Possible
2nd Transformer.	Primary Δ,	Secondary ⅄	

21. **Tests on Three-phase Transformers.** The determination of the core loss and magnetizing current is made by means of an open-circuit test as in the case of single-phase transformers. The power can be measured by the two-wattmeter or the three-wattmeter method, or, alternatively, a single wattmeter used in conjunction with a transfer switch can be used. If the reader will work out the vector diagrams for the two-wattmeter and three-wattmeter methods, he will find that, owing to the low transformer power factor on open circuit, the *meter* power factor will be lower in the three-wattmeter than in the two-wattmeter method, and consequently the latter method is preferable.

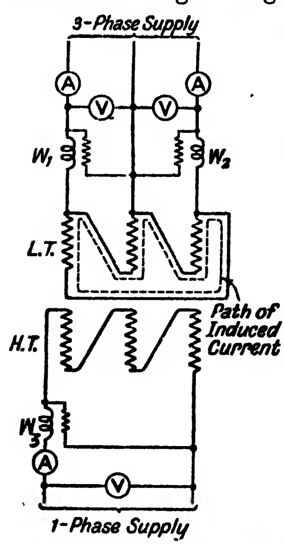


FIG. 239. CONNECTIONS FOR DELTA-DELTA RUN

The short-circuit test is also carried out by the two-wattmeter method, and the resistance, reactance, and impedance are worked out per phase.

For the purpose of a heat run the short-circuit test can also be used, the applied voltage being increased so that the transformer intake is equal to the sum of the normal iron and copper losses. As

indicated in the section dealing with single-phase transformers, this method has the defect that the temperature rises of the iron and copper are not normal. This difficulty can be overcome in the case of three-phase transformers by what is called a delta-delta run. Whatever the normal connections of the phases, both primary and secondary phases are connected in delta, and the iron losses are supplied from a three-phase supply of normal phase voltage and frequency, while the copper losses are derived from a single-phase supply. The connections for the test are shown in Fig. 239. Since

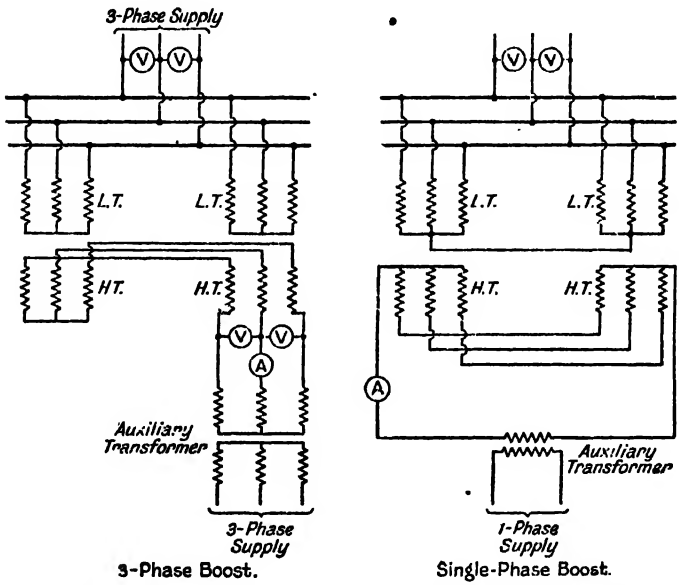


FIG. 240. ALTERNATIVE CONNECTIONS FOR REGENERATIVE TEST ON TWO THREE-PHASE TRANSFORMERS

the three-phase supply is at correct voltage and frequency, the sum of the wattmeter readings, W_1 and W_2, will be the iron loss. The high-tension winding is connected in open delta, and is connected to a single-phase supply, whose voltage, to produce normal full-load current, must be equal to three times the impedance voltage per phase referred to the high-tension side. The ammeter in this circuit will then indicate full-load current, and this current will induce full-load current in the low-tension winding, this induced current being a circulating current, as indicated by the dotted path, and therefore not affecting the readings of wattmeters W_1 and W_2. Hence, wattmeter W_3 will indicate the total copper loss. This test is suitable for low reactance transformers, but not for high reactance transformers, because the single-phase loading will set up abnormal

leakage fluxes, which, in turn, will set up additional losses due to abnormal eddy current in the windings, ironwork, and tank.

When two three-phase transformers are available, the Sumpner test can be carried out as with two single-phase transformers, the iron loss is supplied to one side, usually the low-tension, while the copper loss is supplied to the other side by means of a boosting transformer. In the three-phase case this boost can be carried out either single-phase or three-phase, the connections for both tests being given in Fig. 240. The single-phase boost is, of course, the simpler, but it should only be used with transformers of low impedance, or with banks of three single-phase transformers, for the reasons given in connection with the delta-delta run.

EXAMPLES ON CHAPTER XV.

(1) An alternate current transformer has its primary winding connected with mains whose voltage varies according to a sine law, the frequency being 50. The secondary coil has 50 turns and gives 100 volts on open circuit. The section of the transformer core is 20 sq. in. Determine the maximum value of the induction density in the core. Prove the formula used. (London Univ., 1916.) *Ans.*—45,200 lines per sq. in.

(2) A 100 kVA transformer with a secondary voltage of 400 has a resistance referred to the secondary of 0·016 ohm, the iron loss of 1,000 watts. Plot its efficiency against secondary current for power factors of 1·0 and 0·8 lagging. Calculate the secondary current at which the efficiency is a maximum. *Ans.*—250 amps.

(3) A 100 kVA transformer stepping down from 2,000 to 400 volts has a primary resistance of 0·17 ohm and a secondary resistance of ·0068; the reactances are ·25 and ·01 ohm respectively. Calculate the resistance, reactance, and impedance referred to the secondary, and hence find the per cent regulation on full secondary load of 250 amp. at a power factor of 0·8 lagging. Check the regulation by the Kapp diagram.

Ans.—·0136, ·02, ·024 ohms., 1·43 per cent.

(4) Give the theory of the open and short circuit method of testing a transformer, and show how from the measurement taken it is possible to calculate the efficiency and percentage drop of secondary voltage for a load of known magnitude and power factor.

A transformer to convert 10 kW from 2,000 to 100 volts when tested by the above method showed losses which for the open circuit test at normal voltage were 200 watts at power factor 0·7, and which, for the short circuit test at full load current, were 250 watts at power factor 0·25. Calculate the efficiency of the transformer for an output consisting of a full load secondary current (lagging) at power factor 0·6. (London Univ., 1915.)

Ans.—95·5 per cent.

(5) Calculate the percentage regulation of the transformer in question 4 when working on the stated load. *Ans.*—9·3 per cent.

(6) Two 100 kW single phase transformers are connected in parallel both on the primary and secondary. One transformer has an ohmic drop of ½% at full load and an inductive drop of 8% at full load current, zero power factor. The other has an ohmic drop of ¾% and an inductive drop of 4%.

Show how they will share the following loads: (*a*) 180 kW at 0·9 power factor; (*b*) 120 kW at 0·6 power factor; (C. and G.)

Ans.—$68\frac{1}{3}$ and $111\frac{2}{3}$ kW.; (*b*) 42 and 78 kW.

(7) How would you test the suitability for banking of two transformers (i.e. connecting in parallel) of which the no load ratios of transformation are equal and the regulation is unknown. (C. and G., 1922.)

(8) A system supplies a load which varies in a period of 24 hours as follows: 6 a.m. to 12 noon, 1,000 kW; 12 noon to 1 p.m., 100 kW; 1 p.m. to 6 p.m., 1,000 kW; 6 p.m. to 6 a.m., 300 kW. Energy is transmitted over a line in which the losses at full load are 5,000 watts, and is transformed by a transformer having a no load loss of 6,000 watts, and a full load copper loss of 8,000 watts. Calculate the total losses in transmission and conversion during the 24 hours (C. and G., 1922.)

Ans.—301·2 kW. hours.

(9) A 1,000 kVA and a 500 kVA single phase transformer are connected to the same bus-bars on the primary side. The secondary pressures at no load are 500 and 510 volts respectively. The impedance voltage of the first transformer is 3·4 per cent and of the second 5 per cent. What cross current will pass between them when the secondaries are connected together in parallel? Assuming that the ratio of resistance to reactance is the same in each, what currents will flow in the secondary windings of the transformers when supplying a total load of 1,200 kVA? (C. and G., 1923.)

Ans.—294, 1,750, and 730 amps. (assuming load of unit power factor).

(10) Two electric furnaces are supplied with single-phase current at 80 volts from a three-phase 11,000 volt system by means of two single-phase Scott connected transformers, with similar secondary windings. When the load on one transformer is 500 kW and on the other, 800 kW, what current will flow in each of the three-phase lines (1) at unity power factor; (2) at 0·5 power factor? Neglect phase displacement in, and efficiency of, the transformers. (C. and G., 1921.)

Ans.—(1) 73, 73, and 53 amp.; (2) 146, 146, and 105 amp.

CHAPTER XVI

ROTATING MAGNETIC FIELDS

1. THE magnetic field produced by a single phase current is an alternating field, and we have seen that this can be resolved into two rotating fields travelling in opposite directions. Hence, if one of these fields can be eliminated, or if it is not produced at all, then a pure rotating field will be left. This necessitates polyphase currents.

2. Consider two coils placed 90° apart and connected, one to each phase of a two phase supply, as in Fig. 241. Each coil sets up an alternating magnetic field along its own axis, and each of these fields can be split up into forward and backward travelling components, having an angular velocity of ω, where $\omega = 2\pi f$. Consider the instant the current in coil I is a maximum. Then its magnetic field is a maximum and in consequence the two rotating components, 1 and 2, are each directed along OX. At the same instant the current in coil II is 90° behind its maximum, so that its component fields, 3 and 4, have each to travel 90° before they point together along OY. Hence, they are along OX, but pointing in opposite directions. Now components 2 and 4 are both travelling in a clockwise direction with the same angular velocity. They are thus always opposed, so that they neutralize each other at every instant.

We are thus left with components 1 and 3, which combine together to produce a pure rotating field. It will thus be seen that the direction of rotation of the resulting field is the same as the direction of phase sequence in the two coils; in fact, the resulting field points along the axis of any coil at the instant the current in that coil is a maximum. Thus, to reverse the direction of the field, it is necessary to reverse the phase sequence, which can be done by reversing the connections to one of the coils.

3. Now consider three coils arranged 120° apart, each supplied from one phase of a three phase supply, as in Fig. 242. Consider the instant the current in coil I is a maximum. Then its two rotating fields, 1 and 2, will at that instant be directed along the axis OA. The current in coil II is 120° behind its maximum value, and therefore, each of its component rotating fields, 3 and 4, has to travel 120° before they point together along the axis OB. Hence, one of them will at that instant point along OA, and the other along OC, as shown. Similarly, the rotating fields 5 and 6, due to the current in coil III, have each to travel 240° before they point together along OC. Hence, field 5 is along OA, and 6 along OB.

Collecting the clockwise fields we see that these are 120° apart,

and therefore neutralize one another at every instant. The counter-clockwise fields all point in the same direction and they therefore combine to produce a pure rotating field.

In this case also, the direction of rotation is the same as the direction of phase sequence, first coil I has its maximum current, then II, and then III. Thus, to reverse the direction of rotation of the field the phase sequence must be reversed. The three coils will be either star or mesh connected, and all that is necessary is to reverse two of the connections to the three terminals.

4. The above fields are bi-polar fields, that is, they have one N. and one S. pole. In a polyphase armature winding the various

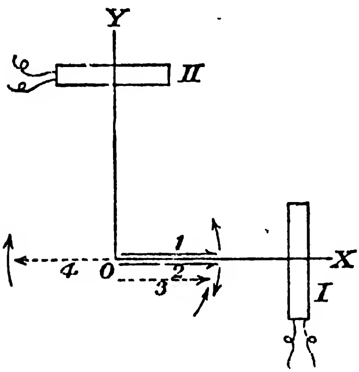


Fig. 241

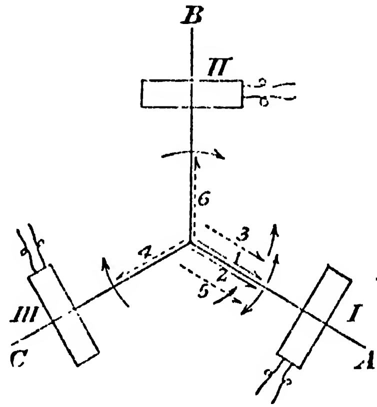


Fig. 242

Production of a Rotating Magnetic Field

phases are separated by an angular distance in space equal to the electrical phase difference between the currents in them, so that such windings, when traversed by polyphase currents, produce a rotating field. In this case the rotating field has as many poles as the machines for which the winding is designed. It will thus be seen that the armature reaction in polyphase alternators consists of a rotating magnetic field which keeps pace with the poles of the rotor. The armature reaction is therefore uniform, instead of pulsating as in a single phase alternator. Similarly, the torque produced by a polyphase synchronous motor is uniform, because the pure rotating field set up by the polyphase armature currents has no backward rotating component. This property of the motor enables it to be rendered self-starting by means of the dampers on the pole faces, the action of which in this connection is explained fully in Chapter XXI.

5. Let each of the coils in Fig. 242 produce an alternating field of maximum strength H along its axes, then, as with alternating current

and voltages, we have the fundamental equation giving the instantaneous value with respect to time

$$h = H \sin \omega t$$

Take as zero time the instant that the rotating vector giving the field H_1 is directed along OA, then we have

$$\left.\begin{aligned} h_1 &= H_1 \sin \omega t \\ h_2 &= H_2 \sin (\omega t + 120) \\ h_3 &= H_3 \sin (\omega t + 240) \end{aligned}\right\}$$

or

$$h_1 = H_1 \sin \omega t$$

$$h_2 = - H_2 \left(\tfrac{1}{2} . \sin \omega t - \frac{\sqrt{3}}{2} \cos \omega t \right)$$

$$h_3 = - H_3 \left(\tfrac{1}{2} \sin \omega t + \frac{\sqrt{3}}{2} \cos \omega t \right)$$

Now resolve these along OA and a perpendicular axis, and denote by $\overline{X}$ and $\overline{Y}$ the total components in these two directions. Then

$$\overline{X} = h_1 + h_2 \cos 120 + h_3 \cos 240$$

$$\overline{Y} = 0 + h_2 \sin 120 + h_3 \sin 240$$

Now put $H_1 = H_2 = H_3 = H$, and we have

$$\overline{X} = \frac{3}{2} H \cos \omega t$$

$$\overline{Y} = \frac{3}{2} H \sin \omega t$$

Hence, for the resultant field, we have

$$H_r = (\overline{X}^2 + \overline{Y}^2)^{\frac{1}{2}}$$

$$= \frac{3}{2} H$$

showing that the field is of constant strength, since its magnitude is independent of time.

For its direction θ with respect to the OA axis, we have

$$\tan \theta = \frac{\overline{Y}}{\overline{X}} = \frac{\sin \omega t}{\cos \omega t} = \tan \omega t$$

$$\therefore \quad \theta = \omega t$$

In other words, the angle θ is proportional to the time, and therefore the resultant field H_r must rotate in space with a uniform angular velocity of ω. In the example taken the direction of rotation of this field is counter-clockwise, because the sequence with which the component alternating fields go through their changes in magnitude is in the order I, II, III. If it is required to reverse the direction of rotation of the field it is necessary to reverse this sequence, e.g.

to change it to I, III, II. This is very easily accomplished by changing over two of the connections to the coils. Thus, if the coils are star connected, the leads to coils *B* and *C* can be changed over.

The nature of the rotating magnetic field produced by the windings of a polyphase machine is discussed on p. 310.

6. The Three Phase Power Factor Meter. This instrument is an application of the principle of the rotating field, one form of the instrument being shown in Fig. 243. Three fixed coils are arranged radially, each coil being connected to one line conductor, if necessary, through current transformers. The currents flowing through these

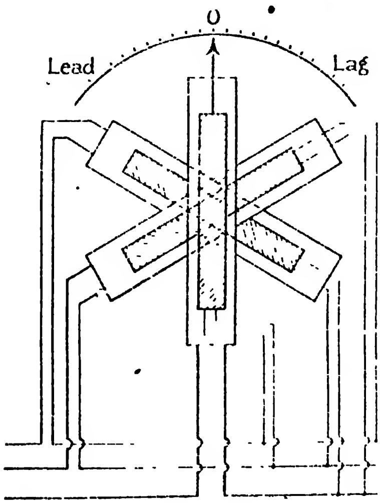


FIG. 343
POWER FACTOR METER

coils produce a rotating magnetic field, say, in a clockwise direction. Inside the fixed coils, and free to rotate, is a similar set of high resistance pressure coils connected, each across one pair of lines, through a series resistance, or, if necessary, through a pressure transformer. The pressure coils also set up a rotating field which is in the same direction as the field produced by the fixed coils. Now if the pressure coils were clamped in the position in which they are drawn in the figure, there would be an angular distance between the two rotating fields equal to the angle of lag of the current. As the pressure coils are free to move, the two fields come into line, thus turning the pressure coils through an angle equal to the angle of lag of the current. If the current is leading, then the pressure coils will be turned in the opposite direction. If the current is in phase with the voltage, then, with the pressure coils in the position shown, the two rotating

fields are already in phase, and there is no couple set up on the moving system, which remains in this position. Hence, the dial is graduated as shown.

A disadvantage of the instrument just described is that the ligament necessary for the leading of the current to the moving coils limits the deflection to about 300 degrees, instead of giving perfect freedom of rotation. This difficulty has been overcome in the moving-iron power-factor meters in which, as the name implies, the moving system consists of iron vanes. In the Ockenden instrument, as manufactured by Messrs. Everett Edgcumbe & Co., Ltd.,

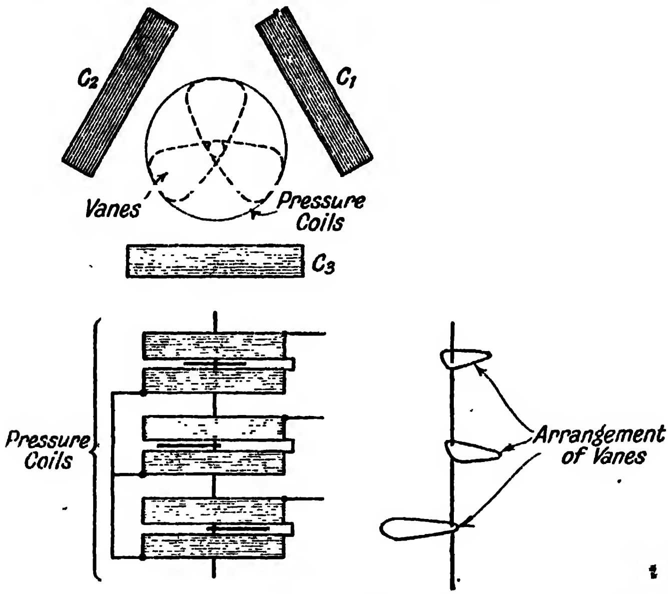


FIG. 244. PRINCIPLES OF THE OCKENDEN MOVING-IRON POWER-FACTOR METER

the rotating field principle is retained, the principle of the instrument being indicated by Fig. 244. C_1, C_2, and C_3 are the fixed current coils which, in virtue of their geometrical spacing, produce a rotating magnetic field as shown previously. The pressure coils are flat-fixed coils arranged in a stack with their axis perpendicular to the view of Fig. 244, so that in plan they appear as a circle, as shown. Each phase comprises two of these coils placed close together, and in the gap an iron vane can rotate. The pair of coils is wound so that unlike poles come together, the vane thus moving in an intense radial field. The result of this is that magnetic interference, both from neighbouring coils in the same instrument and also from external bodies, is eliminated.

CHAPTER XVII

THE ALTERNATOR

1. **In Direct Current Generators** the armature rotates under the poles of an external field system. In modern alternators the construction is the reverse of this, the armature being stationary and the field rotating. The advantage of this construction is that no difficulty is experienced in insulating the stationary armature winding for very high voltages, e.g. as high as 30,000 volts in some cases. Also, the current from the armature is collected from fixed terminals. The field winding of course requires direct current excitation, and since the magnet rotates, this current has to be led to the winding by means of two slip rings. This is not a serious matter, as the excitation voltage is low and the power required for excitation is also small.

The stator core is laminated like a D.C. armature core, ventilating ducts being provided in order to assist cooling. It is not necessary to laminate the field, since this carries a continuous flux, but sometimes the poles or the pole tips are laminated. A typical alternator construction is illustrated in Fig. 245.

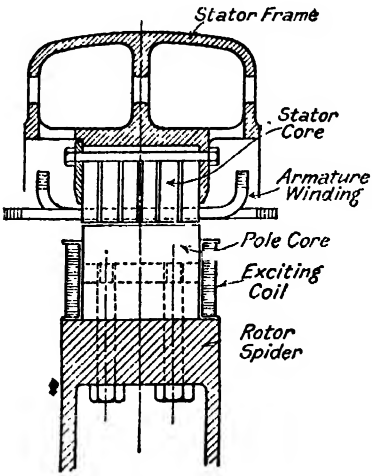


FIG. 245
ALTERNATOR CONSTRUCTION

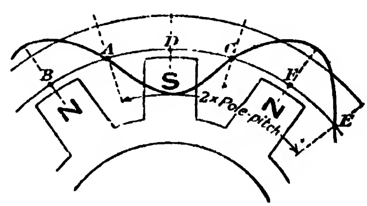


FIG. 246
CALCULATION OF FREQUENCY

2. **Frequency.** When a conductor is opposite the neutral planes, as at A, C, and E (Fig. 246), its induced E.M.F. is zero. Opposite the middles of the poles, at B, D, F, its induced E.M.F. is a maximum, its direction depending on the name of the pole influencing the conductor at any given instant. The E.M.F. induced in a conductor therefore goes through one complete cycle in an angular distance equal to twice the pole pitch.

$\therefore$ No of cycles per revolution $= \frac{p}{2}$ where p = No. of poles.

Let N = r.p.m.

$$\therefore \quad \frac{N}{60} = \text{r.p.s.}$$

$\therefore$ Frequency in cycles per second

$$f = \frac{p}{2} \times \frac{N}{60} = \frac{Np}{120}$$

Hence, if the speed and frequency are specified, the number of poles required is fixed. In a D.C. generator designed to a given specification the number of poles is not fixed, and it would be possible to design, say, 10 and 12 pole machines of equally good performance to the same specification. This definite relationship between f, p, and N has an important bearing on the choice of speed of an alternator. Thus if $f = 50$, then for $p = 2$, $N = 3{,}000$ r.p.m., while for $p = 4$, $N = 1{,}500$ r.p.m., and speeds intermediate between 3,000 and 1,500 are inadmissible.

The possible speeds of alternators working on a frequency of 50 cycles per second are, therefore, as given in the following table. These are also the speeds at which synchronous motors run, and are slightly greater than the speeds of induction motors running without any speed-regulating plant in circuit.

No. of poles $= p$	$N = \frac{120f}{p} = \frac{6{,}000}{p}$	Successive Differences
2	3,000	1,500
4	1,500	500
6	1,000	250
8	750	150
10	600	100
12	500	72
14	428	53
16	375	42
18	333	33
20	300	27
22	273	23
24	250	

It will be seen that when p is large the successive differences are so small that they are of little consequence, but when p is so small as to bring the speed within the range of turbo speeds, the successive differences are very large.

3. Single-phase Armature Windings. Most alternator windings are modifications of the simple wave windings, several of these

being shown in Fig. 247. No. I is the simplest possible winding and can be referred to as a "skeleton" wave winding. In winding

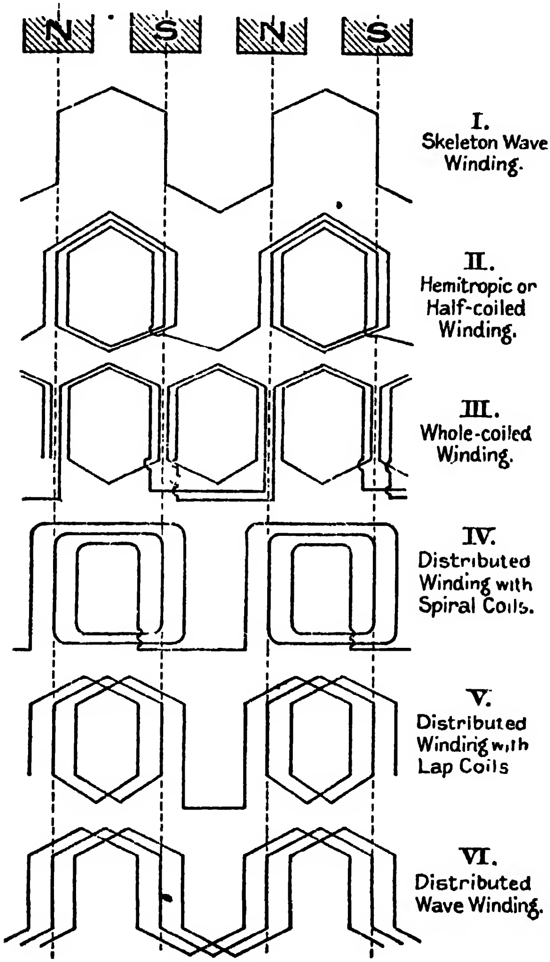


Fig. 247
Single Phase Armature Windings

II, the single-turn coils of the skeleton winding are replaced by multi-turn coils, so that a higher E.M.F. can be obtained from the winding. This winding is called "hemitropic," or half-coiled, because the coils only cover one-half of the armature periphery.

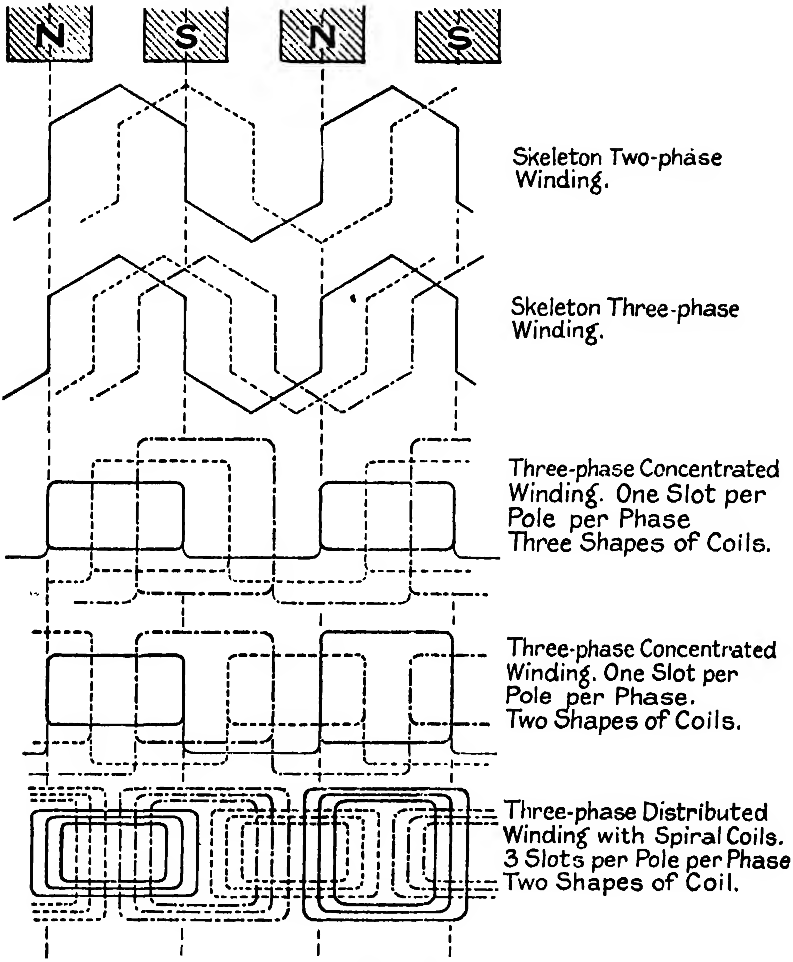


FIG. 248

POLYPHASE ARMATURE WINDINGS

In winding III, the coils are distributed over the whole of the periphery, the winding therefore being called whole-coiled. All the above windings are "concentrated," because there is only one slot per pole. Concentrated windings give the maximum voltage for a given number of conductors, but the wave form of the voltage departs considerably from the desired sinusoidal form. Better wave form, at the sacrifice of a certain amount of output, is obtained by distributing the winding in several slots per pole. The windings IV, V, and VI are examples of distributed windings, the first two

having spiral and lap coils respectively. Winding VI is carried out in wave fashion throughout.

The above windings are "single-layer" windings, because each slot contains one coil side only. Two-layer windings arranged with two coil sides per slot, exactly like D.C. armature windings, are frequently used. These are considered in the next paragraph.

4. **Polyphase Armature Windings.** These windings are arranged similarly to single-phase windings, the only difference being that in a two-phase alternator there are two separate single-phase windings, and in a three-phase alternator there are three separate windings. In a two-phase alternator the separate windings are 90 electrical degrees apart, thus giving the required phase difference. A skeleton two-phase winding is shown in Fig. 248. In a three-phase winding the individual windings are arranged 60° apart for convenience, so that the actual phase differences in the induced E.M.F.s are, between phases I and II, 60°; between II and III, 60°; between III and I, 240°. The second phase is therefore reversed when connected to the terminals, the correct phase relations in the external circuit being thus obtained. Several types of concentrated and distributed winding are illustrated in Fig. 248, their arrangements being the same as for the single-phase windings.

One of the most commonly used windings is the hemitropic winding with spiral coils, as shown in the last example in

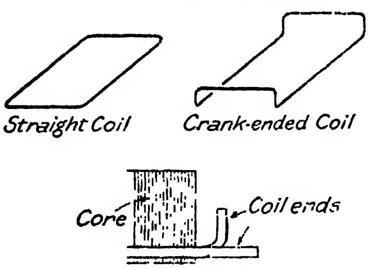


Fig. 249
Arrangement of Coils

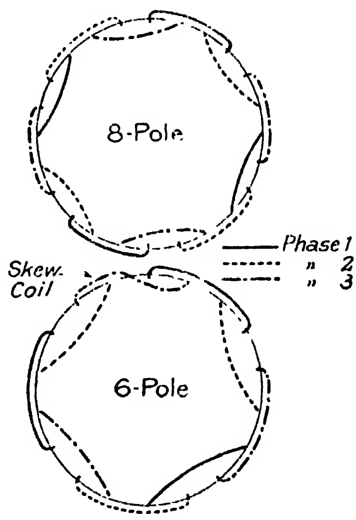


Fig. 250
Winding with Skew Coil

Fig. 248. Two shapes of coil are required to carry out this winding, a plane coil and a crank-ended coil, as in Fig. 249 The coil ends for this winding are in two "ranges" to prevent fouling. If this winding is used with an alternator having an odd number of pairs of poles, then one of the coils has to be "skew" shaped,

that is, one side straight and the other side crank-ended. This is easily proved by tracing out the ends of the coils, as in Fig. 250.

A winding of this type can be used either as a three-phase or a two-phase winding. Consider the winding in Fig. 251. If the coil width is equal to the pole pitch, the winding is three phase, each phase consisting of straight and crank-ended coils alternately. Suppose that the pole pitch is increased to $1\frac{1}{2}$ the coil width. Then all the straight coils can be joined together in one phase, and all the crank-ended coils, joined together in a second phase. If we consider any two consecutive coils in one phase, e.g. A and B, the

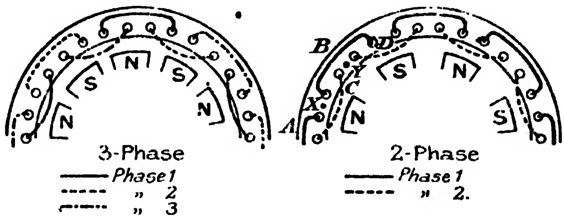
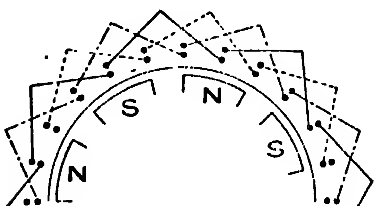


Fig. 251

Winding arranged for either Two- or Three-phase Working

E.M.F.s in them are not in phase, but since they are joined in series they combine to give an E.M.F. in phase with that which would be induced in a conductor placed in the position X. Similarly, the coils C and D in the second phase give together an E.M.F. in phase with that which would be induced in a conductor placed at Y. But Y is 90 electrical degrees from X, and therefore, the winding is a true two phase winding.



The Front End Connections can be carried out either Lap or Wave.

Fig. 252

Two-layer Winding

At one time alternator windings were almost invariably of the single-layer type, i.e. with one coil side per slot. At the present time two-layer windings are commonly used in cases where there are only a few conductors per slot. Fig. 252 shows such a winding.

5. Rotating Field Produced by a Polyphase Winding. Since the total M.M.F. of a three-phase winding is made up of the separate effects of each of the phases it is necessary to consider, first of all, one phase by itself. Fig. 253 (a) shows three conductors of a single-phase concentrated winding having one slot per pole. Any point between conductors A and B is enclosed within a single turn and therefore the M.M.F. of the winding is constant from A to B. Similarly the M.M.F. is constant from B to C, of the same magnitude as before, but

opposite in sign. Hence the M.M.F. of a single-phase concentrated winding has a rectangular distribution, as shown in the figure. It can thus be regarded as a rectangular wave, but it is a stationary wave because its position is fixed in space with respect to the winding itself. On the other hand its amplitude varies sinusoidally with respect to time because the M.M.F. at any instant is proportional to the magnitude of the current at that instant.

Fig. 253 (*b*) shows a single-phase winding distributed over two slots per pole. The winding consisting of conductors *A*, *B*, and *C* will produce the rectangular M.M.F. wave No. 1, while that consisting of conductors *A'*, *B'*, and *C'* will produce the M.M.F. wave No. 2. Adding the waves 1 and 2 together we obtain No. 3 for the total M.M.F. wave of the distributed winding. We see that the

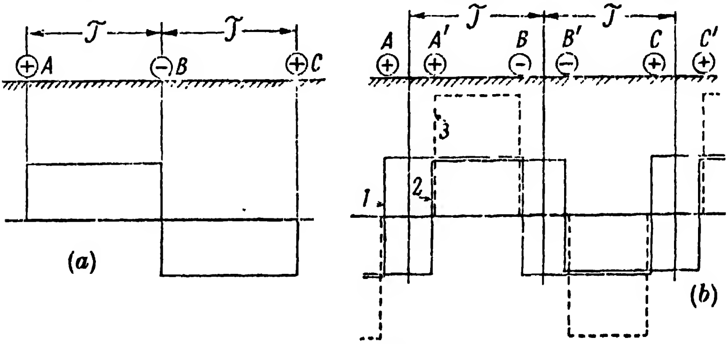


FIG. 253

M.M.F. OF A SINGLE-PHASE WINDING

effect of distributors is to change the shape of the M.M.F. wave from rectangular to stepped.

In the case of a three-phase winding it is necessary to consider the state of affairs at several instants during one cycle. The first of these instants can be that at which the current in phase *I* is a maximum. Then, since we obtain the instantaneous value of a vector by projecting on to the *OY* axis, the vector of I_1 will be directed along *OY*, the diagram therefore being as in Fig. 254 (*a*). Suppose we call the maximum value of the current 100 per cent, then for the instantaneous current in the three phases we have

$$i_1 = +100, \ i_2 = -50, \ i_3 = -50$$

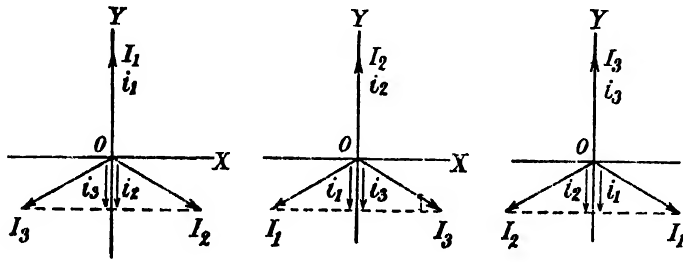
Fig. 254 (*b*) shows the state of affairs one-third of a period later, during which interval the vector diagram has rotated 120 degrees. At this instant we have

$$i_1 = -50, \ i_2 = +100, \ i_3 = -50$$

After another one-third period, we have, from Fig. 254 (*c*)

$$i_1 = -50, \ i_2 = -50, \ i_3 = +100.$$

Now consider an actual three-phase winding. Fig. 255 (1) shows a four-pole three-phase winding with two slots per pole per phase. For convenience in drawing the diagram diamond-shaped coils are used, but the argument applies to any three-phase winding having a coil width equal to the pole pitch, i.e. having full pitch coils. The starts of the three windings are represented by A, B, and C, and the ends by a, b, and c. We will now adopt the convention that a current is positive when it enters a winding at the start, and leaves it at the end. With this convention we have at instant 1 a current of 100 entering at A and leaving at a, a current of 50 entering at b and leaving at B, and a current of 50 entering at c and leaving at C.



A. Instant 1. B. Instant 2. C. Instant 3.

Fig. 254

These currents are indicated by arrows in the winding diagram, and by the conventional crosses and dots in the sectional diagram of Fig. 255 (2). Considering the phases separately we use the method of Fig. 253 (b) to draw the three M.M.F. waves, these being shown in Fig. 255 (3). Adding these together we obtain for the total M.M.F. *at the instant considered* the wave shown in Fig. 255 (4). We see that this M.M.F. is not sinusoidal but that it is stepped, and a little reflection will show that the greater the number of slots per pole per phase, the less important will be the individual steps.

Now consider instant 2. At this instant a current of 50 is flowing into phase I at a and out at B, a current of 100 is flowing in at B and out at b, and a current of 50 in at c and out at C. The cross-section of the winding for this instant is given in Fig. 255 (5), and comparing it with Fig. 255 (2) we see that the current distribution is exactly the same except that it has been displaced to the right by 120 electrical degrees, or two-thirds of a pole pitch. Since the M.M.F. distribution is produced by the current distribution it is clear that at instant 2 the M.M.F. wave will be identical in shape with that of Fig. 255 (4), and that it will have moved two-thirds of a pole pitch. Similarly at instant 3 the M.M.F. wave will have moved another two-thirds of a pole pitch.

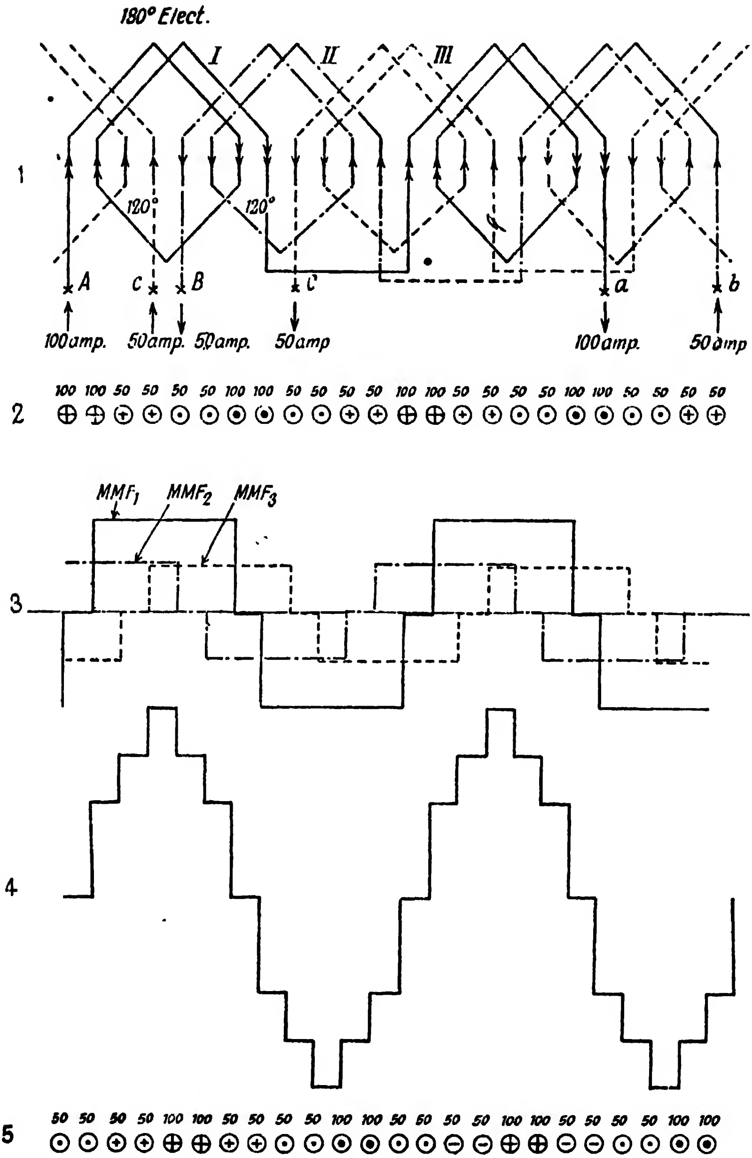


Fig. 255

M.M.F. Wave of a Three-phase Machine

After still another one-third of a period, that is, one complete period from instant 1, the M.M.F. wave will have advanced altogether six-thirds of a pole pitch, or two pole pitches. From § 2 we have seen that this is the same as the speed of rotation of the rotor of the alternator, and this shows that the speed of the M.M.F. wave in space is the same as the speed of rotation of the rotor. Another way of expressing this fact is to say that the M.M.F. wave rotates at synchronous speed.

At the three intervals considered the current distributions in the conductors are identical and the M.M.F. waves therefore identical in shape. At intermediate intervals there are different current distributions, and these cause a difference in the M.M.F. wave, but this difference is small.

6. **Synchronous Impedance.** Owing to magnetic leakage fluxes, confined mainly to those portions of the winding which project beyond the armature core, an alternator armature possesses reactance in addition to ohmic resistance. If L is the armature inductance*, then the reactance is $L\omega$. Now if L is measured with the field stationary, it will be found to vary with the angular position of the field, because this angle decides the relative dispositions of the armature coils and the poles. The curve of L against θ will therefore be a wavy line and of little practical importance. The inductance must therefore be determined under working conditions. The simplest method of doing this is the Behn-Eschenberg method, necessitating the determination of the open circuit and short circuit characteristics. The open circuit characteristic is the curve of induced voltage against exciting current, with the alternator running at normal speed. The short circuit characteristic is the curve of armature current under short circuit conditions against exciting current. To determine this, the armature terminals are short circuited through an ammeter, and a very reduced excitation applied. The excitation is then carefully increased until full armature current is flowing, readings of excitation and armature current having been taken. The curve is a straight line and it can therefore be projected, as shown dotted in Fig. 256, into the normal working portion of the diagram.

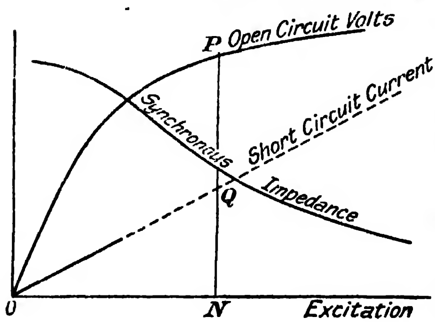


FIG. 256
ALTERNATOR CHARACTERISTICS

* In a polyphase machine L will be the inductance per phase.

Erect a perpendicular PQN at normal excitation ON; then the E.M.F. induced in the armature at this excitation is PN. On short circuit the whole of this E.M.F. is used in driving the short circuit current through the armature winding against the impedance, because the terminal voltage on short circuit is zero.

$$\therefore \text{ Impedance } Z = \frac{PN \text{ (volts)}}{QN \text{ (amp.)}} = \frac{E_o}{I_s}$$

This is called the "synchronous" impedance to indicate that it refers to working conditions.

Now $$Z = \sqrt{R_a^2 + (L_a\omega)^2}$$

$$\therefore \quad L_a\omega = \sqrt{\left(\frac{E_o}{I_s}\right)^2 - R_a^2}$$

The quantity $L_a\omega$ is called the "synchronous reactance." The synchronous impedance and reactance as determined by this method are rather higher than the values under absolutely normal conditions, due to the fact that on short circuit such a very low excitation has to be applied that the field is unsaturated. Other methods of determining the synchronous impedance are given in Chapter XIX.

If the synchronous impedance for different excitations is determined and plotted against excitation the curve has the form shown in Fig. 247.

It is to be noted that the armature impedance obtained by this method is not a true impedance, but is a composite quantity made up of the effects of the true impedance and of the effects of armature reaction, which, during the short-circuit test, is almost entirely demagnetizing. A method of separating this quantity into its two components is given in Chapter XIX.

Example. An alternator has an armature resistance of 0·3 ohm. When a small excitation is applied the open circuit voltage is 50, the short circuit current for the same excitation being 40 amp. Find the synchronous impedance and reactance.

$$Z = \frac{\text{open circuit volts}}{\text{corresponding short circuit current}}$$

$$= \frac{50}{40}$$

$$= 1{\cdot}25 \text{ ohm}$$

$$L_a\omega = \sqrt{Z^2 - R_a^2} = \sqrt{(1{\cdot}25)^2 - (0{\cdot}3)^2}$$

$$= 1{\cdot}22 \text{ ohm.}$$

7. Voltage Characteristics. Since an alternator possesses resistance and reactance there is a drop in terminal voltage with increase of load, exactly as in the case of a transformer.

Let E_0 = E.M.F. induced in the armature = terminal voltage at no load; V = terminal voltage on load. In a polyphase machine E_0 and V will be volts per phase.

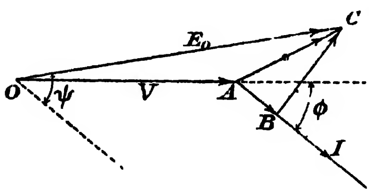


Fig. 257
Alternator Vector Diagram

Then if the load current I lags at an angle φ behind the terminal voltage V, the vector diagram is as in Fig. 257, where $AB = R_aI$ the resistance drop, $BC = L_a\omega I$, or XI the synchronous reactance drop, and $AC = ZI$ the total drop. In modern machines R_a is small compared with X, so that very little error is introduced in neglecting the resistance drop, especially on inductive loads. From the diagram, we have

$$E_0^2 = V^2 + (ZI)^2 - 2V \times ZI \cos(90 + \varphi) \text{ approx.}$$
$$= V^2 + (ZI)^2 + 2V \times ZI \sin\varphi$$

Now $E_0 = ZI_s$

where I_s = short circuit current with normal excitation

$$\therefore \quad (ZI_s)^2 = V^2 + (ZI)^2 + 2V \times ZI \sin\varphi$$
$$1 = \left(\frac{V}{E_0}\right)^2 + 2 \cdot \frac{V}{E_0} \cdot \frac{I}{I_s} \cdot \sin\varphi + \left(\frac{I}{I_s}\right)^2$$

The variables in this equation are V, I, and φ, and by taking different values for I a family of V/I curves can be drawn, each curve corresponding to a definite value of φ. It will be seen that whatever the value of φ all the curves will have the same intercept on the voltage axis, namely, E_0, and the same intercept on the current axis, namely, I_s, the short circuit current The curves for various values of the power factor, $\cos\varphi$, are shown in Fig. 258. An examination of the curves shows that the drop in volts depends upon the power factor, being small when $\cos\varphi$ is unity, and large when $\cos\varphi$ is less than unity, lagging, i.e. with an inductive load. If the load contains capacity then there may actually be a rise in terminal voltage, this rise increasing as the power factor (leading) decreases.

The phase angle ψ of the current I with respect to the induced E.M.F., E_0, is called the "internal" phase angle.

It is usual to express the voltage regulation of an alternator in terms of the *rise* in voltage when full load is thrown off. Thus, if the terminal voltage on load is V, it will rise to E_0 on throwing off the load, and we have

$$\% \text{ regulation} = \frac{E_0 - V}{V} \times 100$$

The rise of voltage on throwing off the load is not the same as the fall of voltage on applying the load, for the following reason. In Fig. 259 the magnetization characteristics for no load and full load are drawn, and a horizontal drawn to give the normal terminal voltage. If the alternator is on no load the working point will be A, and on throwing on the load without increasing the excitation, the new working point will be B, the drop in volts being therefore AB. If working on full load, the excitation has to be increased in order that the terminal voltage may now be normal, the working point being C. On throwing off the load the rise of voltage is CD. Since the two curves are not exactly parallel, AB and CD are not equal.

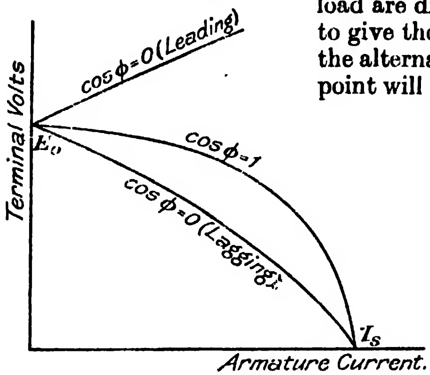


Fig. 258

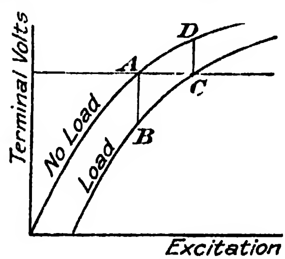


Fig. 259

Alternator Characteristics

It is now usual to express the regulation of an alternator as the percentage rise of voltage when full load is thrown off, and not by the percentage fall when full load is thrown on. The former value is called "regulation up" and the latter "regulation down."

A few years ago it was common practice to build alternators with a " close " regulation, that is, the variation in terminal voltage with load was small. Such machines were costly, and suffered from the serious disadvantage that their internal reactance was so small that in the event of accidental short circuit the initial value of the short circuit current was so high that the mechanical forces set up on the projecting ends of the windings invariably wrecked the machine. At the present time alternators are designed to have a fair amount of internal reactance, and they can be repeatedly short-circuited with impunity. They suffer in consequence from

a big variation of terminal voltage with change in load, but this is looked after by automatic voltage regulators. (*See* Chap. XIX.)

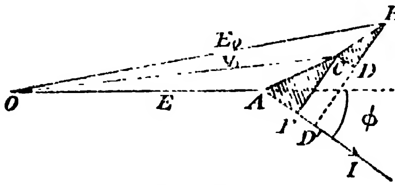


FIG. 260

Average values of the percentage regulation are 8 or 10 per cent on full non-inductive load, and 20 per cent on an inductive load of power factor 0·8.

Example. The alternator in the example on page 315 supplies 100 amp. to a feeder of resistance 1·5 and reactance 2·0. The voltage at the far end of the feeder is 3,000, the load current having a power factor of 0·8 (lagging) with respect to this voltage. Find the terminal voltage of the alternator and also the E.M.F. generated.

Resistance drop in feeder, AB in Fig. 251 $= 100 \times 1{\cdot}5 = 150$ volts

Reactance drop in feeder, BC in Fig. 251 $= 100 \times 2{\cdot}0 = 200$ volts

$\therefore$ Taking X and Y component of the alternator terminal voltage V_1, we have

$$(V_1)_X = 3{,}000 + AB\cos\varphi + BC\sin\varphi$$
$$= 3{,}240$$
$$(V_1)_Y = BC\cos\varphi - AB\sin\varphi = 70$$
$$\therefore \quad V_1 = \sqrt{3{,}240^2 + 70^2} = 3{,}240 \text{ approx.}$$

Resistance drop in armature, $CD = 100 \times 0{\cdot}3 = 30$ volts

$\therefore$ Total resistance drop, $AD' = 150 + 30 = 180$ volts

Reactance drop in armature, $DE = 100 \times 1{\cdot}22 = 122$ volts

$\therefore$ Total reactance drop, $D'E = 200 + 122 = 322$ volts

$\therefore$ Taking X and Y component of the total induced voltage

$$(E_o)_X = 3{,}000 + AD'\cos\varphi + D'E\sin\varphi$$
$$= 3{,}360$$
$$(E_o)_Y = D'E\cos\varphi - AD'\sin\varphi = 150$$
$$\therefore \quad E_o = \sqrt{3{,}360^2 + 150^2} = 3{,}363 \text{ volts approx.}$$

8. **Armature Reaction.** Consider an individual armature coil, as shown in Fig. 261. Each coil side has the maximum E.M.F. induced in it when it is opposite a pole centre, and therefore, the coil as a whole has the maximum E.M.F. when it is opposite an interpolar gap. The curve of E.M.F. with respect to the position of the centre of the coil is therefore a sine wave whose crest and trough values lie in the interpolar gaps. This curve is indicated by a heavy line.

(*a*) PURE RESISTANCE LOAD. The current is in phase with the voltage, and therefore the position of the coil for maximum current

will be the same as the position for maximum voltage; in other words, the current curve, shown dotted, and the voltage curve will be in phase. (Diagram (*a*).)

In the position shown the coil therefore carries maximum current, whose direction at the instant illustrated is inwards under the N. pole and outwards under the S. pole. The magnetic flux set up by the coil is therefore downwards and produces a magnetization of the trailing half of a main pole and demagnetization of the leading half. The effect of armature reaction in this case is therefore one of distortion of the field form, as in the case of the cross-magnetization in a D.C. generator. The effect in the alternator is not very strong because the coil, when carrying maximum current, is opposite the interpolar gap, and therefore, in the position of maximum reluctance.

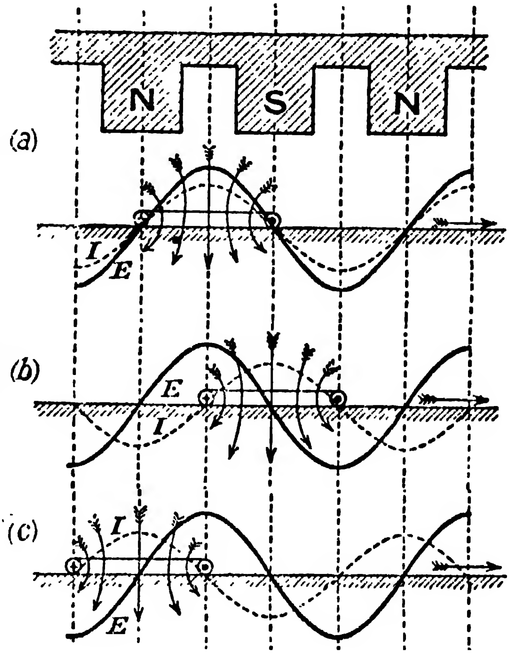


FIG. 261
TO ILLUSTRATE ARMATURE REACTION

(*b*) PURELY INDUCTIVE LOAD. The current lags 90° behind the voltage, and therefore, the coil has to advance 90° (electrical degrees) from the position shown in diagram (*a*) before the current reaches its maximum value. The current is then in the same direction as before, and comparing the directions of the armature and main fluxes, we see that with an inductive load the armature reaction is demagnetizing. Also, the coil is in a favourable position for producing a strong flux because it is directly opposite a main pole. Thus there is a strong demagnetizing action.

(*c*) PURE CAPACITANCE LOAD. In this case the current leads the voltage by 90°, so that the coil will carry its maximum current 90° before it reaches the position shown in diagram (*a*). Hence, in this case the armature reaction exerts a strong magnetizing action.

It is to be remembered that the flux produced by a single-phase armature is an alternating flux, and is therefore pulsating in nature.

We have seen that the armature flux of a polyphase alternator is a rotating flux which keeps pace with the poles of the rotor. It is therefore uniform instead of pulsating.

In the above discussion the angle of lag or lead referred to is the *internal* phase angle, the armature itself being included in the load.

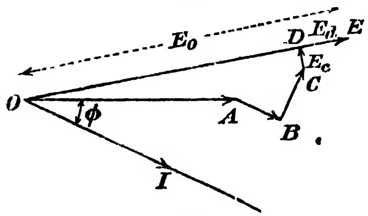


FIG. 262
ALTERNATOR VECTOR DIAGRAM

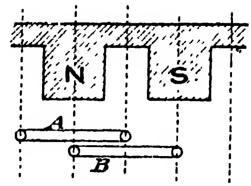


FIG. 263
CALCULATION OF E.M.F.

With a load which is partially inductive or partially capacitive, there is cross-magnetization in addition to direct demagnetization or magnetization. The demagnetizing flux induces a voltage E_d in the armature in direct opposition to that induced by the main flux, and the excitation has to be increased in order that the total induced E.M.F. may have an additional component equal and opposite to E_d. The cross flux produces a voltage E_c, which is in quadrature with the main E.M.F., since the cross and main fluxes are at right angles in space. The complete vector diagram of an alternator when armature reaction is taken into account is therefore as shown in Fig. 262.

OA = terminal voltage V
AB = resistance drop in armature
BC = synchronous reactance drop
CD = E.M.F. induced by cross flux
DE = component of the total E.M.F. to neutralize E_d
OE = total induced E.M.F., E_0

9. **The E.M.F. Equation.** Consider a full pitch coil, i.e. one of width equal to the pole pitch. As this coil moves from the position A to the position B (Fig. 263), the flux linking with it changes from Φ to zero, Φ being the flux per pole. Hence, if the coil has n turns, the change of linkage is Φn, and the *average* E.M.F. induced in it is

$$E_{av} = \frac{\Phi n}{t} \times 10^{-8} \text{ volts}$$

where t is the time in seconds required to move a distance equal to AB. Obviously t is one-quarter of the periodic time T.

$$\therefore \quad \frac{1}{t} = \frac{4}{T} = 4f$$

$$\therefore E_{av} = 4\Phi n f \times 10^{-8}$$

Hence, assuming a sinusoidal wave form,

$$E_{eff} = 1{\cdot}11 \times E_{av}$$
$$= 4{\cdot}44\ \Phi nf \times 10^{-8}$$

If there are C coils, then total number of turns in all the coils $= Cn$, and for the total voltage induced in all the coils, we have

$$E_o = 4{\cdot}44\Phi(Cn)f \times 10^{-8}$$

Again, the product Cn is equal to $Z/2$ where Z is the number of conductors, since each coil has two sides.

$\therefore$ Finally $E_o = 2{\cdot}22\Phi Zf \times 10^{-8}$ volts

If the wave form is not sinusoidal, then the form factor will probably be different from 1·11. If we put k_1 for the form factor, we have

$$E_o = 2k_1\Phi Zf \times 10^{-8} \text{ volts}$$

This equation is correct for concentrated windings, in which the E.M.F.s induced in all the individual turns of any one coil are in phase. If the winding is distributed, then the E.M.F.s induced in the various parts of a coil are not in phase, and this has to be taken into account. Fig. 264 shows the vector diagram for the distributed coil, in which the coil sides are displaced at an angle ψ. The E.M.F.s, E, induced in each coil are equal in magnitude, but there is a progressive phase difference of ψ. The vector diagram for the complete coil is therefore a polygon, and the closing side gives the total E.M.F. in the coil. If there are m sections in the coil, there are m sides, and therefore the closing side is

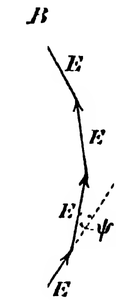


FIG. 264
CALCULATION OF BREADTH FACTOR

$$AB = E \times \frac{\sin \frac{m\psi}{2}}{\sin \frac{\psi}{2}}$$

If all the E.M.F.s had been in phase, the total E.M.F. in the coil would have been mE; hence, by distributing the winding, the E.M.F. is reduced in the ratio

$$k_2 = \frac{\sin \frac{m\psi}{2}}{m \sin \frac{\psi}{2}}$$

This ratio is called the "breadth factor." Taking this factor into account, the complete expression for E_o becomes

$$E_o = 2k_1k_2\Phi Zf \times 10^{-8}$$

Example. A six-pole alternator rotating at 1,000 r.p.m. has a single-phase winding housed in three slots per pole, the slots in groups of three being 20°

apart. If each slot contains 10 conductors, and the flux per pole is 2×10^6 lines, calculate the E.M.F. generated, assuming a sinusoidal flux form

$$\Phi = 2 \times 10^6 ; \; Z = 18 \times 10 = 180 \text{ conductors}$$

$$f = \frac{Np}{120} = \frac{1{,}000 \times 6}{120} = 50$$

$k_1 = 1{\cdot}11$ for a sinusoidal field form

$$k_2 = \frac{\sin \frac{m\psi}{2}}{m \sin \frac{\psi}{2}}, \text{ where } m = 3 \text{ and } \psi = 20$$

$$\therefore \quad k_2 = \frac{\sin 30}{3 \sin 10} = {\cdot}96$$

$$\therefore \quad E_o = 1{\cdot}11 \times {\cdot}96 \times 2 \times 2 \times 10^6 \times 180 \times 50 \times 10^{-8}$$
$$= 382 \text{ volts.}$$

Examples on Chapter XVII.

(1) If a single-phase alternator has 8 slots per pole uniformly spaced, but the winding is arranged with the middle two left empty, find the breadth coefficient. *Ans.*—0·79.

(2) A single-phase alternator of P poles has a large number S slots per pole, each containing one conductor. If the R.M.S. value of the E.M.F. induced in each conductor is 10 volts, what is the total voltage induced ?

Ans.—$\frac{20}{\pi} . SP$.

(3) A single-phase alternator has an armature resistance of 0·1 ohm. When excited to give 50 volts and then short circuited, the short circuit current is 200 amp. To what induced voltage must the alternator be excited if it is to deliver 100 amp. at a power factor 0·8 lagging, with a terminal voltage of 200 ? *Ans.*—222 volts.

(4) A single-phase alternator is excited to 1,000 volts at a frequency of 50. Its armature resistance is 0·8 ohm, and synchronous reactance, 3 ohms. Calculate what length of unloaded cable, of capacity ¼ microfarad per mile, must be connected to its terminals to produce resonance. Calculate the terminal voltage when this occurs. If the flux per pole is 10^6 lines, and the product of form factor and breadth factor 1·0, calculate the number of armature conductors. *Ans.*—3,200 miles ; 3,750 volts ; 1,000 conductors.

(5) The field form of an alternator measured from the neutral plane to the middle of a pole is as shown in the table. It is

Distance in cm. .	0	1	2	3	4	5 to 10
Induction in gap lines per sq. cm.	0	400	2,100	4,500	6,600	8,000

symmetrical about the centre. The active length of conductor is 40 cm. All the 2000 conductors are housed in equally spaced slots 4 per pole, all the slots being filled. If the frequency is 50, find the induced E.M.F.

Ans.—8,400 volts.

CHAPTER XVIII

THE SYNCHRONOUS MOTOR

1. **The Synchronous Motor** is the alternator run as a motor. Direct current excitation is supplied to the field, exactly as when working as a generator; but there is no prime mover, the necessary driving torque being obtained by armature current flowing from the line to which the armature is connected.

Fig. 265 shows two poles of the rotor and stator, the stator poles being indicated as salient poles for convenience. The rotor poles retain the same polarity throughout, but the stator poles advance at synchronous speed, since they are magnetized by polyphase currents. Imagine first of all that the rotor is stationary. Then, if at any instant A has N. and B has S. polarity, the rotor will tend to move in a counter-clockwise direction. Half a period later the stator polarity will be reversed and the rotor will now attempt to rotate in a clockwise direction. The torque on the rotor is thus an alternating one, and owing to its inertia the rotor does not move in any direction. The motor is therefore not self-starting. Suppose now, that the rotor is travelling in a clockwise direction. Then, to maintain this motion, poles A and B must have S. and N. polarity respectively in the position shown. When the rotor has moved a distance equal to the pole pitch, its N. pole will be under the stator pole B, which must therefore have reversed its polarity from N. to S. if the motion is to be maintained. Hence, as the rotor moves a distance equal to the pole pitch, the stator polarity must reverse, that is, the alternating current supplied to the stator must have passed through one-half cycle. But this takes place when the machine is functioning normally as an alternator, and therefore, for both alternator and synchronous motor, we have the same definite relationship between N, p, and f, namely

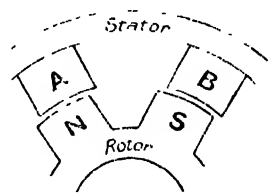


FIG. 265
TO ILLUSTRATE SYNCHRONOUS MOTOR OPERATION

$$f = \frac{Np}{120} \text{ or } N = \frac{120f}{p} \text{ r.p.m.}$$

Such a motor, working on a supply of fixed frequency, will only run at that particular speed at which it would have to be driven if working as an alternator, to give a frequency equal to the supply frequency. Hence the name, synchronous motor.

2. Nature of the Torque. If single-phase current is supplied to the stator it produces an alternating magnetic field which, by Ferrari's principle, can be split up into two rotating fields of half its amplitude, and travelling in opposite directions at synchronous speed. For, consider two such rotating fields both starting from the X axis at zero time, and rotating with angular velocity ω (Fig. 266). Then, after any time t

$$\begin{aligned}
&X \text{ component of No. 1} &&= H\cos\omega t\\
&X \text{ component of No. 2} &&= H\cos\omega t\\
\therefore\ &\text{Total } X \text{ component} &&= 2H\cos\omega t\\
&Y \text{ component of No. 1} &&= H\sin\omega t\\
&Y \text{ component of No. 2} &&= -H\sin\omega t\\
\therefore\ &\text{Total } Y \text{ component} &&= 0
\end{aligned}$$

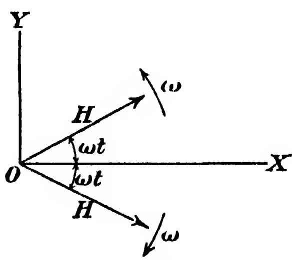


FIG. 266
COMPOSITION OF ROTATING FIELDS

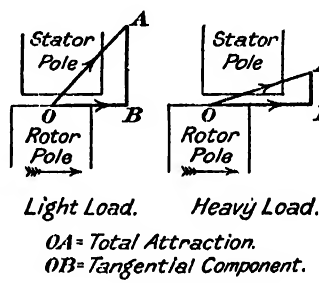


FIG. 267
SYNCHRONOUS MOTOR UNDER VARYING LOAD

Hence, the two rotating fields resolve into an alternating field, $2H\cos\omega t$, of twice their amplitude. Conversely, an alternating field can be resolved into two rotating fields travelling in opposite directions at synchronous speed.

If the motor is run up to synchronous speed in any direction and then switched on to the supply, it will continue to run at that speed under the influence of the rotating component travelling in the same direction. Call this component the forward component, and the other the backward component. Then the forward component will produce a uniform torque on the rotor, but the backward component, having an angular velocity relative to the rotor of *twice* the synchronous velocity, will produce an alternating torque. The resulting torque is therefore pulsating, the frequency of the pulsations being twice the supply frequency. This is a very serious disadvantage of the single-phase synchronous motor, but we shall see that it does not apply to the polyphase motor.

If, however, polyphase current is supplied to the stator then a pure rotating field will be produced and the rotor will run at synchronous speed in the same direction as this field.

We have seen that the motor runs at a fixed speed. If the load is gradually increased, then the rotor poles fall back more and more behind the poles of the forward component of the field, in order that the tangential component of the magnetic attraction between the two sets of poles shall increase. This is illustrated in Fig. 267. It is obvious that as the load torque is gradually increased, a point will be reached at which the increasing distance between the rotor and stator poles causes the tangential component to decrease instead of increase. The motor then stops, since it can only run at synchronous speed or not at all. This value of the torque is called the "pull-out" torque.

3. Vector Diagram of the Motor. In order that a direct current machine may act as a motor, we have seen that its induced E.M.F. must act in opposition to the armature current. In the case of an alternating current machine, it will function as a motor if the induced E.M.F. has a component in opposition to the armature current. Bearing this in mind, we can construct the vector diagram as follows. Let OA (Fig. 268) represent the applied voltage E. and let AB represent the motor induced E.M.F. Then OB, the resultant of OA and AB, is the resultant E.M.F. acting in the armature circuit. Calling this resultant E.M.F. E_r, we have current

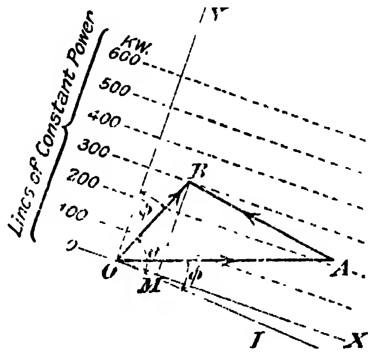


FIG. 268
SYNCHRONOUS MOTOR VECTOR DIAGRAM

$$I = \frac{E_r}{Z}, \text{ lagging an angle } \theta, \text{ where}$$

$$\tan \theta = X/R, \text{ the various quantities referring to one phase.}$$

Represent the current in phase by OI. Then $\angle AOI = \varphi$, the phase of the current with respect to the applied voltage. Draw OY inclined φ to OB. Then $\angle YOB = \varphi$, and since OB ($= E_r$) is proportional to the current, the vector OB, when referred to OY, represents the current in both magnitude and phase.

Then intake of the motor, or intake per phase with a polyphase motor

$$W = EI \cos \varphi$$
$$= E \times BM$$

where BM is the perpendicular dropped from B on to OX. But E is constant. Hence, intake is proportional to BM. Thus, if the

motor is working with constant intake, the locus of the working point B is the parallel to OX through B, since the parallel is at a constant distance from OX. Hence, we can draw a series of parallel lines on the diagram, each line corresponding to a definite value of the intake. These "power" lines are shown dotted in Fig. 260, and it is obvious that for equal increments of intake they will be equally spaced.

The length AB is the induced E.M.F., and since the speed is constant, this length also represents the excitation of the motor. We thus have

OA = applied voltage
AB = excitation
OB = current
Cos $\angle YOB$ = power factor
$\angle BAO$ = angle in electrical degrees between the centre of the stator and rotor poles

Perpendicular distance from working point B to the zero power line is proportional to the intake.

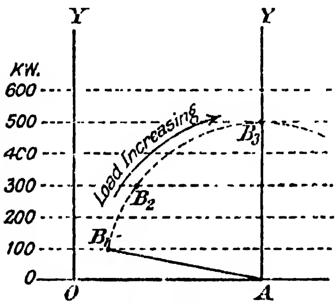


FIG. 269
OPERATION AT CONSTANT EXCITATION

If the working point B is above the horizontal OA the machine will be working as a motor, but if it is below OA the machine will be generating. Since the angle θ is very nearly 90° because R is so small, little error is made in taking θ equal to 90°. This is done in subsequent diagrams.

4. Operation with Fixed Excitation. If the excitation is fixed, then AB is fixed, and the locus of B is a circle, shown dotted (Fig. 269), with A as centre and AB as radius. Suppose the load is such that the intake is 100 kW; then the working point will be at B_1. If the load is increased, the intake will have to increase, and therefore, the working point will have to reach a line of greater power. To do so it must travel along the circle in a clockwise direction. Thus it might travel to B_2, and it will remain there as long as the load remains at the new value. If the load is gradually increased, then the working point will travel round the circle until it reaches B_3, at which the intake is obviously a maximum. For a further increase in load, the point will travel on to a line of *smaller* power. Such a condition is impossible, since an increase in load naturally demands an increase in power, and the whole of the diagram to the right of AY' therefore represents unstable operation of the motor. The intake B_3A is thus the

maximum intake for the given excitation, and if the motor losses are deducted from this, and the resulting output divided by the speed, the pull-out torque is obtained. We see that the pull-out torque depends upon the excitation, and the maximum possible value of this torque is fixed by that value of the exciting current which causes the maximum allowable temperature rise of the field windings.

5. Operation at Constant Power. The locus of the working point in this case is obviously one of the lines of constant power, say, 300 kW. The independent variable is the excitation. If the excitation is small, the working point will be at B_1, say (Fig. 270). The current OB_1 will be large and the angle of lag φ_1 will be large. The power factor will therefore be ow. As the excitation is increased, the working point will move along the 300 kW line towards the left, and for one particular value of the excitation, will be at B_2 on OY. For the given power, the motor current is now a minimum, and the power factor is unity, the current being in phase with the applied voltage. A further increase in excitation will bring the working point to B_3, to the left of B_2. The current has now increased again, but it is leading

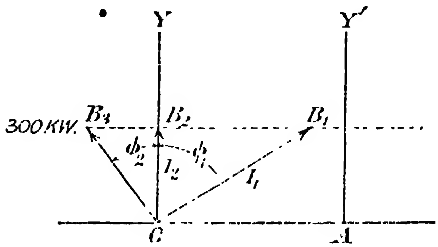


FIG. 270
OPERATION AT CONSTANT POWER

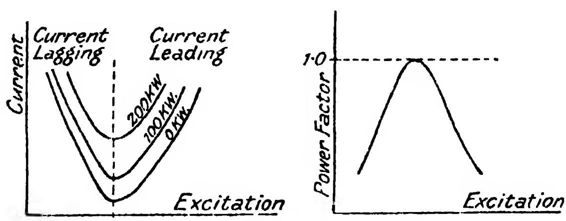


FIG. 271
SYNCHRONOUS MOTOR CHARACTERISTICS

instead of lagging, the angle of lead increasing as the excitation is increased. We thus see that an over-excited synchronous motor takes a leading current, a property which renders the motor extremely valuable as a phase advancer or power factor corrector, that is, a machine which takes a leading current, thereby bringing the total current taken from the supply more nearly in phase with the voltage. A condenser will, of course, perform the same function, since, neglecting the very small dielectric losses in it, it

takes a current leading by 90°. Phase advancers will be considered in more detail in Chap. XXIV.

If the motor current is plotted against excitation, the intake being constant, the characteristic obtained (Fig. 271) is called the "V" characteristic, from its shape. There is a family of such characteristics, each curve of the family corresponding to a definite intake. The curve of power factor against excitation is an inverted V, as shown.

Example. An alternator has an armature resistance of 0·5 ohm and synchronous reactance of 0·866 ohm. It is running as a synchronous motor on a 200 volt supply, the mechanical load on the shaft, including iron and friction losses, being 6·5 kW. The current taken is 50 amp. Find two possible phase angles of the current and two possible induced E.M.F.s.

$$\text{Armature copper loss, } I^2R = 50 \times 50 \times 0{\cdot}5 = 1{,}250 \text{ watts}$$

$$\therefore \text{ Intake} = 6500 + 1250 = 7{,}750 \text{ watts}$$

$$\therefore \text{ Power factor} \quad \cos\varphi = \frac{W}{EI} = \frac{7750}{200 \times 50} = 0{\cdot}775$$

$$\therefore \quad \varphi = 39° \text{ lagging or leading}$$

$$\text{The angle } \theta \text{ (Fig. 240)} = \tan^{-1}\frac{X}{R} = \tan^{-1} 1{\cdot}732 = 60°$$

$$\therefore \text{ Phase angle, } \angle BOA = 60 - 39 = 21°; \text{ or } 60 + 39 = 99°$$

$$\text{Also, resultant E.M.F., } OB = IZ = 50 \times \sqrt{R^2 + X^2} = 50 \text{ volts}$$

$$\therefore AB^2 = OA^2 + OB^2 - 2\,OA \times OB \,.\, \cos \angle BOA$$

$$= 40{,}000 + 2{,}500 - 20{,}000 \,(\cos 21° \text{ or } \cos 99°)$$

$$\therefore AB = 154 \text{ volts or } 214 \text{ volts.}$$

The method of drawing the "V" characteristic is as follows. The vector diagram with the lines of constant power is first drawn, and a series of concentric circles with centre at A then drawn to give a series of back E.M.F.s. The intersections of these circles with the lines of constant power will then locate the working points for specified powers and specified back E.M.F.s. The next step is to convert from back E.M.F.s to exciting current, and this necessitates a knowledge of the magnetization characteristic, a very convenient construction being to incorporate this characteristic in the diagram as shown in Fig. 272. Thus, for a power of 600 kW, the working points for the range of back E.M.F.s shown in the figure are B_1, B_2, B_3, . . . B_7, the current being OB_1, OB_2, OB_3, . . . OB_7; the back E.M.F. vectors are not drawn as there is no necessity to confuse the diagram by their presence. The various back E.M.F.s are projected on to the magnetization characteristic, giving for the corresponding exciting currents $O'M_1$, $O'M_2$, . . . $O'M_7$. The above line currents are now plotted against these values of the exciting current, thereby

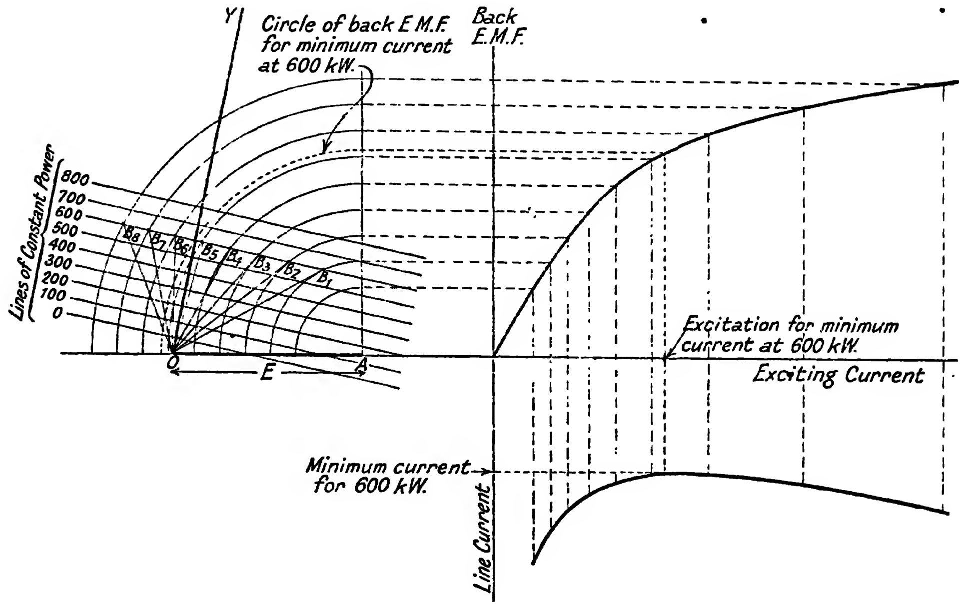


FIG. 272. GRAPHICAL CONSTRUCTION FOR "V" CHARACTERISTICS

giving the "V" characteristic for the particular power of 500 kW. Similarly, the characteristics for other powers can be drawn. If the angle of lag or lead of the various currents are measured the power factors can be calculated, and the curves of power factor against exciting current also drawn. By drawing a semicircle with O as centre and calling its length 100, the intersection of the various current vectors with this semicircle can be projected on to OY, thereby giving the percentage power factors graphically, but this construction is not shown in Fig. 272.

6. **Phase Swinging or Hunting.** We have seen that when a synchronous motor is loaded, the rotor poles fall back a certain angle behind the poles of the forward rotating field, in order that sufficient tangential force may be set up to produce the necessary torque to cope with the load. If the load is suddenly thrown off, the rotor fields are pulled into almost exact opposition to the poles of the forward field, but because of the inertia of the rotor the rotor poles travel too far. They are then pulled back again, and so on; an oscillation thus being set up about the position of equilibrium corresponding to the new conditions of load. It will be seen that these oscillations will be set up whenever the load varies, and if the variations in load are periodic and synchronize with the natural period of oscillation of the rotor, mechanical resonance will be set up and the amplitude of the swing of the rotor poles relative to the poles of the rotating field will become so great that the motor will fall out of step. It is therefore evident that some damping couple must be introduced, so that, in the event of oscillations being set up, they will be immediately damped out and no large amplitude of swing obtained. These dampers take the form of heavy copper grids housed in the pole faces. When the motion is uniform, there is no relative motion between the rotor and stator forward rotating poles, and no currents induced in the dampers. If hunting takes place, the relative velocity of the two sets of poles induces heavy currents in the dampers, and the kinetic energy of oscillation is thus damped down by being converted into heat energy in the dampers.

An expression for the period of swing of the rotor can be determined as follows. Suppose the motor is working on no load, and imagine for simplicity that there are no losses. Then the supply voltage E and the motor induced voltage E_1 will be equal and opposite so long as the motion is steady. If for some reason the rotor is displaced relatively to the stator field by a small angle θ, then one vector will swing round through the angle θ, and a resultant voltage E_r will be set up (Fig. 273). For small values of θ, E_r is proportional to θ. We have, in fact, $E_r = E \times \theta$.

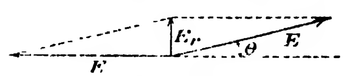


FIG. 273
PRODUCTION OF SYNCHRONIZING TORQUE

Now E_r will set up a current through the armature, of magnitude $I = E_r/Z$, lagging very nearly 90° behind E_r in the case of a modern machine with high reactance. Thus I will be nearly in phase with E, so that the power conveyed by I will be

$$W = EI = E \times \frac{E_r}{Z} = \frac{E}{Z} \times E\theta$$
$$= EI_s\theta$$

where I_s is the short-circuit current corresponding to normal excitation. If the angular velocity of the rotor is Ω radians per second, the torque set up by the current I will be

$$\text{Torque} = \frac{\text{power}}{\text{angular velocity}} = \frac{EI_s\theta}{\Omega} = A \,.\, \theta, \text{ say.}$$

If K is the moment of inertia of the rotating masses this torque will be utilized in accelerating the rotor so long as there is no damping.

$$\therefore\ K\frac{d^2\theta}{dt^2} + A\theta = 0.$$

This is the differential equation of a simple harmonic oscillation, and comparing it with the equation

$$\frac{d^2x}{dt^2} + n^2x = 0$$

we see that the frequency of the oscillations set up is

$$f = \frac{n}{2\pi}, \text{ or in the case of the motor}$$

$$f = \frac{1}{2\pi} \times \sqrt{\frac{EI_s}{K\Omega}}$$

This is the frequency of the oscillations when there is no damping. If the pole faces are provided with damping windings, then there will be a retarding torque set up proportional to the angular velocity of phase swing. The effect of this torque is to damp out the oscillations and also to diminish their frequency.

These pole-face damping windings are also used in starting up the synchronous motor, but as their action is then that of an induction motor this method of starting will be dealt with in Chapter XX.

Examples on Chapter XVIII.

(1) A single-phase synchronous motor rotates at 1,000 r.p.m. on a 500 volt supply. Its armature resistance is 0·1 ohm and reactance 0·5 ohm. If it can be excited up to an induced voltage of 2,000, calculate its pull-out torque. Draw the vector diagram for the motor and from it deduce the "V" curves.

(2) A single-phase synchronous motor for use on a 500 volt circuit gives, when running as a generator, a short-circuit current of 150 amp. with normal excitation applied. The armature resistance is 0·2. To what voltage must

the motor be excited so as to develop 40 h.p. at unity power factor, the mechanical losses being 5 h.p.? What will be the armature current?

Ans.—549 volts, 67·2 amps.

(3) A single-phase alternator, when driven at 1,000 r.p.m., has a magnetization characteristic as follows—

Exciting current . .	30	60	90	120	150.
Open circuit volts . .	460	820	1,040	1,200	1,300

A short-circuit current of 200 is produced when the exciting current is 40 amp. The armature resistance is 0·4 ohm. Plot the curve of current against exciting current with a constant intake of 100 kW at 1,000 volts when running as a synchronous motor at 1,000 r.p.m.

(4) A three-phase synchronous motor, when working on constant pressure mains at constant load, takes a current which varies with the excitation. Explain this result, and show how the "V" curve of the motor may be estimated approximately from the magnetization curve. (London Univ., 1924.)

CHAPTER XIX

VOLTAGE REGULATION AND PARALLEL OPERATION OF ALTERNATORS

1. **Preliminary Data.** In order to determine the voltage regulation of an alternator, without actually putting the machine on load, it is necessary to know the armature synchronous impedance (*see* Chapter XVII). In the case of a polyphase alternator the synchronous impedance per phase is required. The open circuit and short circuit method of determining this was considered. This is the method usually adopted in practice, mainly because it does not present any great experimental difficulty. Two other methods are explained below, but these are not so frequently used.

2. **The Ampere-turn Method.** This method, which was invented by Rothert, necessitates the determination of the open circuit and short circuit characteristics, but the data are used differently

On open circuit, the excitation required to produce E volts at the terminals is equal to the abscissa OM ($= i_1$, say) (Fig. 274). With full load current I circulating in the armature, the necessary excitation during the short circuit test is ON ($= i_2$, say). This excitation (a) induces an E.M.F. which is used entirely in overcoming the synchronous impedance of the armature, (b) overcomes the demagnetizing effect of the armature reaction. Also, during this test the current lags nearly 90° because of the small resistance of a modern alternator, and therefore, the armature reaction is almost entirely demagnetizing. The excitation necessary to produce a terminal voltage of E when the armature is delivering a current I under actual working conditions is equal to the *vector* sum of the separate excitations i_1 and i_2. It is necessary to take the vector sum of i_1 and i_2 because the demagnetizing effect of the armature reaction depends upon the power factor of the load. Now when the internal angle of lag is 90°, the armature reaction is wholly demagnetizing. As the angle of lag is decreased, the demagnetizing

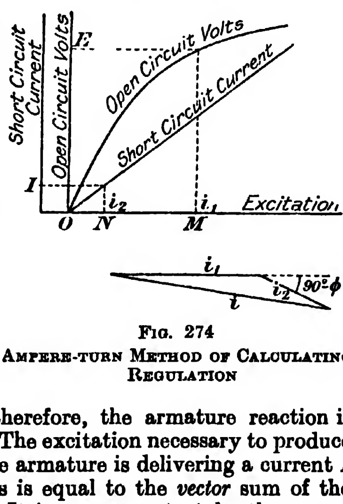


FIG. 274

AMPERE-TURN METHOD OF CALCULATING REGULATION

effect is decreased, being, in fact, proportional to the sine of the phase angle. Hence, if the regulation for a load current I of power factor $\cos\varphi$ is required, the excitation i_1 is set off, and i_2 is inclined at an angle $(90-\varphi)$ to i_1, as shown. The resultant, i, gives the total excitation necessary under actual working conditions.

This method is sometimes called the "optimistic" method, because it gives results which are somewhat more favourable than the characteristics under normal working conditions.

3. The Wattless Current Method. This is due to Potier. The magnetization characteristic, i.e. the curve of terminal volts against excitation, is plotted first at no load, and then when the armature is delivering full load current *at zero power factor* (Fig. 275). The point A on this curve obviously corresponds to the armature short-circuited. The excitation OA consists of two parts, (a) that required to overcome the demagnetizing effect of the armature reaction, which, as explained before, is almost entirely demagnetizing on short circuit; (b) the excitation required to supply the voltage drop due to the synchronous impedance.

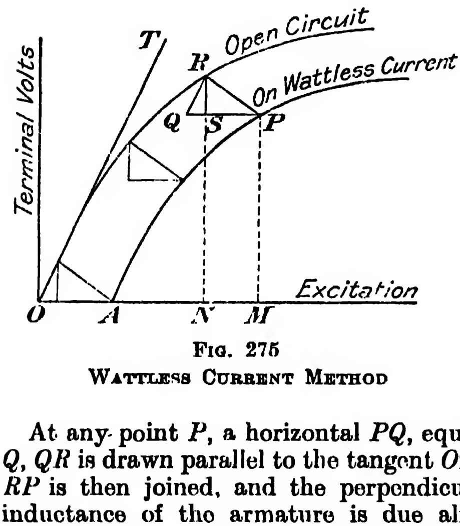


Fig. 275
Wattless Current Method

At any point P, a horizontal PQ, equal to OA, is drawn. From Q, QR is drawn parallel to the tangent OT at O to the no load curve, RP is then joined, and the perpendicular RS drawn. Since the inductance of the armature is due almost entirely to magnetic leakage fluxes which take paths through the air, the portion QS of the total excitation represented by QP is equal to the excitation required to supply the reactance drop. The perpendicular SR is therefore equal to the reactance drop, V, say. If the armature current is kept throughout at I amp. during the determination of the wattless current characteristic, then the reactance drop is equal to $L\omega I$. Hence, $L\omega I = V$.

$$\therefore \text{ Synchronous reactance } = \frac{V}{I}.$$

The length SP is the excitation required to overcome the armature demagnetizing effect. This method therefore enables this effect to be determined experimentally. If there were no armature reaction, the excitation could be reduced from OM to ON, the terminal volts remaining equal to PM. If the armature had no

reactance, the terminal voltage would rise from $SN(=PM)$ to RN. Now both armature reaction and reactance drop in volts are proportional to the current, and since the current is kept constant during the test, it follows that the triangle RSP fits the two curves at all points, as shown. Hence, it is necessary to determine two points only on the wattless current curve, namely A and some point such as P. The no load magnetization curve can then be shifted parallel to itself in the direction RP until it fits the two points A and P. It then gives the wattless current characteristic.

The disadvantage of this test lies in the fact that it is very difficult to obtain an inductive load of large capacity. Chokers can be

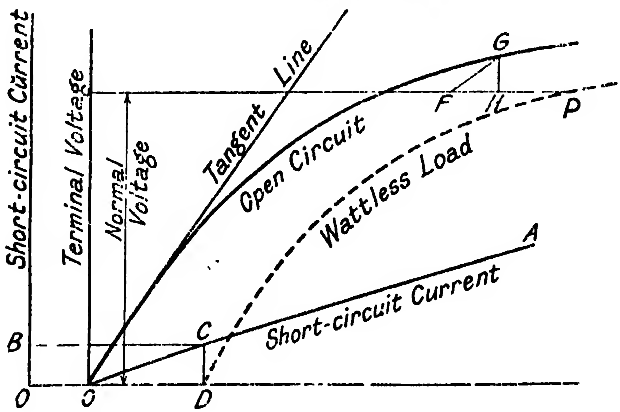


FIG. 276

used for small machines, but the most convenient load for a large machine consists of a number of synchronous motors running idle with their excitations reduced as far as possible. Such a load is not absolutely wattless, but it is found that when the power factor is very low, the magnetization curve is almost exactly coincident with that for zero power factor.

We will now consider a practical method, which is a combination of the three previous methods. The open-circuit and short-circuit characteristics are determined, as in the Behn-Eschenberg method. Next a small portion of the wattless load characteristic is then determined, or, since the method only requires one point on this curve, the wattless load can be adjusted to give normal full-load current at normal voltage. This point is represented by the point P in Fig. 276. Mark off OB equal to the current flowing in the wattless load test, draw the horizontal BC, and then the perpendicular CD. Then OD represents the excitation required to produce this current on short-circuit, and since the short-circuit current is very nearly wattless, the point P is on the zero power-factor characteristic for this particular current, as shown dotted. From P mark off PF equal

to OD, and then draw FG parallel to the tangent at the origin to the open-circuit characteristic, to cut this characteristic at G. Then draw the perpendicular GH. The length PH is the demagnetizing ampere-turns due to a wattless load of amperage equal to that in the wattless current test, while GH is the reactance drop per phase in the armature. Hence, GH measured in volts, divided by the current, gives the reactance per phase of the machine. The excitation for the same current loading during short-circuit is OD, hence PH subtracted from OD ($= FP$) gives the excitation required to supply the reactance drop in the machine. The regulation is now determined as follows. Draw OA, Fig. 277, to represent the normal

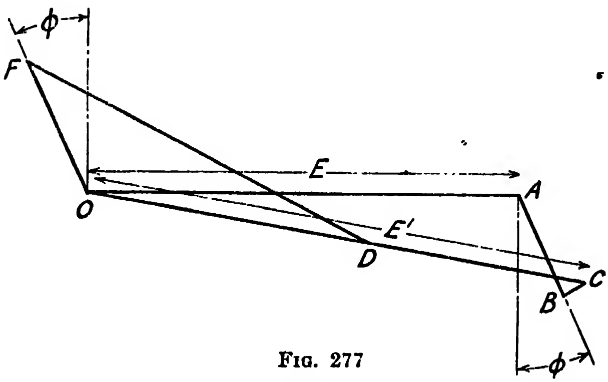


FIG. 277

voltage per phase, and set off AB equal to the reactance drop, and inclined φ to the perpendicular to OA; cos φ is the power factor of the load at which the regulation is required. Next set off the perpendicular BC equal to the resistance drop per phase. Joining OC we obtain the E.M.F. generated per phase and, referring back to the open-circuit characteristic, we obtain the excitation required to generate this voltage. Mark this excitation off along the direction of OC; let it be OD. Then mark off OF equal to the demagnetizing ampere-turns, or the equivalent exciting current (whichever is plotted in Fig. 276) as shown, and complete the triangle OFD. Then FD is the excitation required to maintain the normal terminal voltage at the load current used in the construction, and at a power factor equal to cos φ. Referring back to the open-circuit characteristic, the voltage corresponding to this excitation is read off. Calling this E_0, we have for the regulation up

$$\text{Percentage regulation up} = \frac{E_0 - E}{E} \times 100$$

It is very common practice to use a calculated, instead of a measured, value of the demagnetizing ampere-turns. As an example of the method, consider a 600-kVA, 3,300-volt, 3-phase, 50-cycle alternator whose open-circuit and short-circuit characteristics are

given in Fig. 278. The calculated value of the demagnetizing ampere-turns per pole at full-load current and zero power factor is 2,780, and the stator resistance per phase is 0·127 ohm.

$$\text{Full-load current} = \frac{600 \times 1{,}000}{\sqrt{3} \times 3{,}300} = 105 \text{ amp.}$$

and on referring to the short-circuit characteristic, we see that the excitation on short-circuit to give this current is 3,080 ampere-turns per pole. Now the armature reaction has been shown to be almost

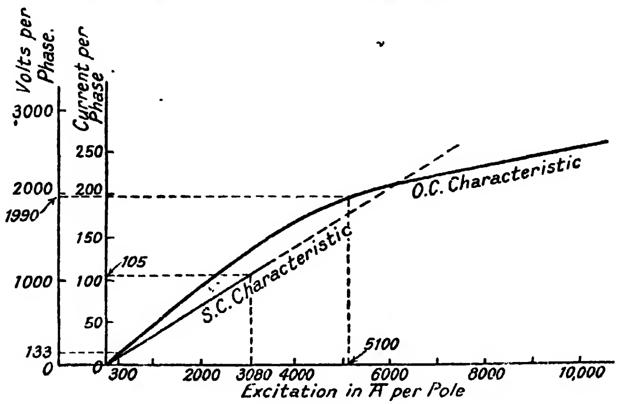


FIG. 278

entirely demagnetizing during short-circuit conditions, so that the excitation to generate the armature voltage required to overcome the reactance drop is (3,080 – 2,780) = 300 ampere-turns per pole. Referring this to the open-circuit characteristic, we see that the reactance drop per phase at full-load current is 133 volts. Also

$$\text{Resistance drop per phase} = 105 \times 0{\cdot}127 = 13{\cdot}3 \text{ volts}$$

$$\text{and normal terminal volts per phase} = \frac{3{,}300}{\sqrt{3}} = 1{,}910 \text{ volts.}$$

There is now sufficient data to make use of the construction of Fig. 277. The combined voltage and ampere-turn diagram is drawn to scale in Fig. 279 for a load current of the normal value of 105 amperes per phase, at a power factor of 0·8 lagging.

$$\therefore\ \varphi = \text{arc cos } 0{\cdot}8 = 37° \text{ (very approx.)}$$

The diagram gives a generated voltage of 1,990 per phase, which by reference to the open-circuit characteristic, requires an excitation of 5,100 ampere-turns. The demagnetizing ampere-turns of 2,780 are set off along *OF*, as explained previously, the resultant excitation therefore being 7,240 ampere-turns per pole. This corresponds to a

voltage per phase of 2,240 when the load is thrown off, the regulation up, therefore, being

$$\frac{2{,}240 - 1{,}910}{1{,}910} \times 100 = 17{\cdot}6 \text{ per cent}$$

4. **Automatic Voltage Regulators.** Modern quick acting voltage regulators are based on the "overshooting the mark" principle. When the load on the alternator increases and more excitation is required to keep the voltage constant, the regulator produces an

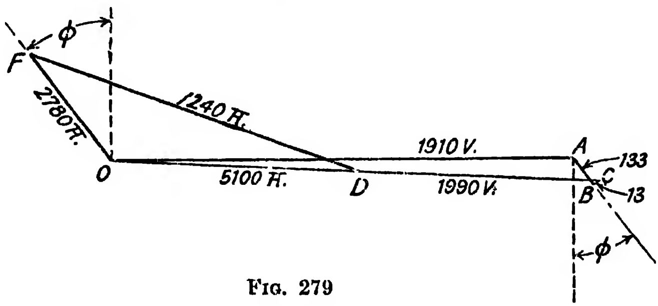


Fig. 279

increase in excitation more than is ultimately necessary. This is because the fluctuations in load may be very rapid, whereas the inductance of the alternator field will cause the flux to increase to the desired value very slowly, unless there is a large increase in exciting current. Before the voltage has time to increase to the value corresponding to the increased excitation, the regulator reduces the excitation again. There are two main types—

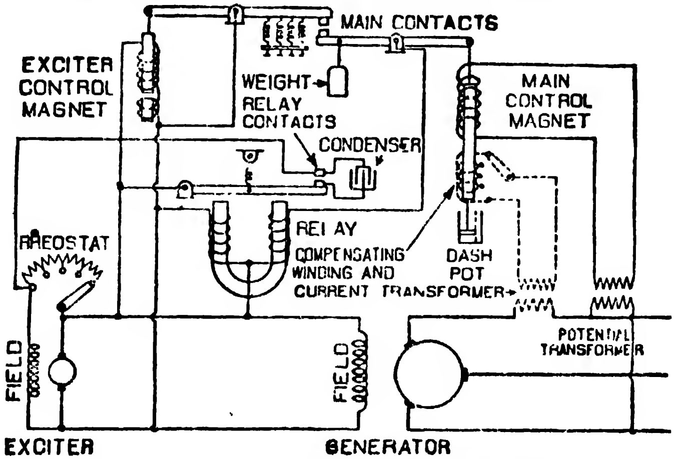
(*a*) The vibrating type, in which a fixed resistance is cut in and out of the exciter field circuit.

(*b*) The rotary type, in which a variable resistance is used.

5. **The Tirrill Regulator.** This is an example of the vibrating type. The essential parts are shown in Fig. 280. At the top there are two levers, that on the left operated by a solenoid energized in proportion to the exciter voltage, and that on the right by an A.C. magnet having both shunt and series excitations. This magnet is so adjusted that with normal load and voltage at the alternator, the pulls of the two coils are equal and opposite. In the event of an increase in load the series coil predominates, and so pulls down the A.C. magnet. This closes the contacts between the two levers and so de-energizes the relay horse-shoe magnet, which thereby releases its armature and short-circuits the rheostat in the exciter field. There is in consequence a sudden increase in excitation, which causes the alternator voltage to rise very rapidly. At the same time the excitation of the exciter control magnet is increased, thus pulling down the left-hand lever, energizing the

relay, and re-inserting the field rheostat before the alternator voltage has had time to increase too far. The reverse action takes place when the load on the alternator decreases.

It will be seen that because of the overshooting the mark principle the terminal voltage does not remain absolutely steady, but oscillates



(*By courtesy of the British Thomson Houston Co., Ltd.*)

FIG. 280. TIRRILL AUTOMATIC VOLTAGE REGULATOR

rapidly between maximum and minimum values. The regulator is so quick acting that the variation in voltage is less than 1 per cent.*

6. **Parallel Operation of Alternators.** Before an incoming machine can be switched on to the bus-bars the following conditions have to be fulfilled—

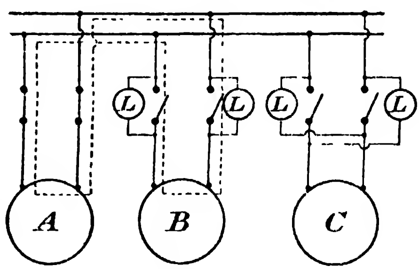


FIG. 281

ALTERNATORS IN PARALLEL

(*a*) The voltage of the incoming machine must be the same as the bus-bar voltage.

(*b*) The phase of the machine voltage must be identical with the phase of the bus-bar voltage relative to the feeders, i.e. opposite in phase relative to the local circuit through the armatures and bus-bars. This circuit is shown dotted in Fig. 281.

(*c*) The frequency of the incoming machine must be the same as the bus-bar frequency.

* For fuller information see Garrard, *Electric Switch and Control Gear*.

Condition (*a*) is indicated by a voltmeter, and conditions (*b*) and (*c*) are both indicated by synchronizing gear.

Consider first of all the case of single phase alternators. The simplest form of synchronizer consists of two lamps, *LL*, connected across the main switch as indicated in Fig. 272. If the frequencies of the alternators *A* and *B* are not equal, the phase angle between the voltages of *A* and *B* will be continually changing, and therefore, the current through the lamps and through the local circuit shown dotted will be changing. The resultant voltage will undergo changes similar in character to the beats produced when two sources of sound of slightly different frequencies are sounding together. This is indicated in Fig. 282. In consequence, the lamps will flicker, the alternations in brightness being rapid when there is a large difference in the frequencies, and slow when the frequencies are nearly equal. In the middle of a dark period the two voltages will be in opposition with respect to the local circuit. Hence, the speed of the incoming machine is adjusted until the lamps go in and out very slowly, the incoming voltage is adjusted equal to the bus-bar voltage, and then the switch is closed in the middle of a dark period. It is somewhat easier to judge the middle of the bright, than the middle of the dark, period, and some engineers prefer to synchronize "lamps bright." This necessitates the crossing over of the lamp connections, as in the case of machine *C*.

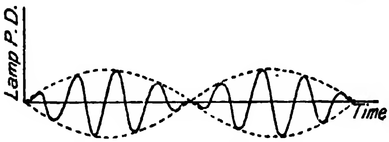


FIG. 282
VOLTAGE VARIATIONS IN SYNCHRONIZER

7. Conditions Necessary for Successful Parallel Operation. Since, with respect to the local circuit, the E.M.F. of an alternator is in phase opposition to the E.M.F. of another alternator with which it is working in parallel, the machines run as synchronous motors relative to one another. Hence, if one machine gets into difficulties, say, through a failure in steam supply, it must receive wattful motoring current from the other.

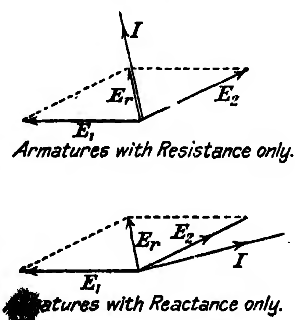


FIG. 283
SYNCHRONIZING CURRENT

(*a*) Consider two machines having resistance but no reactance. Their E.M.F.s E_1 and E_2 (Fig. 283) will be practically in phase opposition, so that their resultant E_r will be almost in quadrature with both E_1 and E_2. The synchronizing current *I* will be in phase

with E_r and therefore, practically in quadrature with E_1 and E_2, so that it will be an idle current and will therefore convey no real power to the machine needing help.

(b) Suppose that the armatures have reactance only. Then the synchronizing current I will be in quadrature with E_r, and therefore, practically in phase with one of the machine voltages; E_2 in Fig. 283. Thus machine II will supply real power to machine I, so that the latter will keep running. This shows that for successful parallel operation, reactance in the armatures is absolutely necessary.

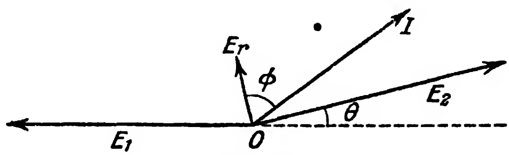


FIG. 284

Now consider actual machines with both resistance and reactance, let the angular phase difference of the two induced E.M.F.s be θ, and let the circulating current I lag an angle φ behind E_r, as shown in Fig. 284. Then so long as θ is small and the E.M.F.s are equal, we can write

$$E_1 = E_2 = E, \text{ and } E_r = E\theta$$

$\therefore$ Circulating current

$$I = \frac{E_r}{Z} = \frac{E\theta}{Z}$$

where Z is the combined impedance per phase of the two armatures. Now so long as θ is small, the angle between OI and OE_2 is very nearly equal to $(90 - \varphi)$, and therefore

$$\text{Synchronizing power } W_s = E_2 I \cos(90 - \varphi)$$

$$= \frac{E^2}{Z} \cdot \theta \sin \varphi$$

$$\therefore \qquad \frac{dW_s}{d\theta} = \frac{E^2}{Z} \cdot \sin \varphi$$

$$= \frac{E^2}{\sqrt{R^2 + X^2}} \times \frac{X}{\sqrt{R^2 + X^2}}$$

$$= E^2 \times \frac{X}{R^2 + X^2}$$

and this is a maximum when $X = R$, showing that the maximum synchronizing power would be given with an armature reactance equal to the armature resistance. This condition is, of course, never fulfilled in actual practice, the resistance always being small in

comparison with the reactance. Nor is it desirable that this condition should be fulfilled, since it would give rise to an excessively high restoring torque whenever a machine deviated from the steady angular position.

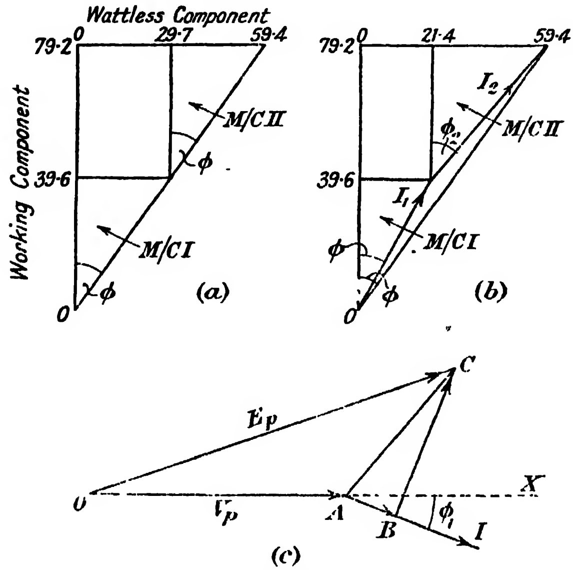


FIG. 285

Example. Two identical three-phase alternators are coupled in parallel to a total load of 1,500 kW at 11,000 volts, power factor 0·8 lagging. The synchronous reactance of each machine is 60 ohms per phase, and resistance 2·8 ohms per phase. The power supplied by each machine being maintained the same, the excitation of the first alternator is adjusted so that its armature current is 45 amp. lagging. Calculate—

(*a*) The armature current of the second alternator.
(*b*) The power factor at which each alternator operates.
(*c*) The E.M.F. of the first alternator.

Total load current

$$I = \frac{\text{watts}}{\sqrt{3}\,V\cos\phi}$$

$$= \frac{1{,}500{,}000}{\sqrt{3} \times 11{,}000 \times \cdot 8} = 99 \text{ amp.}$$

Working component $= I\cos\phi = 99 \times \cdot 8 = 79 \cdot 2$

Wattless component $= I\sin\phi = 99 \times \cdot 6 = 59 \cdot 4.$

When the conditions are identical, each machine delivers one-half of each component, so that the current diagram is as shown in Fig.

285 (*a*), and each machine has the same power factor as the load. Also each total machine current = 49·5 amp.

Since the steam supplies are not altered, the working components will remain the same at 39·6 amp. per machine, but an adjustment of the excitations will alter the division of wattless current. The total current taken by machine I is reduced from 49·5 to 45 amp. and therefore its wattless current is reduced, the new value being

$$\sqrt{45^2 - 39{\cdot}6^2} = 21{\cdot}4 \text{ amp.}$$

Hence the wattless current delivered by machine II is

$$59{\cdot}4 - 21{\cdot}4 = 38 \text{ amp.}$$

The current diagram is now as in Fig. 276 (*b*), and we have

(*a*) $$I_2 = \sqrt{39{\cdot}6^2 + 38^2} = 55 \text{ amp.}$$

(*b*) $$\cos\phi_1 = 39{\cdot}6/45 = {\cdot}88$$

$$\cos\phi_2 = 39{\cdot}6/55 = {\cdot}73$$

(*c*) Terminal P.D. per phase

$$V_p = \frac{11{,}000}{\sqrt{3}} = 6{,}350 \text{ volts.}$$

For machine I

$$\begin{aligned} \text{Resistance drop per phase} &= RI_1 = 2{\cdot}8 \times 45 = 126 \\ &= AB, \text{ Fig. 276 } (c) \\ \text{Reactive drop per phase} &= XI_1 = 60 \times 45 = 2{,}700 \\ &= BC, \text{ Fig. 276 } (c). \end{aligned}$$

Denoting by $\overline{X}$ and $\overline{Y}$ the OX and OY components of the E.M.F. per phase, E_p, we have

$$\begin{aligned} \overline{X} &= V_p + RI_1 \cos\phi_1 + XI_1 \sin\phi_1 \\ &= 6{,}350 + 126 \times {\cdot}88 + 2{,}700 \times {\cdot}475 \\ &= 7{,}741 \text{ volts.} \\ \overline{Y} &= XI_1 \cos\phi_1 - RI_1 \sin\phi_1 \\ &= 2{,}700 \times {\cdot}88 - 126 \times {\cdot}475 \\ &= 2{,}310 \text{ volts.} \\ \therefore\ E_p &= \sqrt{7{,}741^2 + 2{,}310^2} \\ &= 8{,}060 \text{ volts} \\ \therefore\ E &= \sqrt{3} \times 8{,}060 \\ &= 14{,}000 \text{ volts.} \end{aligned}$$

8. Division of Load between Two Machines. When a machine is switched on to the bus-bars its steam supply is small, so that it may be actually taking a small amount of motoring power from the bus-bars. At the most it will only supply a small amount of power to them. Its working point may therefore be in some position such as P_1 on the diagram (Fig. 286). To bring it into the generating half of the diagram, it is necessary to increase the steam supply. This causes the excitation vector BP_1 to swing round until the working point reaches, say, P_2. The machine is now generating, but its output may not be sufficient, and its power factor will be low because of the large angle of lag, $\angle P_2AZ$, of the current AP_2 behind the voltage. The next adjustment is therefore to increase the excitation until the power factor is in the neighbourhood of unity. During this adjustment the working point travels along a power line from P_2 to P_3. The steam supply is again increased, thus bringing the working point to P_4 on a line of greater output, and the excitation is then increased again. Thus the adjustments are made in steps, until the machine is delivering the required output. When adjusting the steam supply, the driver watches the wattmeter, whereas the adjustment of the excitation is made by means of the ammeter, or of a power factor meter.

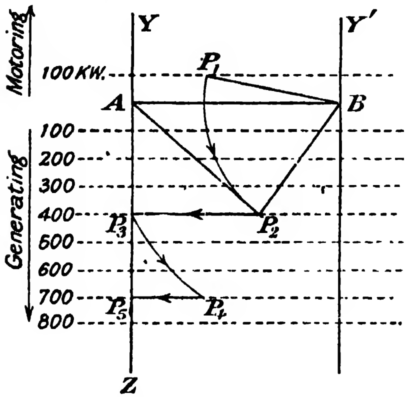


FIG. 286
TO ILLUSTRATE DIVISION OF LOAD

Compare this with the case of a D.C. shunt generator in which the load is adjusted by varying the excitation, the engine governor automatically admitting the required amount of steam.

9. Synchronizing Gear for Three-Phase Machines. By arranging three lamps across the poles of the main switch as in the case of machine B (Fig. 287), it is possible to synchronize with lamps dark. A better arrangement is to cross connect two of the lamps as in the case of machine C. Suppose that the voltage star ABC (Fig. 288) refers to the bus-bars, and $A'B'C'$, to the incoming machine C. Then the instantaneous voltages across the three lamps in the case of machine C are given by the vectors AB', $A'B$, and CC'. Now both vector diagrams are rotating in space, but they will only have the same angular velocities if the incoming frequency is equal to the bus-bar frequency. Suppose the incoming machine is too slow. Then diagram $A'B'C'$ will rotate more slowly

than ABC, so that at the instant represented AB' is decreasing, $A'B$ is increasing, and CC' is increasing.

If the incoming machine is too fast, then AB' is increasing, $A'B$ is decreasing, and CC' is decreasing.

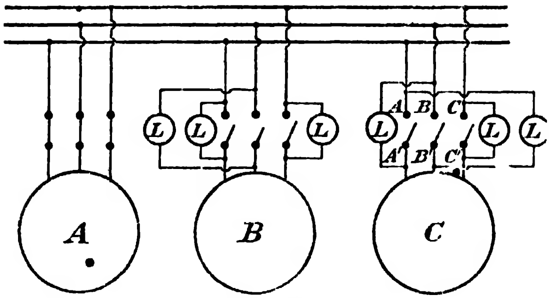


FIG. 287
THREE-PHASE ALTERNATORS IN PARALLEL

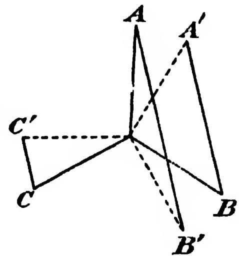


FIG. 288
VOLTAGE RELATIONS IN SYNCHRONIZER

Hence, if the three lamps are placed in a ring a wave of light will travel in a clockwise or counter-clockwise direction round the ring, according as the incoming machine is fast or slow. This arrangement therefore indicates whether the speed must be decreased or increased. The switch is closed when the changes in light are very slow, and at the instant the lamp connected directly across one phase (CC' in the figure) is dark. Lamp synchronizers are only suitable for small low voltage machines.

For large machines a rotary synchroscope is almost invariably used. This synchroscope, which is based on the rotating field principle (*see* Chap. XVI), consists of a small motor with both field and rotor wound two phase. The stator is supplied by a pressure transformer connected to two of the main bus-bars, while the rotor is supplied through a pressure transformer connected to a corresponding pair of terminals on the incoming machine. Two-phase current is obtained from the phase across which the instrument is connected by a split-phase

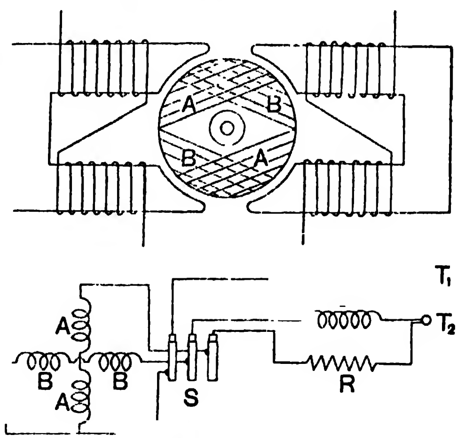


FIG. 289
ROTARY SYNCHROSCOPE

device, as shown in Fig. 289. One rotor, phase A, is in series with a non-inductive resistance R, and the other, B, in series with an inductive coil C, the two then being connected in parallel. The phase difference so produced in the currents through the two rotor coils causes the rotor to set up a rotating magnetic field. By a similar device, the stator produces a rotating magnetic field. Now if the incoming machine has the

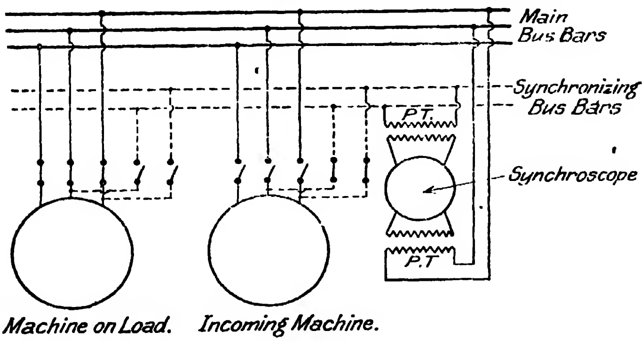


FIG. 290

ARRANGEMENT OF SYNCHRONIZING GEAR

same frequency as the bus-bars, the two fields will travel at the same speed, and therefore, the rotor will exhibit no tendency to move. If the incoming machine is not running at the correct speed, then the rotor will tend to rotate at a speed equal to the difference in the speeds of the rotating fields set up by its rotor and stator. Thus it will tend to rotate in one direction if the incoming machine is too slow, and in the opposite direction if too fast.

In practice, it is usual to perform the synchronizing on a pair of auxiliary bars, called synchronizing bars. The rotor of the synchroscope is connected permanently to these bars, and the incoming machine switched on to these bars during synchronizing. In this way, one synchroscope can be used for a group of alternators. The arrangement of the synchronizing bars and switch gear is illustrated in Fig. 290.

CHAPTER XX

THE INDUCTION MOTOR

1. THE polyphase induction motor is used more extensively than any other form of A.C. motor. It derives its name from the fact that the current in its rotor is not drawn from the supply, but is *induced* by relative motion of the rotor conductors and the magnetic field produced by the stator currents. The stator, which is built exactly like an alternator stator, carries a polyphase winding similar to an alternator winding. The stator winding is connected to the supply, and the polyphase currents circulating through it produce a magnetic field which rotates at synchronous speed. The lines of force of the stator field cut the rotor conductors and induce currents in them; hence, by Lenz's Law, the rotor follows after the stator field Now it is on the relative velocity of the rotor and stator field that the magnitude of the induced rotor current, and therefore, of the torque, depends; hence, even on no load the rotor speed must be *less* than synchronous speed, because of the friction and iron losses. When extra load is put on the motor, the torque must increase in order to cope with it; hence, the induced rotor current must increase, and therefore, the relative velocity of rotor to stator field must increase. But the speed of the stator field is fixed by the frequency of the supply, and therefore, as the load on the motor increases, the actual speed of the rotor decreases. Its speed characteristic is therefore somewhat similar to that of a D.C. shunt motor.

2. For a unidirectional torque to be set up, the rotor field produced by the induced rotor currents must have the same angular velocity as the main stator field, and therefore, the lines of force "slip" past the rotor conductors.

Let ω_1 = angular velocity of stator field, and therefore of rotor field

ω_2 = actual angular velocity of rotor

The difference $(\omega_1 - \omega_2)$ is called the "absolute" slip.

The ratio $\frac{\omega_1 - \omega_2}{\omega_1}$ is called the "fractional" slip, and $\frac{\omega_1 - \omega_2}{\omega_1} \times 100$ is the percentage slip.

The full load slip of modern motors varies from about 4 per cent for small motors to 1·5 or 2 per cent for very large ones. The actual speed is thus very close to the synchronous speed under normal working conditions.

When the rotor is stationary, the stator field sweeps past the rotor conductors with full synchronous speed, and the frequency of the rotor current is equal to the line frequency, f. The motor then functions like a transformer with distributed windings and a narrow air gap in the magnetic circuit. When running at a speed near to synchronism the relative velocity of stator field and rotor is $\sigma\omega_1$, where σ is the fractional slip. But the frequency of the rotor currents is proportional to this relative velocity. Hence, the rotor currents have a frequency of σf.

3. There are two types of rotor winding, the "squirrel-cage" and the "phase wound." The squirrel-cage rotor consists of a number of copper bars threaded through slots in a laminated rotor core, with their ends connected to stout copper end rings. These windings are thus permanently short-circuited on themselves, and external resistance cannot be connected in series for starting purposes.

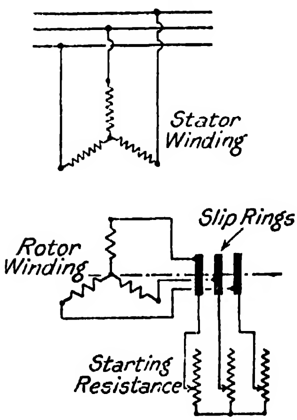


FIG. 291
INDUCTION MOTOR CONNECTIONS

The phase wound rotor is provided with a distributed winding ; it is nearly always a two layer wave winding in large motors. The separate phases are generally connected in star, and the three free ends, connected each to one of three insulated slip rings mounted on the motor shaft. When running normally these slip rings are short-circuited, but for starting purposes they are connected to a three phase star-connected starting resistance, as in Fig. 291.

The squirrel-cage rotor is as a rule used only for comparatively small motors, because, as we shall see, its starting torque is very small and it takes a large current from the line at the moment of switching on. The rotor resistance is very low, and therefore, the motor is very efficient, and the simple construction of the rotor renders it mechanically robust, and comparatively cheap to manufacture. The small starting torque is the result of the low rotor resistance. When squirrel-cage motors are used for crane or hoist work, where a large starting torque is more important than high efficiency, because of the intermittent nature of the load, it is usual to increase the rotor resistance by making the end rings, and sometimes the bars as well, of high resistance metal, e.g. German silver.

4. **Relation between Slip and Rotor I^2R Loss.** When an induction motor is operating normally, the slip is so small that the frequency of the magnetic reversals in the rotor core is only of the order of

one or two per second, the rotor iron loss therefore being quite negligible. Hence

$$(\text{Power of rotating field}) = \text{Output} + \text{rotor copper loss}$$

If T is the motor torque, then

$$\text{Power of rotating field} = T\omega_1$$

$$\text{Output} = T\omega_2$$

$$\therefore \quad T\omega_1 = T\omega_2 + \text{rotor copper loss}$$

$$\therefore \quad \omega_1 - \omega_2 = \frac{\text{rotor copper loss}}{T}$$

$$\therefore \quad \frac{\omega_1 - \omega_2}{\omega_1} = \frac{\text{rotor copper loss}}{T\omega_1}$$

$$= \frac{\text{rotor copper loss}}{\text{input power}}$$

$$\therefore \quad \text{Fractional slip} = \frac{\text{rotor copper loss}}{\text{input power}}$$

$$\text{and percentage slip} = \frac{\text{rotor copper loss}}{\text{input power}} \times 100$$

$$= \text{rotor copper loss expressed as}$$

a percentage of the input power of the motor.

This relation shows why it is that motors with a very low rotor resistance have a small slip, while those with a high rotor resistance have a large slip.

5. Torque. The mechanism of torque production in the induction motor is essentially the same as in the D.C. motor. Fig. 292 shows one rotor conductor which, for convenience, is again shown midway between the stator and rotor surfaces. The stator field is travelling in a clockwise direction, and therefore, since the rotor speed cannot be equal to the stator field speed, the rotor is travelling counter-clockwise *relative to the stator field.* Hence when applying the right-hand rule to the rotor conductor its motion must be considered as being from right to left. This shows that the E.M.F. in the conductor acts outwards, and therefore the current acts outwards. The directions of the lines of force in the two component fields are therefore as shown in the first figure, while the resultant field distribution is as given in the second figure. Thus lines of force are bent round the conductor and a torque set up in the same direction as the stator

field. In this case also the forces which exist in an actual machine act on the tops of the rotor teeth and not on the conductors themselves, and the student is advised to draw the figure corresponding to Fig. 91.

In the D.C. motor the torque is proportional to the product of the flux per pole and the armature current, and similarly in the induction motor the torque is proportional to the flux per stator pole and the rotor current. There is, however, an additional factor to be taken into account, namely the rotor power factor. We have seen that the rotor frequency at any fractional slip σ is σf, and

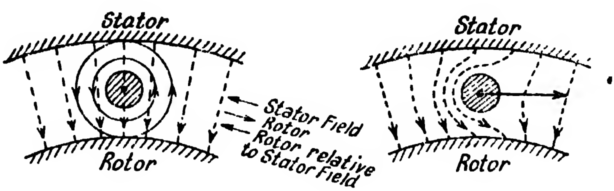


FIG. 292

TORQUE PRODUCTION OF AN INDUCTION MOTOR

therefore if X_2 is the rotor standstill reactance, its reactance at any slip σ is σX_2. First of all imagine the rotor at rest, then its reactance is X_2 and its impedance therefore

$$Z_2 = \sqrt{R_2^2 + X_2^2}$$

while $$\cos \phi_2 = \frac{R_2}{Z_2}$$

Now in a normal motor X_2^2 is much larger than R_2^2 with the result that

$$Z_2 \simeq X_2$$

and $\cos \phi_2$ is small.

Thus the current will lag the rotor induced voltage by an angle of the order of 90°.

Now imagine that the rotor could travel at exactly synchronous speed, then at this speed

$$\sigma = 0 \quad \therefore \quad \sigma X_2 = 0$$
$$\therefore \quad Z_2 = R_2$$
$$\text{and } \cos \phi_2 = 1$$

Actually, an unregulated motor cannot run exactly at synchronism but the slip is so small that the condition $\cos \phi_2$ equal to unity is very nearly fulfilled. We will therefore consider the two extreme cases of $\cos \phi_2 =$ unity and $\cos \phi_2 = 0$. Fig. 293 shows two pole

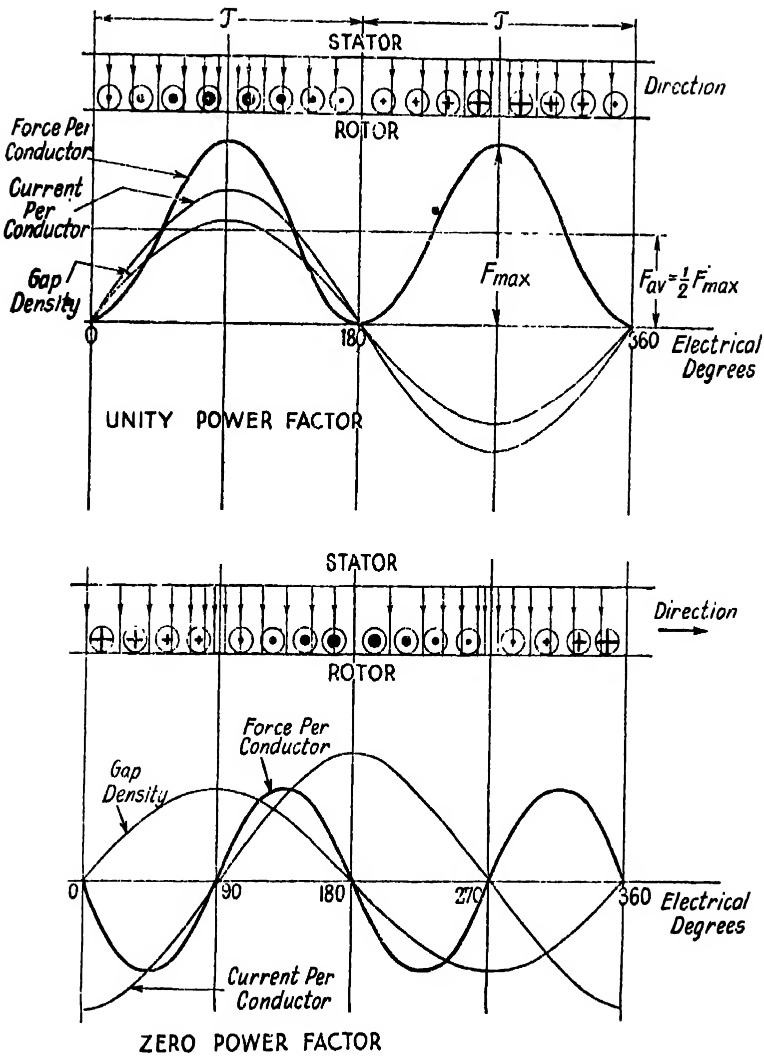


FIG. 293

EFFECT OF ROTOR POWER FACTOR ON THE TORQUE OF AN INDUCTION MOTOR

pitches of an induction motor. We have already seen that the distribution in space of the M.M.F. of a three-phase winding is stepped and is not given by a smooth curve. But owing to the tendency of a magnetic flux to spread out laterally the flux density distribution produced by this M.M.F. will have the corners rounded off, so that when considering the general properties of the motor we can assume that the distribution of air-gap flux density in space is sinusoidal. This will result in a sinusoidal distribution of rotor induced E.M.F. in space, and therefore a sinusoidal distribution of rotor current in space. Hence for the case of unity power factor the gap density and rotor current distributions are in space phase. Now the force on a conductor is given by the product of its current and the gap density at its centre, and consequently by plotting the product of gap density and corresponding rotor current we obtain a curve giving the distribution of force round the rotor periphery. We see that for unity power factor this force is always positive but that at intervals of a pole pitch (τ) it falls to zero. The average force is one-half of the maximum force.

The second diagram of Fig. 293 shows the operating conditions for a rotor power factor of zero. The current lags 90 degrees behind the induced voltage, and therefore the rotor will have travelled 90 electrical degrees, or half a pole pitch, before the current in any particular bar will be the same as in the previous figure. In other words, the rotor current distribution will be shifted 90 electrical degrees to the right. On plotting the curve of force per conductor we now see that the force is reversed at intervals of half a pole pitch, and that the average force is zero. Hence the torque is zero. These curves are analogous with those of Figs. 165 and 166, giving the curves of power in a single-phase circuit, and it therefore follows that the torque developed by the rotor is proportional to the rotor power factor.

We thus see that in the induction motor the flux is proportional to

$$\left(\begin{matrix}\text{Flux entering rotor}\\ \text{from stator}\end{matrix}\right) \times \text{rotor current} \times \left(\begin{matrix}\text{cos of angle between}\\ \text{flux and current}\end{matrix}\right)$$

The flux is proportional to the stator applied voltage E_1, exactly as in a transformer.

$$\text{Rotor current per phase} = \frac{\text{rotor E.M.F.}}{\text{rotor impedance per phase}}$$

When the rotor is stationary, there will be a certain transformation ratio K between stator and rotor, and for the rotor E.M.F. we have

$$E = KE_1$$

When the motor is running, the relative velocity of rotor and stator field is proportional to the fractional slip σ, and since the fractional slip at standstill is unity, we have for the rotor E.M.F. when running,

$$E_2 = \sigma K E_1.$$

Again, the frequency of the rotor current is σf; hence, if X_2 is the rotor reactance per phase to currents of line frequency, its reactance to currents of slip frequency will be σX_2. Hence, the rotor current is given by

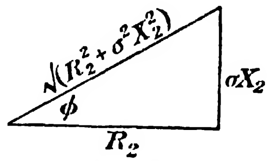


FIG. 294
ROTOR IMPEDANCE TRIANGLE

$$\frac{\sigma K E_1}{\sqrt{R_2^2 + \sigma^2 X_2^2}}$$

Again, from the impedance triangle (Fig. 294), we have

$$\cos \varphi = \frac{R_2}{\sqrt{R_2^2 + \sigma^2 X_2^2}}$$

The expression for the torque, T, therefore becomes

$$T \propto E_1 \times \frac{\sigma K E_1}{\sqrt{R_2^2 + \sigma^2 X_2^2}} \times \frac{R_2}{\sqrt{R_2^2 + \sigma^2 X_2^2}}$$

$$\propto \frac{\sigma K E_1^2 R_2}{R_2^2 + \sigma^2 X_2^2}$$

Several important properties of the motor can be deduced from this expression—

$$(a)\ T \propto \frac{1}{\sigma K} \times \left\{\frac{\sigma K E_1}{\sqrt{R_2^2 + \sigma^2 X_2^2}}\right\}^2 \times R_2$$

The quantity in the brackets is the rotor current, and we therefore have

$$T \propto \frac{I_2^2 R_2}{\sigma}, \text{ since } K \text{ is a constant}$$

$$\propto \frac{\text{rotor copper loss}}{\text{slip}}$$

(*b*) When the motor is running under normal conditions σ is very small, and we can neglect $(\sigma X_2)^2$ in comparison with R_2^2. Hence, under these conditions,

$$T \propto \sigma E_1^2, \ K \text{ and } R_2 \text{ being constants}$$

$$\propto E_1^2(\omega_1 - \omega_2)$$

Now rotor intake $\propto$ to $T\omega_1$

$$\therefore \quad (\omega_1 - \omega_2) \propto \frac{\text{rotor intake}}{E_1^2 \omega_1}$$
$$\propto \text{rotor intake}$$

so long as E_1 is constant. This shows that the *fall in speed* is proportional to the rotor intake, and therefore, very approximately to the output. We also see that the torque is proportional to the square of the applied voltage, a property of the motor which renders it very susceptible to changes in voltage. It is necessary that the voltage regulation of a line supplying induction motors should be very good.

(*c*) At the moment of starting, the rotor is stationary, and therefore, $\sigma = 1$. Hence

$$\text{starting torque} \propto \frac{R_2}{R_2^2 + X_2^2}$$

This is a maximum when $R_2 = X_2$, R_2 being regarded as the variable. Therefore, to obtain the maximum starting torque, sufficient resistance should be put in series with the rotor to make the total resistance per phase equal to the rotor reactance per phase, to currents of line frequency. The variation of starting torque with rotor resistance is shown graphically in Fig. 295. A squirrel-cage or short-circuited rotor has a small starting torque because its rotor resistance is very small, and can not be increased for starting purposes by the addition of an external resistance.

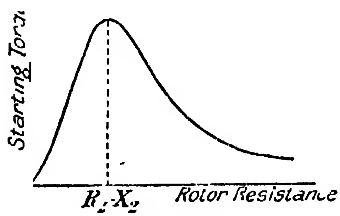


FIG. 295

EFFECT OF ROTOR RESISTANCE ON STARTING TORQUE

(*d*) Under running conditions,

$$T \propto \frac{\sigma R_2}{R_2^2 + \sigma^2 X_2^2}$$

This is a maximum when $R_2 = \sigma X_2$, σ now being the variable. Substituting R_2/X_2 for σ, we have for the maximum torque under running conditions

$$T_{max} \propto \frac{\frac{R_2}{X_2} R_2}{R_2^2 + \frac{R_2^2}{X_2^2} X_2^2} \propto \frac{1}{X_2}$$

Since X_2 is a constant for a given motor, we see that the maximum torque attainable is a constant and is independent of the rotor resistance. Hence, if we draw a family of torque/slip or torque/speed curves for a series of values of the total rotor resistance, they will all have the same maximum value. Such a family of curves is shown in Fig. 296. The perpendicular OY is drawn at unity slip

or zero speed, and the intercepts on this axis give the starting torque of the motor with a rotor resistance equal to that corresponding to the particular curve. If R_2 is equal to X_2 the maximum torque occurs at starting, as shown by curve III, and for any other resistance the torque at starting is less than the maximum value. The shaded portion of the diagram represents the normal working range, and it will be seen that although a high resistance is an advantage for starting purposes, it is a disadvantage when running normally. Curve I is for a normal rotor of small resistance: this gives a greater torque than any of the others within the working range, and the maximum torque is not far removed from this range, a disadvantage in the case of a motor

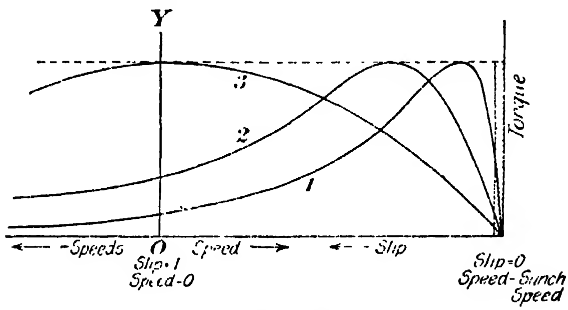


FIG. 296

TORQUE SLIP CHARACTERISTICS

subjected to sudden violent overloads, even if of short duration. Curve II is for a high resistance rotor, and we see that within the working range the increase in resistance causes a decrease in torque. For a given torque the motor represented by curve II suffers a larger slip than the normal motor, so that its speed characteristic is similar to that of a D.C. cumulatively compounded motor. For certain classes of work this characteristic is an advantage, since it relieves the motor both electrically and mechanically when the opposing torque is heavy. Also, the position of the maximum torque is farther removed from the normal working range, so that the motor is not so liable to pull out in the event of a serious overload. For average motors the pull out torque is about 2½ times the normal full load torque.

As an actual example consider the case of a 250-b.h.p. induction motor having a rotor resistance per phase of ·016 ohm and standstill reactance per phase of ·15 ohm. We have seen that the torque and rotor current can be written

$$T = K_1 \frac{\sigma R_2}{R_2^2 + \sigma^2 X_2^2}; \; I_2 = K_2 \frac{\sigma}{(R_2^2 + \sigma^2 X_2^2)^{\frac{1}{2}}}$$

where K_1 and K_2 are constants which need not be determined. Substituting the numerical values for R_2 and X_2, we have

$$T = K_1 \frac{\cdot 016\sigma}{(\cdot 016)^2 + (\cdot 15)^2\sigma^2}$$

$$I_2 = K_2 \frac{\sigma}{\{(\cdot 016)^2 + (\cdot 15)^2\sigma^2\}^{\frac{1}{2}}}$$

We will take the values of K_1 and K_2 to be unity, the units in which T and I_2 are expressed then being arbitrary units. Taking a series of values of σ from zero up to 2·0 in steps of 0·1, we can tabulate as follows, this being more expeditious than making an individual calculation for each point on the curves.

$R_2 = \cdot 016$. $X_2 = \cdot 15$.

σ	σ^2	σX_2^2	$R_2^2 + \sigma X_2^2$	$\sqrt{R_2^2 + \sigma^2 X^2}$	I_2 $\dfrac{\sigma}{\sqrt{R_2^2 + \sigma X_2^2}}$	T $\dfrac{\sigma}{R_2^2 + \sigma X_2^2}$
·1	·01	·000225	·000481	·0244	4·16	208
·2	·04	·0009	·001156	·034	5·88	180·4
·3	·09	·00203	·002256	·0477	6·3	131
·4	·16	·0036	·003856	·0621	6·42	101·6
·5	·25	·0056	·005856	·0765	6·55	86

It is not thought necessary to give the whole of the table, as the reader can make the necessary calculations for himself. The curves of T and I_2 against σ are plotted in Fig. 297, and we see that the maximum torque occurs for a slip of $\dfrac{\cdot 016}{\cdot 15} = \cdot 107$ or 10·7 per cent.

In the same figure the torque curves for a resistance R_2 of ·083 and for a resistance of ·16 ohm per phase are given, and it will be seen that these also exhibit a maximum at a slip of R_2/X_2, namely, for $\sigma = 1$, when $R_2 = X_2$, and for $\sigma = \dfrac{\cdot 083}{\cdot 15} = \cdot 553$, when $R_2 = \cdot 083$. For all three curves the magnitude of the maximum torque is the same.

6. **Crawling of Induction Motors.** Squirrel-cage motors sometimes exhibit a tendency to run at a speed very much smaller than synchronous speed, usually one-seventh. This is due to the fact that since it is impossible to distribute the stator winding in more than 3 or 4 slots per pole per phase, even in large machines, the wave of stator flux, instead of being a pure sine wave, is stepped. This wave, when analysed, gives, in addition to the fundamental wave which rotates at synchronous speed, odd numbered harmonics,

third, fifth, seventh, and so on. The third harmonic waves produced by the three phases neutralize one another as will be seen later, and the most important of the remaining harmonics are the fifth and the seventh. Now the field set up by the fifth harmonic, considering this separately, rotates backwards, while that due to the seventh harmonic rotates forwards. Hence, the torque/slip characteristic will have three components, that due to the fundamental, that due to the fifth harmonic, and that due to the seventh. These separate characteristics are drawn in Fig. 298, the

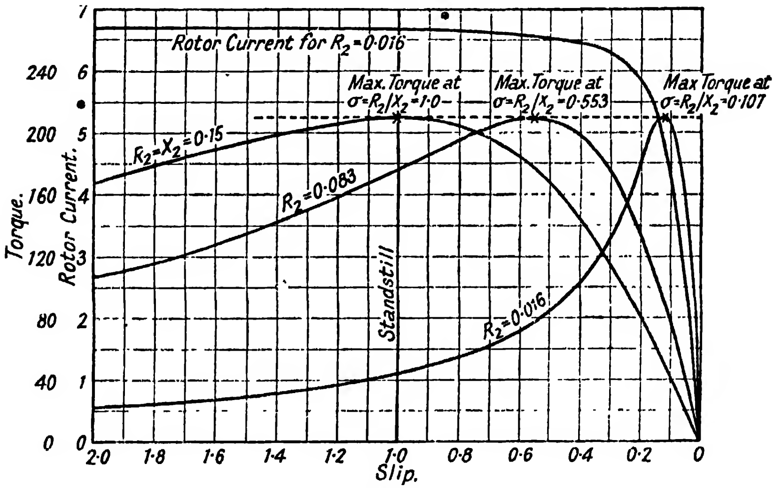


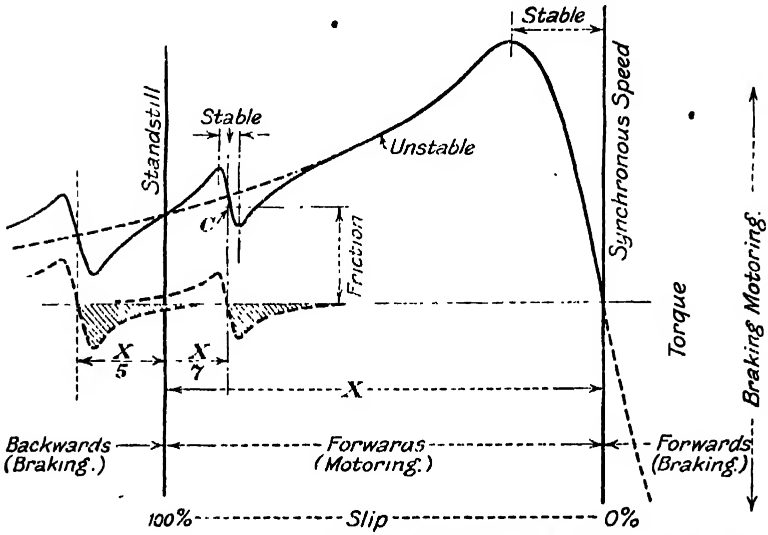
FIG. 297. INDUCTION MOTOR CHARACTERISTICS

total torque being obtained by adding corresponding ordinates of the three separate curves. The curve for the seventh harmonic crosses the slip axis at one-seventh fundamental speed. Similarly, the curve for the fifth harmonic crosses the slip axis at one-fifth fundamental synchronous speed, but in this case the point of intersection is to the left of the standstill axis. The curve of total torque is given by the heavy line, and from its shape we see that, as the motor starts up from rest, the torque increases up to a maximum, then decreases, and finally reaches another maximum. Also, the position of the first maximum is just below one-seventh synchronous speed. If the frictional torque of the motor is such that it requires the motor to develop a torque greater than that in the neighbourhood of one-seventh full speed, the motor will " crawl " at this speed and will be perfectly stable so long as the working point is in the neighbourhood of C. If the field form of the motor is such that a fairly large seventh harmonic is produced, then the only way to avoid this trouble is to make the rotor resistance sufficiently high to

enable the motor to develop a torque at one-seventh speed greater than the frictional torque.

This phenomenon is not exhibited by motors with wound rotors, because the inserting of external resistance for starting purposes gives the motor a torque very much greater than the frictional torque. In fact, such motors will start against full load torque without difficulty

7. **Magnetic Locking.** The rotor of squirrel-cage motors sometimes exhibits a tendency to remain in one place, and not to rotate



(*By courtesy of the Institution of Electrical Engineers*)

FIG. 298

EFFECT OF HARMONICS ON THE TORQUE SLIP CURVE

at all: this happens when the numbers of rotor and stator slots have a common factor. In such a case there will be certain rotor positions which make the air gap reluctance a minimum, and it is in such positions that the rotor tends to remain. This is easily overcome by making the number of rotor slots prime to the number of stator slots.

8. **Methods of Starting Squirrel-Cage Motors.** An induction motor can be regarded as a transformer with a small air gap in the magnetic circuit. Hence, a squirrel-cage motor, when at rest, and therefore having no back E.M.F. induced in it, can be regarded as a transformer with a short-circuited secondary. Such a transformer would take a very high current if the full voltage were applied to its primary winding, and therefore, for starting it is necessary to apply a reduced voltage. When the rotor is well

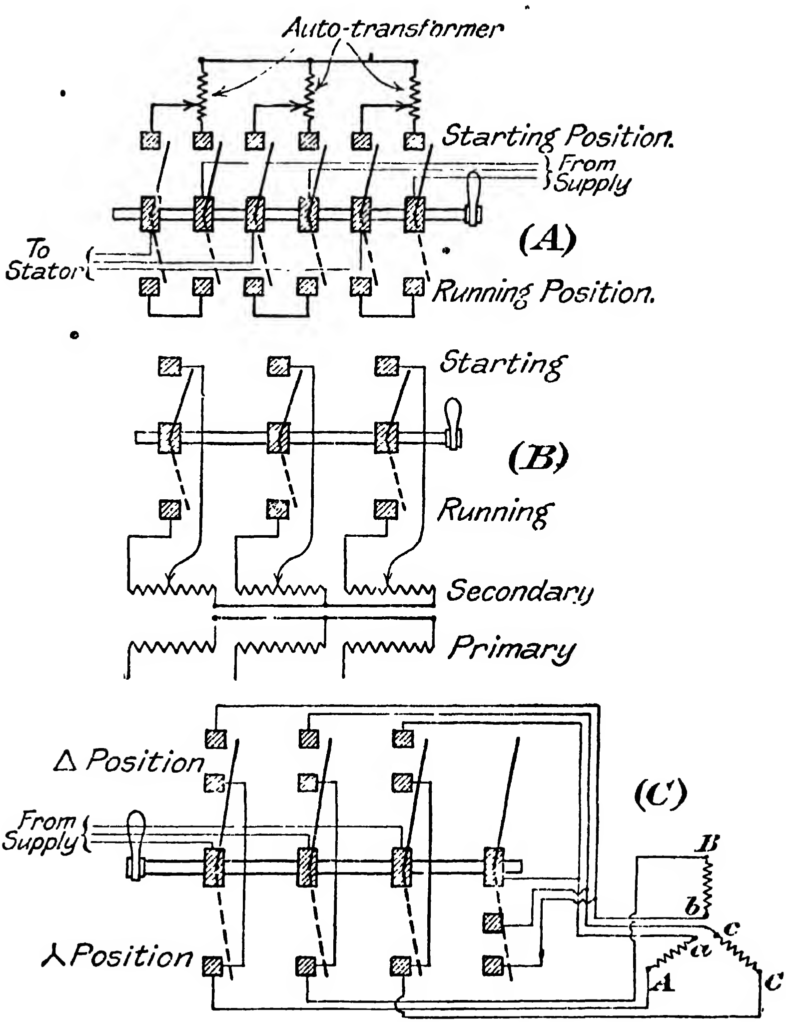


Fig. 299
Starters for Squirrel-cage Motors

under way, the full voltage can be applied. There are several methods of obtaining this reduced voltage.

In the auto-transformer starter, the reduced voltage is obtained by taking tappings from a three phase auto-transformer, as shown in Fig. 299 (A). In the running position full line voltage is applied, and the auto-transformer left out of circuit.

If the motor is supplied through a step-down transformer, then the secondary of this transformer can be used as an auto-transformer as shown in Fig. 299 (B). This considerably simplifies the switch-gear, but the extra connections from the transformer are a disadvantage.

In the "star-delta" method the motor is designed to operate normally with the stator phases mesh connected, so that the normal phase pressure is equal to the line pressure. For starting, the phases are connected up in star, the pressure applied to each phase being thus $1/\sqrt{3}$ of the normal pressure. Fig. 299 (C) shows the arrangement of a star-delta starter and the stator phases; the phases are each brought out to two terminals. A four-pole throw-over switch is used, the object of the fourth pole being to provide the star point when the phases are connected in star. It has no contacts on the delta side.

Example. A squirrel-cage motor has a ratio of rotor standstill reactance to resistance per phase of 4 to 1, and the maximum torque is $2\frac{1}{4}$ times the normal full load torque. Calculate (1) the full load slip, (2) the ratio of starting torque to normal torque, (a) with direct-on starting, (b) with star-delta starting, (c) with an auto-transformer starter with 70 per cent tapping.

With normal voltage on the stator

$$T = k \times \frac{\sigma R_2}{R_2^2 + \sigma^2 X^2_{2\cdot0}}$$

when $\qquad T = T_{max}, \quad \sigma = R_2/X_{2\cdot0}$

Let $\qquad T_{max} = 100\%$

$$\therefore\ 100 = \frac{kR_2^2}{X_{2\cdot0}} \Big/ R_2^2 + \frac{R_2^2}{X^2_{2\cdot0}} \cdot X^2_{2\cdot}$$

$$\therefore\ k = 200X_2$$

(1) $\qquad T_{normal} = \dfrac{T_{max}}{2\frac{1}{4}} = \dfrac{100}{2\frac{1}{4}} = 44{\cdot}4$

$$\therefore\ 44{\cdot}4 = 200X_{2\cdot0} \cdot \frac{\sigma R_2}{R_2^2 + \sigma^2 X^2_{2\cdot0}}$$

also $\qquad X_{2\cdot0} = 4R_2$

$$\therefore\ 44{\cdot}4 = 800R_2 \cdot \frac{\sigma R_2}{R_2^2 + 16\sigma^2 R_2^2}$$

$$\frac{800\sigma}{1 + 16\sigma^2}$$

This gives $\sigma = 5{\cdot}7$ per cent or 107 per cent, showing that the motor will run normally with a full load slip of 5·7 per cent. In the region of counter-current braking, the slip at normal full load torque will be 107 per cent (Fig. 300).

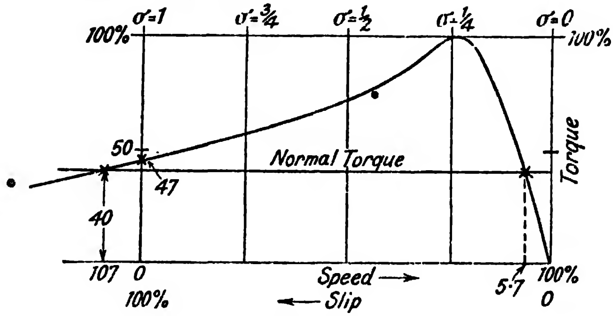


FIG. 300

(2) At starting $\sigma = 1$

$$\therefore\ T_{starting} = \frac{kR_2}{R_2^{\,2} + X^2_{2\cdot0}}$$

Putting $\qquad k = 200X_{2\cdot0}$ and $X_{2\cdot0} = 4R_2$

we have $\qquad T_{starting} = 47.$

(*a*) At full stator voltage, $T_{starting}$ is given by the above value of 47 per cent of the maximum possible torque

$$\therefore\ T_{starting} = \frac{47}{44{\cdot}4} = 1{\cdot}06T_{normal}$$

(*b*) Star-delta starting gives $1/\sqrt{3}$ times full voltage, and therefore 1/3 times the torque at full voltage

$$\therefore\ T_{starting} = \frac{1{\cdot}06}{3} = {\cdot}35T_{normal}$$

(*c*) $\qquad T_{starting} = 1{\cdot}06 \times {\cdot}7^2 = {\cdot}52T_{normal}$

9. **Starting of Synchronous Motors by Induction Motor Action.** We have seen that the synchronous motor will not start by synchronous-motor action, with the result that the motor has to be brought up to speed either (*a*) by the help of a pony motor, or (*b*) by induction-motor action. This second method is the more commonly used, and the necessary torque is developed in damping windings which are housed in the pole faces, and also connected by copper straps across the interpolar gaps, so as to form a squirrel-cage

winding. As with the squirrel-cage induction motor, it is necessary to apply a reduced voltage to the machine at starting so as to avoid excessive rushes of current, and one method of doing this, as used by the B.T.H. Co., is shown in Fig. 301. It consists of an auto-transformer and two oil switches, one a four-pole, the other a three-pole, which are mechanically interlocked so that it is impossible for

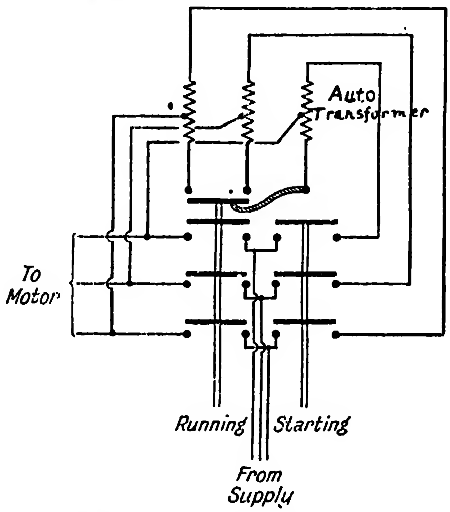


FIG. 301. B.T.H. METHOD OF STARTING A LARGE SALIENT-POLE SYNCHRONOUS MOTOR

both switches to be closed at the same time. When the starting switch is closed the incoming line is connected to the transformer whose neutral is already provided by the fourth pole of the running switch being closed when the other three poles are open. The motor thus starts up on the reduced voltage corresponding to the transformer tap in use. When the running switch is being closed the starting switch is opened just before contacts are made, and when the running switch is definitely closed the full voltage is applied to the motor, and at the same time the neutral point of the transformer is opened so that the transformer will not be taking current permanently.

It is important to appreciate the effect which the starting, or damping, winding has on the performance of the motor while it is synchronizing. The torque developed and the speed attained before synchronism depend, for a damping winding with a given number of conductors, on the resistance of the winding exactly as in the case of an induction motor; in fact, if torque is plotted against speed the curves similar to those of Fig. 296 will be obtained. Three such curves are given in Fig. 302. Curve 1 corresponds to a low

resistance comparable with that of an induction motor rotor, curve 2 to a resistance higher than this, and curve 3 to a resistance of such value that the maximum torque occurs at standstill. From the point of view of mere starting torque a high resistance starting winding

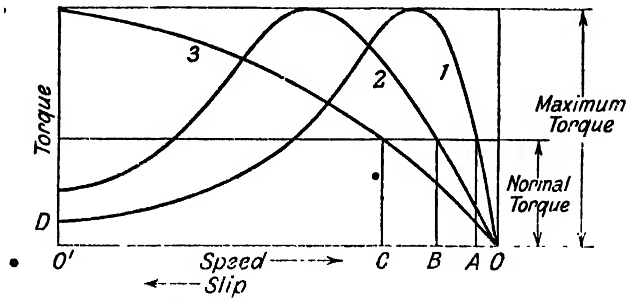


FIG. 302. INFLUENCE OF RESISTANCE OF STARTING WINDING ON STARTING PERFORMANCE

is thus an advantage. Now consider the synchronizing torque, that is, the torque which has to make up for the difference between the speed up to which the motor is brought by induction motor action, and the synchronous speed, i.e. the torque which has to pull the motor into step. Suppose that the motor has to start against a torque equal to the normal full-load value, then with a resistance of starting winding corresponding to curve 3 the speed will only be brought up to $O'C$ which, on an average, will be about 75 per cent of synchronous speed. Consequently the slip will be about 25 per cent, and it will be quite impossible for the motor to synchronize against such a large slip. If the resistance of the winding is very low, as in curve 1, the starting winding will bring the speed up to $O'A$, which will be 95 or 96 per cent of the synchronous speed. The slip will thus be only 4 or 5 per cent, and the motor will consequently have no difficulty in pulling into step. But with such a resistance the motor cannot start against full-load torque, because the starting torque $O'D$ is less than this. From this it follows that the design of the starting winding has to be a compromise between the diametrically opposed *desiderata* of high starting torque and high synchronizing torque. With the ordinary construction it is, therefore, impossible to obtain a starting torque and synchronizing

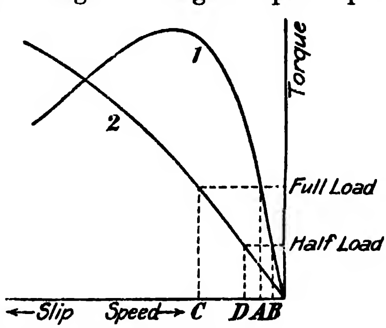


FIG. 303
SPEED VARIATION WITH CHANGE IN LOAD

torque anything like so high as with the synchronous-induction motor, described in paragraph 12. The following figures give the order of the performance obtained, although, naturally, they vary considerably with different motors. For a starting voltage of 60 per cent normal the starting torque is about 40 per cent; while the synchronizing torque varies from 10 per cent for 50 per cent voltage up to 40 per cent for full voltage.

10. Speed Control of Induction Motors. All alternating current motors without commutators suffer from the disadvantage that

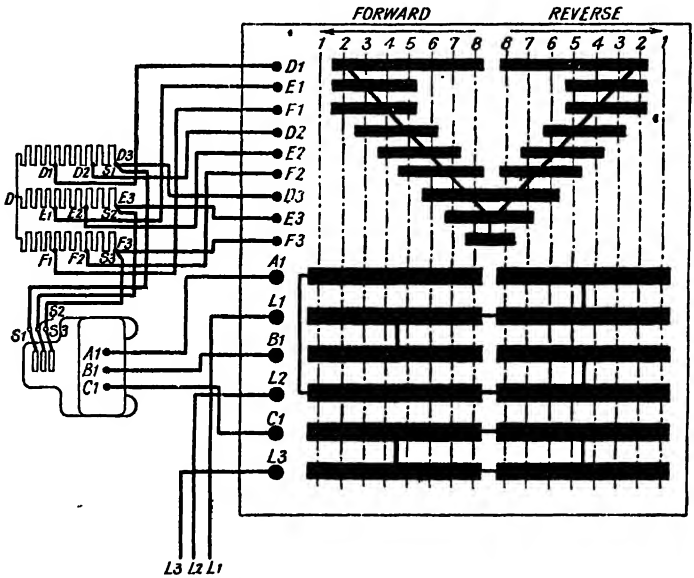


FIG. 304. CONTROLLER CONNECTIONS FOR SLIP-RING MOTOR

they are fundamentally single speed machines, because the magnetic field rotates at synchronous speed. The economical control of the speed of induction motors is, in consequence, much more difficult than that of direct current motors. The various methods of speed control in common use are as follows—

(*a*) RHEOSTATIC CONTROL. This can be applied to motors with wound rotors only. Resistance is included in the rotor circuit, the speed depending on the value of the additional resistance per phase. We have seen that the percentage slip is equal to the percentage rotor copper loss. If the motor is working on a constant torque, the current will be sensibly constant and the drop in speed below synchronism will be proportional to the extra resistance per phase. We thus see that drop in speed is proportional to

the power dissipated in the rheostat, the method being therefore very wasteful if speeds much below synchronism are required. Thus if the speed is half synchronous speed, half the power supplied to the motor will be wasted in the rheostat. A further disadvantage of the method is that the speed is only a function of the external resistance so long as the load torque remains constant. Consider the two

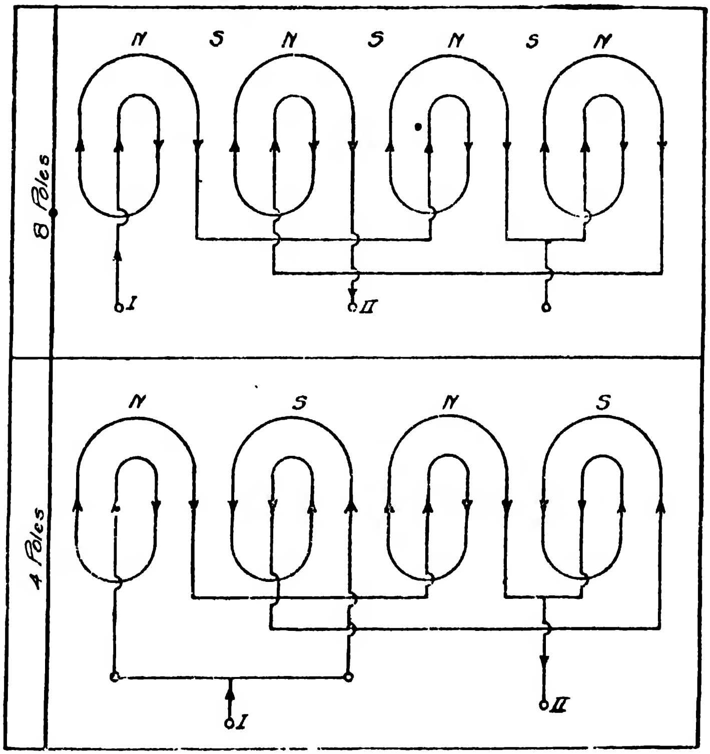


FIG. 305

PORTIONS OF SINGLE PHASE WINDING FOR 8 AND 4 POLES

torque speed characteristics in Fig. 303. Curve I corresponds to a motor running with the slip rings short-circuited. If the torque falls from, say, full load, to half load, then the change of speed will be *AB*. If so much resistance is added that the torque speed characteristic is represented by curve II, then for the same change in torque the change in speed is *CD*, which is very much greater than *AB*.

For very large motors, resistance controllers are almost invariably of the liquid type, but for small and medium sized machines metallic resistances with a controller of the barrel type are common. A

typical connection scheme for such a controller is shown in Fig. 304.

(*b*) POLE CHANGING. The stator is wound so that the number of poles can be varied. If p_1 and p_2 are the numbers of poles, then there are two synchronous speeds, given by

$$n_1 = 120f/p_1, \text{ and } n_2 = 120f/p_2.$$

It is easy to arrange for two numbers of poles in the ratio of 1 : 2, Fig. 305 showing the method of changing from four to eight poles, a single phase winding being shown for simplicity.* The larger number of poles is obtained by making all the poles of one name, consequent poles. This introduces considerable magnetic leakage, the result being that the power factor, when running with the larger number of poles, i.e. at the lower speed, is not very good. The method is therefore best suited to motors of large diameter, in which the pole pitch, even with the larger number of poles, is not so small as to cause excessive magnetic leakage. When more than two speeds are required, it is possible to obtain these by re-arrangement of the same winding, but so many tappings are required that it is simpler to have two separate windings housed in the same slots. The rotor of these motors is usually of the squirrel-cage type, since such a rotor is suitable for any number of poles. If the rotor is phase wound, then its number of poles must also be changed.

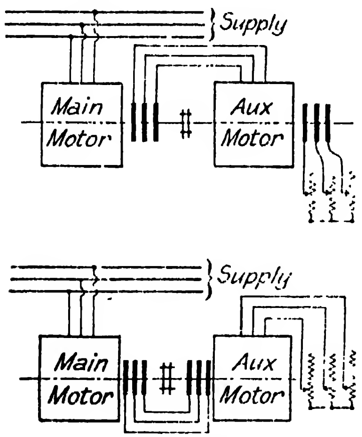


FIG. 306
CASCADE SPEED CONTROL

Additional speeds can be obtained by auxiliary rheostatic control, but this, of course, lowers the efficiency.

(*c*) CASCADE CONTROL. In this method two induction motors are mechanically coupled, and the rotor of the main motor is phase wound. The second rotor can be either squirrel-cage or phase wound. The stator of the first motor is connected to the supply, but the second motor receives its supply from the rotor of the first. Also the supply to this second motor can be taken either to its stator or rotor, as indicated in Fig. 306. It will be seen that, although the main motor is acting as a motor relative to the supply, its rotor is acting as a generator relative to the second

* For complete information on all methods of speed control of induction motors, see Walker, *The Control of the Speed and Power Factor of Induction Motors.*

motor, and the two machines are thus working under very different conditions. The main motor has a supply of line frequency, but the back E.M.F. injected into its rotor by the second machine causes the speed to be considerably below synchronism, and its rotor frequency is therefore much greater than normal slip frequency. The second motor is working on a supply of low frequency, namely, the slip frequency of the main motor, but apart from this its operation is normal. Generally the two motors run at the same speed, but we will consider the case of belted motors in which the ratio of the speed of the main motor to that of the second motor is $1/n$. Let the main motor have p_1, and the second motor p_2, poles. Let σ_1 and σ_2 be the respective fractional slips, and let the line frequency be f.

$$\text{Then synchronous speed of main motor} = \frac{120f}{p_1}$$

$$\text{Actual speed of main motor} = \frac{120f}{p_1}(1 - \sigma_1)$$

$$\text{Synchronous speed of second motor} = \frac{120\sigma_1 f}{p_2}$$

$$\text{Actual speed of second motor} = \frac{120\sigma_1 f}{p_2}(1 - \sigma_2)$$

$$= \frac{120\sigma_1 f}{p_2} \text{ approx.}$$

since this motor is working normally and its slip σ_2 is small.

$$\text{Hence } \frac{120f}{p_1}(1 - \sigma_1) \div \frac{120\sigma_1 f}{p_2} = \frac{1}{n}$$

$$\therefore \sigma_1 = \frac{1}{1 + \frac{1}{n} \times \frac{p_1}{p_2}}$$

$$= \frac{np_2}{np_2 + p_1}$$

Thus, in the particular case in which the motors are solidly coupled, $n = 1$; also, if $p_1 = p_2$, then $\sigma_1 = \frac{1}{2}$, and the set runs at one-half synchronous speed. It is essential that the two motors shall be mechanically coupled, since if they are not, then, if mechanical load is put on any one motor it will stop and act like a transformer while the other motor will run light at full speed.

Substituting the above expression for the slip σ_1 in the expression for the speed of the main motor, we have

$$N_1 = \frac{120f}{p_1}(1 - \sigma_1)$$

$$= 120f \times \frac{1}{p_1 + np_2}$$

Hence, the main motor runs at the same speed as a single motor having $(p_1 + np_2)$ poles.

The usual method of applying the cascade method is to connect the two motors so that their torques act in the same direction, the speed then being obtained from the expressions developed above, and being less than the normal speed of the main motor. If a speed greater than normal speed is required, the torque of the second motor can be reversed by the simple expedient of changing over two of the leads to its stator winding. This is called differential cascading and the speed, for a direct-coupled set, is

$$N = 120f \times \frac{1}{p_1 - p_2}$$

This method of control is peculiar in that the torque developed by the machines varies very considerably with changes in speed. In particular, if the speed is the same as the synchronous speed of the main motor the torque will be zero, because of the fact that there will be no E.M.F. induced in the rotor of the main machine, and therefore no current in either stator or rotor winding of the rear machine. At speeds greater than the synchronous speed of the main machine, the slip of this machine is negative, and therefore the phase rotation of its rotor currents is reversed. This, combined with the fact that the connections to the stator of the second machine have been reversed with respect to normal cascading, explains why stable operation at speeds greater than synchronous speed is possible.

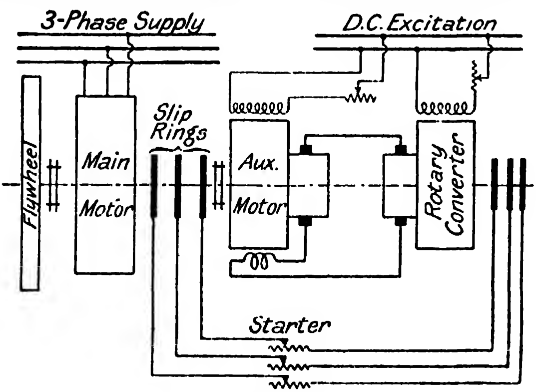


FIG. 307

ROTARY CONVERTOR IN CASCADE WITH INDUCTION MOTOR

The main disadvantage of this method, apart from the necessity of a second motor, is that the magnetizing current for both motors has to be drawn from the line, and since it is a wattless current it

causes a poor power factor. To minimize this lowering of the power factor, commercial cascade sets invariably have a small second motor with very few poles compared with the main motor, the magnetizing current required by the second motor being, therefore, small. The expression for the slip shows that in such a case the cascade speed is not very much below the normal synchronous speed of the main motor. If more than two speeds are required, cascade control can be combined with pole changing; such sets are in fairly extensive use for driving rolling mills and mine ventilating fans.

(*d*) Cascade Control with Rotary Convertor and Auxiliary Direct-Current Motor. This is known as the Kramer system. It will be seen that in the ordinary cascade control, speed variation is obtained by utilizing the slip energy of the main rotor in a second motor, instead of wasting it in a rheostat. In the Kramer method this slip energy is first converted into direct current energy by means of a rotary convertor (Fig. 307), and it is then utilized in a direct current motor which is direct coupled to the main motor. The speed is controlled merely by altering the excitation of the direct current motor. This alters the back E.M.F., which, being approximately the same as the voltage at the commutator of the rotary, alters the voltage at the slip rings of the latter. This slip ring voltage is injected into the rotor circuit of the main motor and causes the required reduction in speed. The advantage of this method is that any speed within the working range can be obtained, instead of only two or three speeds by the other methods. Also, if the rotary convertor is designed to operate with high excitation, it will take a leading current, and thus considerably improve the power factor of the set.

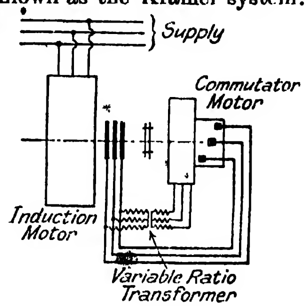


Fig. 308
Commutator Motor in Cascade

The Kramer system can be used with a flywheel load equalizer if means are provided to reduce the speed automatically when the load increases. This is done very simply by compounding the auxiliary D.C. motor. An increase in load causes an increase in rotor current in the induction motor, and this in turn passes through the rotary to the direct current motor, whose back E.M.F., as a result of the increased series excitation, increases. This, as explained previously, brings down the speed, so enabling the flywheel to give up some of its stored energy. With a load equalizer of this kind the system is frequently used for driving continuously running rolling mills. Without the load equalizer, and with a shunt excited auxiliary motor, it is used for driving mine ventilating fans.

(e) CASCADE CONTROL WITH AUXILIARY POLYPHASE COMMUTATOR MACHINE.* Imagine an ordinary induction-motor rotor, without a stator, provided with the usual slip-rings and supplied with alternating current from an external source. If the rotor is stationary a rotating magnetic field will be set up, this field travelling at synchronous speed with respect to the rotor, and therefore at synchronous speed in space. The field will induce in the rotor a back E.M.F. in a manner similar to the production of a back E.M.F. in the primary winding of a transformer on open circuit, and the current which flows will be practically wattless, and its magnitude will be limited almost entirely by the standstill reactance of the rotor windings. If the rotor is provided with a commutator as well as with slip-rings, the arrangement then being that of the rotor of the rotary convertor (see Chap. XXI), there will be an E.M.F. of supply frequency between any brushes pressing on the commutator. This E.M.F. will not vary in magnitude if the brushes are rocked round the commutator and the angular spacing of the various brushes kept constant, but it will obviously vary in time-phase whenever such movement of the brushes is carried out. In fact, if the brushes are capable of movement throughout two pole pitches, phase displacement through 360 electrical degrees will be possible.

Now suppose that the rotor is made to rotate, say, in a direction opposite to that of the rotating magnetic field, then since the tapping points to the slip-rings are points which are fixed relative to the rotor winding itself, the rotating field is compelled to maintain its synchronous speed relative to the rotor itself. Thus, if the rotor itself moves in the above stated direction, the actual velocity of the rotating field in space will be reduced. If the rotor speed is just equal to the synchronous speed, the rotor field will be stationary in space, and the voltage appearing at the commutator end will then be unidirectional, as distinct from alternating. The R.M.S. value of this voltage is, however, unaffected by the movement of the rotor, because it is fixed by the relative velocity of the rotor itself with respect to the rotor field, and as we have seen, this relative velocity is always equal to the synchronous velocity.

If the speed of the rotor is not equal to its synchronous speed, then the E.M.F. at the commutator will have a frequency equal to the difference between the actual and the synchronous speeds, so that in this respect the appliance becomes a frequency convertor. In this form it can be connected in cascade with a large induction motor, and used for the purpose of speed regulation. There are three points to notice in connection with this frequency convertor—

(1) The frequency of the alternating voltage appearing at the

* The best student's book on commutator motors is *The Commutator Motors* by Teago. A larger work is *The A.C. Commutator Motor*, by Olliver. The commutator motor is also discussed at greater length in Chapter XXII.

commutator is not proportional to the speed of the convertor, but to the difference between its speed and the synchronous speed.

(2) For a given angular spacing of the brushes, e.g. three brushes spaced 120° apart for a three-phase two-pole machine, the E.M.F. is independent of the brush position.

(3) The time phase of this E.M.F. can be altered by rocking the brushes round the commutator.

It is also to be noted that although we have only considered three-phase current fed to the slip-rings in the above discussion,

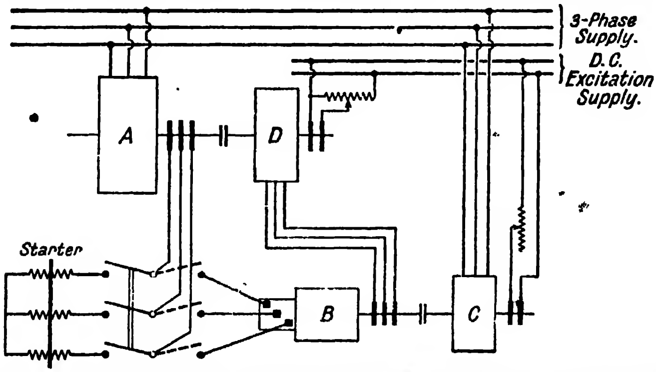


Fig. 309. Speed Control of Large Induction Motor by Frequency Convertor

the number of phases of converted-frequency current taken off at the commutator can be anything we like, being fixed simply by the number and spacing of the brushes.

Now consider a winding having a commutator and having the alternating-current supply brought to the commutator instead of to the slip-rings. It is now to be noted that, as long as the brushes are set in a definite position, the current is supplied to the rotor, not at points which are fixed relative to the rotor winding as when slip-rings are used, but at points which are fixed in space. This is so because, the brushes being fixed, the commutator puts them in electrical connection with the various rotor coils in succession. The result of this arrangement is that the rotor field always travels in space at synchronous speed, no matter what the actual speed of the rotor itself may be. It is therefore possible for a rotor fed in this manner to run at any speed inside a polyphase stator and still have a unidirectional torque acting on it, and this, in brief, is the mode of operation of the polyphase commutator motor. In fact, the rotor can run in any direction, because this direction does not influence the direction of the rotor field, which is decided only by the electrical phase sequence of the brushes pressing on the commutator.

The actual frequency of the currents flowing through the rotor windings is, of course, the slip frequency, this condition holding, no

matter how the current is supplied to the rotor, i.e. whether induced as in the plain induction motor, or whether supplied to a commutator from an external source. Hence, with this method of supply also, the commutator acts as a frequency convertor.

It is also to be noted that in the case of the rotor of an induction motor, the commutator need not be part and parcel of the rotor construction, i.e. not integral with the machine as in the case of a direct-current armature and its commutator, since the action is exactly the same if the commutator and the winding connected to it constitute an entirely separate armature driven by a small motor. A separate frequency convertor of this type can be applied to the speed control of a large induction motor, one such scheme being given in Fig. 309. A is the main motor which, after being started up by an ordinary resistance starter, has its rotor winding connected to a frequency converter B. This, as explained above, is a rotor provided with both commutator and slip-rings; a stator winding is not required, but the iron part of the stator is provided so as to give a low reluctance path for the magnetic flux. It will be seen that the frequency convertor is fed with alternating current from the rotor of the main motor at its commutator end. The current taken off at the slip-ring end of this convertor is used as supply for a synchronous motor D, which is direct-coupled to the main motor A. The frequency convertor is driven by another synchronous motor C. Suppose the main motor is running with a fractional slip of σ, then the frequency of the currents supplied to the commutator end of the frequency convertor is σf, with the result that the rotating field of this machine will rotate at a speed of σN_s. The actual rotor speed due to the synchronous motor drive is N_s, the relative speed of rotor to rotating field thus being $N_s(1-\sigma)$. Hence the frequency of the current taken off at the slip-rings is $f(1-\sigma)$ and the speed of the synchronous motor D is $N_s(1-\sigma)$. But this is also the speed of the main motor A to which it is coupled, showing that the frequency convertor used in the above manner supplies D with current of the correct frequency. The speed is obtained, as in the Kramer control, by varying the excitation of D, while power-factor control is obtained by varying the excitation of the synchronous motor C.

The discussion of polyphase commutator motors as distinct from frequency convertors is somewhat beyond the scope of this book, but it can be said that they have a stator winding of the ordinary kind, while the rotor has a winding like a direct current armature with a commutator. The stator can be shunt, series, or compound excited, as in a D.C. motor. The method of connecting a shunt commutator motor in cascade with an induction motor is shown in Fig. 308. It will be seen that the stator is supplied through a regulating transformer, and the speed of the set is adjusted by means of this transformer. The effect of an alteration of the number of turns on the secondary winding is to alter the back E.M.F. of the commutator

motor, and thus to alter the speed of the set. The speed torque characteristic of a set of this kind is fixed by the characteristics of the commutator motor, being similar to that of a shunt or compound motor, according as the commutator motor is shunt or compound wound.

11. Squirrel-Cage Motors with High Starting Torque. The ordinary squirrel-cage motor suffers from the disadvantage that, owing to its very low rotor resistance, its starting torque is very low. Many attempts have been made to produce a squirrel-cage motor with high starting torque. In the Boucherot motor the rotor has two separate squirrel-cages, one inside the other, as in Fig. 310. The outer one is of high resistance metal, and the inner one of copper. The inner winding is much farther from the periphery of the core, its bars are surrounded by more iron, and therefore, its inductance is higher than that of the outer high resistance winding. At the moment of starting, the rotor induced currents are at full line frequency, and therefore the reactance ($2\pi Lf$) of the inner cage will render its impedance so high that the rotor currents at starting will be confined to the outer cage, in spite of its greater resistance. The starting torque will therefore be high. As the motor speeds up, the frequency of the rotor current decreases, and the reactance of the windings becomes of less importance than the resistance. Hence, when running at full speed, when the rotor frequency is only of the order of one or two cycles per second, the impedance of the windings will depend only on their resistances. The rotor current at full speed is thus confined mainly to the inner low resistance winding, and the efficiency of the motor is therefore high.*

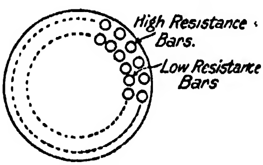


FIG. 310
BOUCHEROT ROTOR

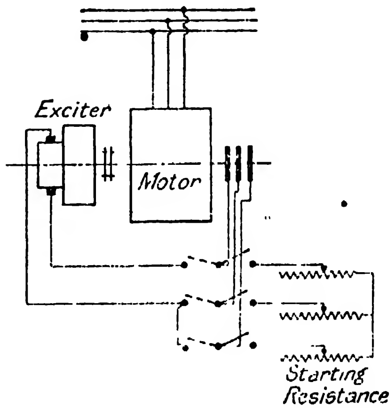


FIG. 311
SYNCHRONOUS INDUCTION MOTOR

In a squirrel-cage motor recently evolved by Dr. Wall,† the rotor

* The theory of the double-cage motor is given in Behrend, *The Induction Motor*, and Punga and Raydt, *Modern Polyphase Induction Motors*.
† See *Journ. I.E.E.*, Vol. 63, p. 287.

bars each consist of a simple form of transformer. Now when the secondary winding of a transformer is closed, the equivalent resistance of the primary winding depends upon the frequency, and increases with increase of frequency. If each rotor bar acts like a transformer, then at starting, when carrying currents of line frequency, the equivalent resistance will be high. When running normally, the rotor frequency is so small that the equivalent resistance is practically the same as the ordinary resistance of the rotor, the efficiency therefore being high. The first rotor bar used consisted of a copper bar surrounded by, but lightly insulated from, a steel tube coated inside and outside with chemically deposited copper. A simpler construction, and one which functions just as well, is to fit a steel tube over the rotor bar without any special insulation, and to copper-plate the whole composite tube. The steel tube and main bar are thus in electrical contact, and the composite bar is an auto-transformer instead of an ordinary transformer.

12. **The Induction Motor run as a Synchronous Motor.** At the present time it is becoming common practice to use synchronous motors for drives where an absolutely fixed speed is not a disadvantage, e.g. for air compressors. The disadvantage of the ordinary synchronous motor is that it will not start under load. By employing an induction motor, starting against full load torque is obtained, and when full speed is attained the motor is converted to a synchronous motor by sending direct current through its rotor winding. This is done by switching over the rotor from the ordinary starting resistance to a D.C. exciter, usually direct coupled, as shown in Fig. 311. It will be seen that one rotor phase carries twice as much direct current as the other two, but although special windings have been evolved to avoid this, it is of little importance, because each rotor phase consists of conductors distributed over the whole periphery, the danger of unequal rotor heating therefore being avoided.

Now when the rotor carries only induced alternating currents, the rotor field slips past the rotor bars, and any motor speed is possible, according to the frequency of these currents. When the rotor carries direct current, the rotor poles are fixed relative to the rotor itself, and therefore, since the rotor field must travel at synchronous speed with the stator field, the rotor itself must also travel at synchronous speed. The induction motor action cannot bring the rotor quite up to synchronous speed, and therefore, when the rotor is switched over from the starting resistance to the exciter, the induction motor slip speed has to be made up. When running uniformly at synchronous speed the rotor will carry direct currents only, but if there is any departure from this speed the rotor will carry in addition induced alternating currents. The rotor being of low resistance, and the exciter also of low resistance, its winding

acts as a damping winding, and it is not necessary to provide a separate damping winding, as in the salient pole synchronous motor.

Consider what happens when the direct current excitation is switched on. A synchronizing torque is suddenly set up, its magnitude being given by an expression $T_m \sin \theta$, where θ is the angle between the rotor and stator fields. Also, there is an induction motor torque which, as we have seen, is proportional to the slip, $d\theta/dt$, as long as the slip is small. If the motor is started under load there is a constant load torque, and finally there is the torque $Id^2\theta/dt^2$ required to accelerate the rotor. The variations in *slip* speed (not actual speed) are therefore represented by the variations in speed of the mechanical model represented in Fig. 312. A disc of moment of inertia I can rotate on a horizontal axis, and a weight W_1 is suspended by a string passed round it. If the radius of the disc is unity and this weight W_1 is numerically equal to the load torque, then the torque produced by W_1 will represent the load torque. It will try to accelerate the disc, just as the load torque in the actual motor tries to accelerate the slip, and therefore, reduce the actual speed. If the disc is acted on by a frictional torque proportional to its speed, then this torque will represent the induction motor torque in the actual motor. If these are the only external torques acting on the disc, then a uniform speed will be attained when they are equal and opposite. This uniform speed corresponds to the uniform slip speed of the induction motor before the direct current excitation is switched on.

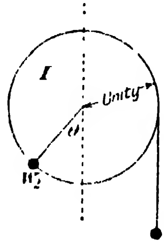


FIG. 312
TO ILLUSTRATE SYNCHRONIZING

Now suppose that another weight W_2 is suddenly clutched to the periphery of the disc. If this weight is numerically equal to the torque T_m, the torque set up by it when it is in any angular position from the vertical will be $T_m \sin \theta$. Thus it will represent the synchronizing torque produced by the sudden application of the direct current excitation. There are now two possible conditions of motion of the model; firstly, it may come to rest with the weight W_2 in such a position that the torque $T_m \sin \theta$ due to it just balances the load torque due to the weight W_1. If this happens the slip velocity is zero, which means that the actual motor has pulled into step and is now running uniformly at synchronous speed. The value of the angle θ, which makes these two torques balance, is the angle by which the rotor poles will lag behind the stator poles. Secondly, the disc may continue to make whole revolutions, its speed being greater when W_2 is at the bottom than when W_2 is at the top. In the actual motor this means that the slip speed is not uniform, the motor being therefore subjected to violent mechanical

stresses. In order that the motor may pull into step, it is necessary that in the model of its slip motion the disc shall come to rest. It is therefore necessary that the weight W_2 shall be sufficiently large; and secondly, that it shall be clutched on to the disc somewhere near the bottom position, so that it immediately begins to exert a retarding torque on the disc. Clutching on the weight W_2 at the bottom is similar to switching on the direct current excitation at the moment the induced rotor fields and stator fields are in line. *This position is indicated by the induced stator current passing through its zero value, and therefore, if a moving coil zero centre ammeter is included in the rotor circuit, this ammeter will indicate when to switch on the direct current excitation.**

The induction motor run as a synchronous motor has the following advantages compared with the ordinary salient pole synchronous motor—

(*a*) It will start and synchronize against full load torque, whereas the ordinary synchronous motor must be started under no load conditions.

(*b*) Its air gap is not much greater than that of an ordinary induction motor, whereas the gap of a salient pole motor is long. The exciter is not therefore of such large capacity as that for a salient pole motor, but, on the other hand, it must be a low-voltage, heavy-current machine owing to the very low resistance of the rotor winding.

(*c*) The rotor winding combines the functions of exciting and damping winding: the salient pole motor requires a separate damping winding.

(*d*) No special starting gear is required. The salient pole motor must be started up either by a separate starting motor or by applying a reduced voltage to the armature, either method involving complication and extra control gear.

The above comparison shows that if a mechanical load is to be driven, but at the same time the phase advancing properties of the synchronous motor are to be utilized, then the autosynchronous motor is a good proposition. If the synchronous motor is to act as a phase advancer only, without doing any mechanical work, then the salient pole type is the best, because owing to its longer air gap it is able to supply a greater amount of wattless leading kVA without becoming unstable.

In practice the choice between the salient-pole type and induction-motor type depends upon the conflicting requirements of high

* For a complete account see Cotton, "The Pulling into Step of the Synchronous Induction Motor," *Journ. I.E.E.*, Vol. 63, p. 211. This paper has a very extensive bibliography.

starting torque and high synchronizing torque, and stability during synchronous motor operation. The salient pole motor cannot develop a starting torque equal to that of the induction-motor type unless elaborate and expensive modifications are made to the machine, and consequently where a high starting torque is necessary the induction-motor type is preferable. On the other hand, the salient-pole machine has a field system which is magnetically strong in comparison with the armature, because of the long air gap, with the result that it is very stable on violently fluctuating loads. The induction-motor type is at a disadvantage in this respect, as the very small air gap means a magnetically weak rotor, and therefore violent phase swinging when the load varies. The best field of application of the synchronous-induction motor is, therefore, for drives requiring a heavy starting torque, but in which the load torque is fairly steady.

13. Circle Diagram of the Induction Motor. The total stator flux produced by the stator current consists of two portions, namely—

Φ_1 = useful stator flux, i.e. the flux from the stator which enters the rotor

F_1 = leakage stator flux, i.e. the flux from the stator which does not enter the rotor.

Similarly, for the flux produced by the rotor currents, we have the useful rotor flux Φ_2, that rotor flux which enters the stator, and the leakage rotor flux F_2, that rotor flux which does not enter the stator. Then the total stator and rotor fluxes are $(\Phi_1 + F_1)$ and $(\Phi_2 + F_2)$ respectively, and we have

$$\frac{\Phi_1 + F_1}{\Phi_1} = \lambda_1, \text{ the stator leakage factor}$$

$$\frac{\Phi_2 + F_2}{\Phi_2} = \lambda_2, \text{ the rotor leakage factor.}$$

Since the induction motor acts like a transformer, the total stator and rotor fluxes will be nearly, but not exactly, in phase opposition. Draw OA (Fig. 313) to represent the total stator flux, and mark off $OC = F_1$, $CA = \Phi_1$. Draw AB to represent the total rotor flux, making $AD = \Phi_2$ and $DB = F_2$. Then the resultant stator flux, which is equal to the vector sum of the total stator flux and the useful rotor flux, is equal to the vector sum of OA and AD, namely, OD.

Now, as in a transformer, we are justified in assuming that the

resultant flux remains constant. Also, since the wattless component of the no load current is proportional to this flux, we see that OD represents, to another scale, the wattless or magnetizing component of the no load stator current.

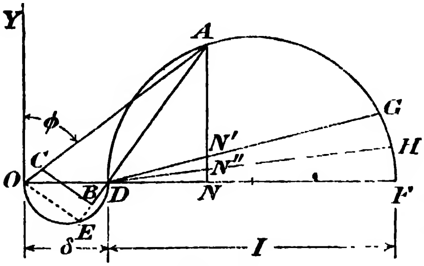


FIG. 313
INDUCTION MOTOR CIRCLE DIAGRAM

Again, the resultant rotor flux is the vector sum of the total flux produced by its own current and the flux entering the rotor from the stator. It is therefore the vector sum of AB and CA, that is, CB. Now the stator and rotor currents are in phase with the fluxes they produce, namely, OA and AB respectively. The rotor current is induced by the oscillations in the resultant rotor flux CB, and therefore, the rotor current is in quadrature with the flux CB. Hence, the angle ABC is a right angle. Producing AB to E, and drawing OE parallel to CB, we have, $\angle OED = 90°$, and therefore, the locus of E is a semicircle. Now

$$\frac{ED}{DA} = \lambda_1 \cdot \lambda_2 - 1 = \text{a constant.}$$

Hence, the locus of A is also a semicircle with its centre on OD produced.

Assume first of all an ideal motor with no losses. OA represents the stator current, and AB the rotor current to the same scale. Hence, AD represents the rotor current to a slightly different scale since

$$\frac{AB}{AD} = \lambda_2, \text{ a constant.}$$

Hence the point D represents the no load working point. The motor will be running at synchronous speed, since we are assuming no losses, and therefore, the rotor current will be zero. The point F at the extremity of the current locus will be the "standstill" point, that is, OF will be the stator current flowing if the rotor is clamped and full voltage applied.

The quantity $(\lambda_1\lambda_2 - 1)$ is called the "dispersion coefficient," δ, say. Hence

$$\frac{OD}{DF} = \frac{ED}{DA} = \delta$$

Calling the diameter DF, unity, we therefore have $OD \doteq \delta$. We shall see that for a given magnetizing current it is desirable that

the diameter of the circle shall be as large as possible, and therefore, δ must be as small as possible. In order to keep δ small the magnetic leakage must be reduced to a minimum, and since the leakage depends mainly on the width of the air gap, induction motors have an exceedingly narrow air gap.

Since the magnetizing current is in quadrature with the applied voltage, the voltage will be represented in phase by the direction of OY.

Hence, for any working point A, we have—

Stator current per phase $I = OA$

Angle of lag $\varphi = \angle YOA$

Power factor $= \cos\varphi$

Intake $= \sqrt{3}\,EI\cos\varphi$

$= \sqrt{3}\,E \times AN$

$\propto AN$

The angle of lag is a minimum, and therefore, the power factor a maximum when the stator current vector is tangential to the circle, as at OT (Fig. 314).

$$\therefore \text{ Max. power factor} = \cos\varphi_{min}$$

$$= \frac{TM}{MO} = \frac{\frac{1}{2}}{\frac{1}{2} + \delta}$$

$$= \frac{1}{1 + 2\delta}$$

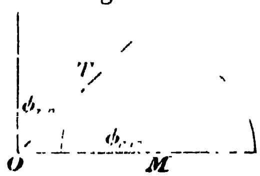


FIG. 314
CONSTRUCTION FOR MAXIMUM POWER FACTOR

Thus, to secure a good power factor it is necessary that δ should be small.

The effect of the various losses taking place in the motor is to modify the diagram. Consider first of all the copper losses. Since the motor acts like a transformer, the ratio of stator to rotor current is constant. Hence

$$\text{Total copper loss} \propto (\text{rotor current})^2$$

$$\propto AD^2$$

$$\propto DN \times DF$$

$$\propto DN, \text{ since } DF \text{ is constant}$$

Draw a line DG of fixed inclination (Fig. 313). Then DN is proportional to NN'; and if the inclination is so chosen that NN',

represents the copper losses to the same scale that AN represents the intake, the standstill point will be at G instead of F.

NN' can be divided at N'', such that

$N'N''$ = rotor copper loss, and $N''N$ = stator copper loss.

Now consider the iron losses. As in a transformer, the resultant stator flux is approximately constant, and therefore, the stator iron losses are sensibly constant. The iron losses in the rotor can be neglected because of the very slow frequency of magnetic reversal under working conditions. Again, the fall in speed within the working range is so small that the friction and windage losses can also be taken as being constant. The stray losses are therefore constant, and the small working component of the stator current required to supply these losses will be constant. The intake of the motor without losses is AN. Hence, marking off NN''' to represent the stray losses (Fig. 315), we have for the intake corresponding to the working point A the length of the vertical AN'''. Drawing a horizontal through AN''', we get O' for the origin of the actual diagram when losses are taken into account. The actual no load current per phase, I_0, is therefore $O'D$, $O'O$ being its working component, and OD the magnetizing component. For any working point A

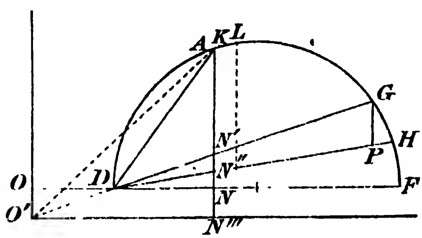


FIG. 315
COMPLETE CIRCLE DIAGRAM

the stator current $= O'A$

Intake $= AN'''$

Total losses $= N'N'''$

$\therefore$ Output $= AN'$

$$\text{Efficiency} = \frac{AN'}{AN'''}$$

The maximum output occurs when the working point is at K, where K is such that the tangent at K is parallel to DG.

The torque is proportional to the rotor intake, i.e. to the sum of the output and the rotor copper loss. It is therefore

represented by AN'', and it is a maximum when the working point is at L, where the tangent at L is parallel to DH. Thus length of the vertical from L on to DH gives the pull-out torque.

$$\text{Slip} = \frac{\text{rotor copper loss}}{\text{rotor intake}}$$

$$= \frac{N'N''}{AN''}$$

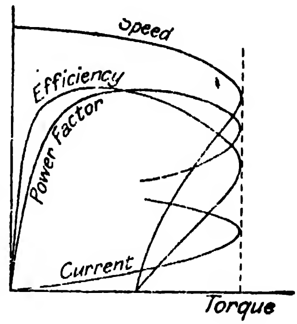


FIG. 316
INDUCTION MOTOR CHARACTERISTICS

at the standstill point G, $N'N''$ becomes GP, and AN'' also becomes GP, the slip then being unity. The starting torque without any added rotor-resistance is obviously the torque at standstill, namely, GP, and we see that this is very small. If resistance is inserted in the rotor circuit this is equivalent to increasing the inclination of DG, but not of DH. Hence, the standstill point is brought more to the left of the diagram. If the resistance added to the rotor is increased, the point G moves more and more to the left, and we see that the length GP first increases, then reaches a maximum, and finally decreases. This agrees with the variation of starting torque with rotor resistance as derived analytically.

It is easily deduced from the circle diagram that if the performance characteristics are plotted against the torque, the various curves will turn back at a value of the torque equal to the pull-out torque. This is shown in Fig. 316.*

14. **Experimental Determination of the Circle Diagram.** The following tests are performed: First, the motor is allowed to run light with full voltage applied to the stator. The stator current per phase, I_0, is measured, and also the no load intake W_0, measured by the two wattmeter method. Then the power factor at no load

$$\operatorname{Cos} \varphi_0 = \frac{W_0}{\sqrt{3}\, EI_0}$$

Thus both I_0 and φ_0 are known, so that the no load current vector OA (Fig. 317) can be drawn.

Next, a brake is put on the motor so that the rotor is either clamped or allowed to crawl round slowly, the slip-rings being short-circuited. This is analogous to the short-circuit test on a

* See also "The Circle Diagram of the Induction Motor," *Journ. I.E.E.*, 1928, p. 1,174.

transformer, and it is therefore necessary that a reduced voltage should be applied, this voltage being adjusted until the stator current is equal to the normal full load current. If the applied volts E_s, the stator current I_s, and power W_s are measured as before, then we have

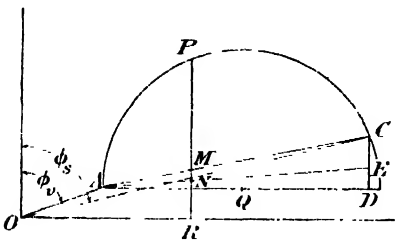


FIG. 317
CONSTRUCTION OF CIRCLE DIAGRAM

$$\cos \varphi_s = \frac{W_s}{\sqrt{3}\, E_s I_s}$$

so that the angle φ_s can be calculated. Now the short-circuit current would be very much greater if the normal voltage were applied, and we have, approximately, standstill current with full voltage applied

$$= I_s \times \frac{E}{E_s}$$

This gives the length of the stator current vector at standstill, and since its phase angle φ_s is known, we can locate the standstill point C.

In making this test it is preferable to allow the motor to crawl round slowly, since in this way the reluctance of the gap, which varies with the relative positions of the rotor and stator teeth, will have its average value. Obviously the speed must be very low or a back E.M.F. would be induced, the resulting point C then not being the true standstill point.

The centre of the circle is on the horizontal through A, and both A and C are points on the circle; hence, bisecting AC at right angles, we obtain Q, the centre of the circle which can therefore be drawn.

In order to draw the torque line it is necessary to separate the stator and rotor copper losses. This can be done in several ways. The power W_s in the standstill test is equal to the total copper losses, because the iron losses at the reduced voltage are negligible. If R_s is the stator resistance per phase, then the stator copper loss during this test is $3I_s^2R_s$, and the rotor copper loss, therefore $(W_s - 3I_s^2R_s)$. Hence, the perpendicular CD in the diagram is divided in the ratio

$$\frac{CE}{ED} = \frac{W_s - 3I_s^2R_s}{3I_s^2R_s}$$

the torque line AE then being drawn. This is the easiest method to apply if the motor has a squirrel-cage rotor. If the rotor is

phase wound, then the stator and rotor resistances per phase can be measured separately. For any stator and rotor currents I_1 and I_2 respectively, we have

$$\frac{\text{Rotor copper loss}}{\text{Stator copper loss}} = \frac{I_2^2 R_r}{I_1^2 R_s} = \frac{R_r}{R_s} \times \left(\frac{I_2}{I_1}\right)^2$$

The transformation ratio (I_2/I_1) can be obtained by including ammeters in the rotor circuit during the short-circuit test.

We have seen how to calculate the power factor, efficiency, and slip for different values of the stator current per phase. We can also find the intake and output in watts, and the torque in lb. foot units, by the following equations. Let E = line volts. Then for any working point P, if the lengths PR, PM, and PN are measured on the current scale,

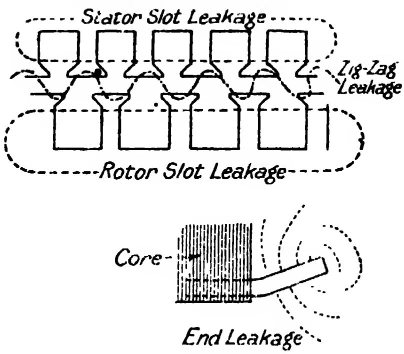


Fig. 318

Leakage in Induction Motors

$$\text{Intake} = \sqrt{3}\, E \times PR$$
$$\text{Output} = \sqrt{3}\, E \times PM$$
$$\text{Torque} = \frac{\sqrt{3}\, E \times 33{,}000}{746 \times 2\pi N_s} \times PN$$

where N_s is the synchronous speed in r.p.m., namely, $120f/p$.

15 It will be seen that the characteristics of an induction motor depend very largely upon the value of the dispersion coefficient δ, and therefore, upon the stator and rotor magnetic leakage fluxes. Magnetic leakage in an induction motor can take place in several ways, as illustrated in Fig. 318.

(*a*) *Slot leakage.* This flux passes across the slots and along the tops of the teeth

(*b*) ***Fluxes which zig-zag backwards and forwards between the tops of the stator and rotor teeth.*** This leakage is called the "zig-zag" leakage.

(*c*) ***End** leakage.* This takes place at the projecting ends of the coils.*

16. **The Equivalent Circuit of the Induction Motor.** We have seen that the induction motor is essentially a transformer, and conse-

* For a more complete discussion of induction motor leakage, see any standard textbook on the design of electrical machinery.

quently it can be represented by an equivalent circuit similar to that of the transformer, as given in Fig. 218. There is, however, one essential difference, and that is, that the motor acts like a transformer whose secondary (rotor) resistance varies with change of speed. This is shown by the following reasoning

$$I_2 = \frac{\sigma K E_1}{\sqrt{R_2^2 + \sigma^2 X_2^2}}$$

$$= \frac{K E_1}{\sqrt{\left(\frac{R_2}{\sigma}\right)^2 + X_2^2}}$$

$$= \frac{K E_1}{\sqrt{\left(\frac{N_s}{N_s - N} \cdot R_2\right)^2 + X_2^2}} .$$

The effect of speed can thus be regarded as changing the secondary resistance from R_2 at standstill to $R_2 \frac{N_s}{N_s - N}$ at any speed N. The change in resistance is therefore

$$R_2 - R_2 \cdot \frac{N_s}{N_s - N} = R_2 \cdot \frac{N}{N_s - N}$$

Hence, if we denote the reciprocal of the transformation ratio K by k we have for the equivalent circuit the accurate form of Fig. 319 (A), and the approximate form of Fig. 319 (B). Using the approximate circuit, we see that the current I_2 flows through a variable resistance $\left\{(r_1 + k^2 . r_2) + k^2 . \frac{N}{N_s - N} . r_2\right\}$ in series with a constant reactance $(x_1 + k^2 x_2)$. The two voltage drops in these will be—

$$\left\{(r_1 + k^2 r_2) + k^2 . \frac{N}{N_s - N} . r_2\right\} I_1'$$

in phase with I_2 and $(x_1 + k^2 x_2)I_1'$, in quadrature with I_2. But the vector sum of these two drops is the applied voltage E_1, and therefore, since the two component voltages are in quadrature, the voltage triangle will lie in a semicircle as shown in Fig. 321. In this triangle $O'A$ is the reactance drop and AB the resistance drop. Also, since the reactance is constant, $O'A$ is also proportional to the current I_1', which is the current induced in the stator by the rotor current, exactly as in the case of a transformer. The total stator current is I_1, which is the vector sum of I_1', and I_0 the no-load current, this latter having a magnetizing component I_μ and a working component I_ω. Hence, the triangle $OO'A$ is a current triangle, OA

being the stator current, and $O'A$ the rotor current (referred to the stator). It is obvious that the diagram thus obtained from the equivalent circuit is the circle diagram as developed previously by another method. The standstill point, D, will be that for which N is zero, the resistance therefore being $(r_1 + k^2r_2)$. By giving r_2 a series of values, as, for example, when a rotor starter is used with a

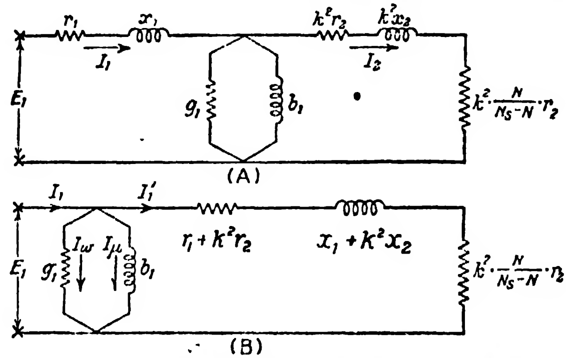


FIG. 319. EQUIVALENT CIRCUIT OF INDUCTION MOTOR

slip-ring motor, any desired position between the extremes O' and D can be secured for this point.*

17. Testing of Induction Motors. The following tests are made on induction motors—

(*a*) *No-load Test at Normal Voltage and Frequency.* This gives the no-load current and power factor and also the sum of the core loss and friction loss.

If the no-load test is carried out at a series of applied voltages, after the manner of the no-load test on a D.C. shunt motor, it is possible to separate the iron and friction losses. Fig. 320 shows the results of a test made on a 5 h.p. 250 volt squirrel-cage induction motor, the watts being plotted against $V^2/1{,}000$, a very convenient unit for low-voltage motors. At low voltage the speed of an induction motor falls off considerably, with the result that the initial portion turns downwards. Also for voltages beyond normal, i.e. for $V^2/1{,}000$ greater than 65 in the case of the 250 volt motor, the curve turns upwards. There is, however, a sufficient range over which the graph is linear to enable a straight line to be drawn and the friction and windage loss to be thus determined with good accuracy.

(*b*) *Test at Reduced Voltage with Rotor Either Stalled or Allowed to Rotate Very Slowly.* This test gives the short-circuit current and

* For further development of the theory of the induction motor, and of other A.C. motors, as well as circuit theory, see Mallett, *Vectors for Electrical Engineers.*

power factor, so that the results of tests (*a*) and (*b*) along with resistance tests on the windings give sufficient data for the circle diagram to be drawn.

(*c*) *Heat Run.* It is possible to perform a regenerative test on two induction motors, but the test is difficult and is rarely carried out. The usual method is to couple the motor to a direct-current generator and to load up this machine by means of a resistance load, if of small size, or to adjust its excitation so that when paralleled on to a D.C. supply it returns energy to this supply. The excitation is,

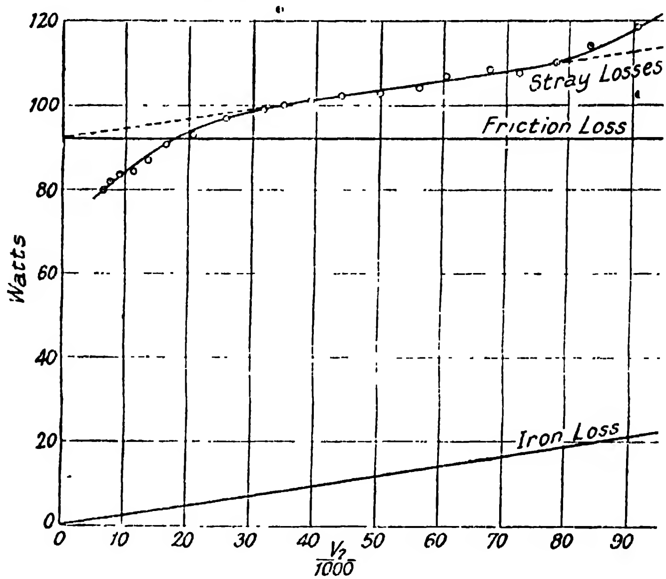


FIG. 230. ANALYSIS OF STRAY LOSSES IN INDUCTION MOTOR

therefore, adjusted so that the induction motor takes its normal full-load current from the A.C. supply. This test gives the temperature rise of the different parts of the motor, usually by direct measurement with thermometer, and also information for the calculation of the efficiency. No attempt is made to calculate the efficiency directly from the intake of the D.C. generator, since this method is liable to considerable errors. Instead, the various losses in the motor are determined separately, and the total losses then obtained by adding together the individual losses. This is called the segregation of losses method, and it is the best method for all classes of electrical machinery. The intake of the motor on load is determined by the two-wattmeter method, and the stator voltage, stator current, and speed, or slip, read. As is shown later, the value

of the slip is used in the calculation of the efficiency, and some method of determining the slip accurately is therefore necessary. The slip is given by

$$\sigma = \frac{N_s - N}{N_s}$$

and therefore it can be calculated from measured values of N_s and N, but since the difference $(N_s - N)$ is so small in comparison with N_s, a small error in either measurement will lead to a considerable error in the calculated value of σ.

In the case of machines with very small slip, a good method is to

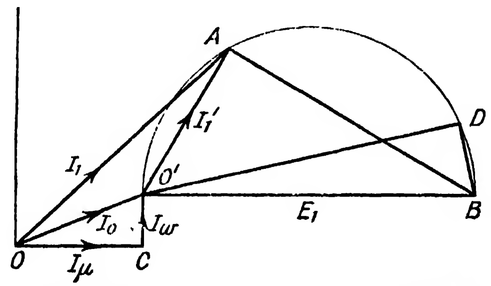


FIG. 321. DERIVATION OF CIRCLE DIAGRAM FROM EQUIVALENT CIRCUIT

connect a zero-centre, moving-coil ammeter in one of the connections between slip-rings and starter. The very low-frequency current through the instrument will cause the pointer to oscillate backwards and forwards at slip frequency, one complete oscillation (i.e. one "swing-swang") of the needle corresponding to one complete cycle of the rotor current. Thus, by timing the needle with a stop-watch, a very accurate determination of the slip can be made.

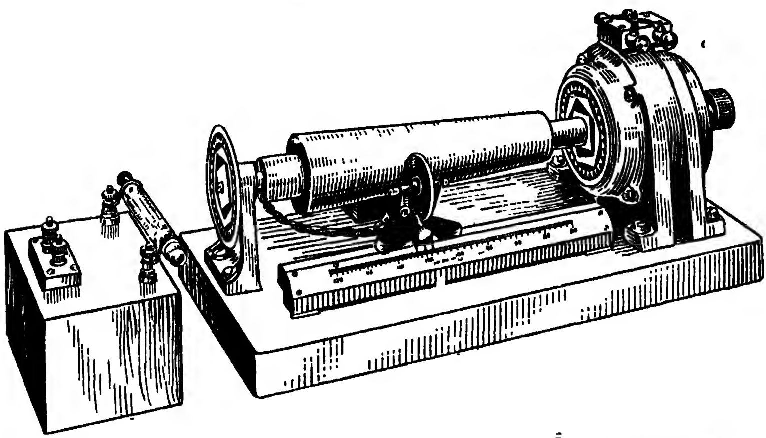
The stroboscopic method is another good method. In this, a white disc with black sectors, the number of sectors being equal to the number of poles, is mounted on the motor shaft and is illuminated by light which is intermittently blotted out, as, for example, by being made to pass through a rotating disc with equidistant slits in the periphery. Suppose this disc is driven by a small synchronous motor, and that there are as many slits as the driving motor has poles, then the interval between the glimpses of the painted disc will be equal to the time taken for the synchronous motor to rotate an angular distance equal to the pole pitch, namely, half the periodic time of the supply. If the induction motor rotated at synchronous speed its disc would rotate through an angle equal to the spacing of the black sectors during this interval, so that at each glimpse the pattern would appear to be in the same position, i.e. stationary in space. But since the motor travels at a speed less than synchronism, the angular distance travelled by a sector during the period

between two glimpses will be less than the sector pitch, and as a result the disc will have the appearance of rotating backwards slowly. The speed of the pattern is timed with a stop-watch, and this speed is the slip speed, the speed at which a synchronous motor would run if supplied at slip frequency.

$$\therefore \text{ Slip frequency} = \sigma f = \frac{np}{120} \text{ cycles per sec.}$$

where n is the observed speed of the pattern in revolutions per minute.

If the disc can be screened from the light, the same phenomenon



(H. Tinsley & Co)

FIG. 322. DRYSDALE STROBOSCOPIC SLIP METER

can be observed if the disc is illuminated by a number of neon lamps in parallel, worked from the stator supply. With either method observations are easy to make so long as the slip is small; but with a large slip the speed of rotation becomes so fast that, combined with the eyestrain caused by the intermittent illumination of the disc, the timing of the disc becomes difficult. This is overcome in the Drysdale slip meter, in which the slotted disc, instead of being driven directly by the synchronous motor, is driven by a conical roller. When making an observation the speed of the disc is varied by sliding it along the roller until a position is reached at which the pattern appears stationary. The slip is then read directly from a scale on the instrument. This instrument is illustrated in Fig. 322.

The method of calculating the efficiency of the motor from the results of the test is best illustrated by a numerical example. It will be seen that the slip is required in order that the rotor I^2R loss may be separated from the total copper losses.

The results of no-load and full-load tests on a three-phase mesh-connected motor were as follows. The intake as measured by the two-wattmeter method gave 3,000 and – 550 watts on no-load, and 39,000 and 23,500 watts on full load. The motor had four poles, the supply frequency was 50, full-load speed 1,460 r.p.m., stator resistance per phase (hot) 0·13 ohm. No-load current 25 amp., full-load current 92 amp. per line.

Stray losses + no-load stator I^2R loss

$$= 3{,}000 - 550 = 2{,}450 \text{ watts}$$

No-load stator I^2R loss $= 3 \times \left(\frac{25}{\sqrt{3}}\right)^2 \times 0{\cdot}13$

$$= 82 \text{ watts}$$

$\therefore$ Stray losses $= 2{,}450 - 82 = 2{,}370$ watts (say)

Full-load intake $= 39{,}000 + 23{,}500 = 62{,}500$ watts

Stator I^2R loss $= 3 \times \left(\frac{92}{\sqrt{3}}\right)^2 \times 0{\cdot}13 = 1{,}100$ watts

$\therefore$ Rotor intake $= 62{,}500 - (1{,}100 + 2{,}370)$

$$= 59{,}030 \text{ watts}$$

Synchronous speed $N_s = 1{,}500$ r.p.m.

Actual speed $N = 1{,}460$ r.p.m.

$$\therefore \% \text{ slip } \frac{N_s - N}{N_s} \times 100 = \frac{1{,}500 - 1{,}460}{1{,}500} \times 100$$

$$= 2{\cdot}66$$

$$\therefore \text{ Rotor } I^2R \text{ loss } = \frac{2{\cdot}66}{100} \times 59{,}030$$

$$= 1{,}570 \text{ watts}$$

$\therefore$ Output = Rotor intake – rotor I^2R loss

$$= 59{,}030 - 1{,}570$$

$$= 57{,}460 \text{ watts}$$

Intake $= 62{,}500$ watts

$$\therefore \text{ Percentage } = \frac{57{,}460}{62{,}500} \times 100$$

$$= 91{\cdot}9\%$$

It will be seen that the above method necessitates that the motor shall be put on full load. If this is not possible, then it is necessary to draw the circle diagram from the results of no-load and locked tests, and to determine the efficiency from that.

18. **The Induction Motor run as a Generator.** If the rotor of an induction motor is driven by another motor, or by a prime

mover, and at the same time the stator winding is connected to a three-phase supply, then so long as the rotor speed is less than synchronous speed, the rotor will develop a motoring torque which, for small values of the slip, is proportional to the slip. If it is driven at a gradually increasing speed, its torque will gradually decrease and will become zero at exactly synchronous speed. The rotor will then carry no induced current. If its speed is now increased above synchronism, its slip will be negative, the back E.M.F. induced in the stator by the pulsations in the useful rotor flux will be reversed, and will no longer have a component in direct opposition to the supply E.M.F. Hence, the induction motor will now be acting as a generator.

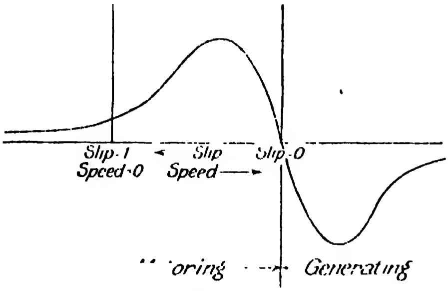


Fig. 323
Torque Slip Curve

The complete torque speed curve for speeds below and above synchronism is as shown in Fig. 323. When the induction motor functions as a generator, it is called an "asynchronous" generator. It differs from the ordinary synchronous generator in the following respects—

(*a*) It has no direct current excitation.

(*b*) It will only generate when its stator is connected to a line of fixed frequency, its exciting current being the wattless magnetizing current drawn from the line. This current, as we have seen, produces a rotating field, and there is obviously no difference between a rotating magnetic field produced by polyphase alternating currents, and one produced by a direct current excited system which is itself driven at synchronous speed.

(*c*) The frequency of the magnetizing current fixes the frequency of alternating current supplied by the induction motor. Thus the frequency is not affected by the speed at which the asynchronous generator is driven. There must be at least one synchronous alternator connected to the system in order to fix the frequency.

(*d*) No synchronizing is required since the machine cannot generate any E.M.F. until it is connected to the line.

It will be seen that such generators have a somewhat limited application, the most promising field probably being the utilization of somewhat small and variable water supplies where there is no hydraulic storage, and where the whole of the available energy must be either utilized or wasted. They are also used in connection with variable speed induction motor sets. Thus, if a

commutator motor is worked in cascade with a main induction motor for speed regulating purposes, it can be direct coupled to an asynchronous generator whose stator is connected to the supply, instead of being coupled to the main motor. In such a case the slip energy of the main rotor is returned (less losses) to the line, instead of being converted into mechanical energy.

Asynchronous generators, or induction generators as they are commonly called, are nearly always provided with squirrel-cage rotors.

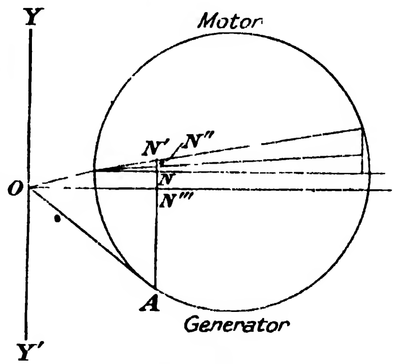


Fig. 324

Circle Diagram of Induction Generator

The characteristics of the generator can be derived from the circle diagram, the working point being now below the horizontal axis. For any working point A (Fig. 324),

$$\text{Total power delivered to generator} = AN'$$

$$\text{Rotor copper loss} = N'N''$$

$$\text{Stator copper loss} = N''N$$

$$\text{Stray losses} = NN'''$$

$$\text{Output} = N'''A$$

$$\text{Slip} = \frac{N'N''}{N''A}$$

$$\text{Power factor} = \cos AOY'$$

19. The Single-phase Induction Motor. This motor is similar in construction to the polyphase induction motor, with the exception that the stator has a single-phase winding. The alternating field produced by the single-phase stator current can be resolved into two rotating components, and if the motor is running at nearly synchronous speed under the influence of the forward component, then its velocity relative to the backward component will be nearly twice synchronous speed. The total torque is the sum of the torques developed by each of these fields assumed as acting independently. It can therefore be obtained by drawing the torque speed curves. The curve for the forward field is exactly similar to that of a polyphase motor. That for the backward field is below the horizontal axis, since this field produces a retarding torque owing to its direction; also, the curve is reversed, since synchronous speed to the forward field is twice synchronous speed

to the backward field, while full speed in the reversed direction is synchronous speed to the backward field. The two component curves are shown dotted in Fig. 325, and the total torque is indicated by the full line curve. It will be seen that this characteristic is similar in shape to that of a polyphase motor with low rotor resistance, with the exceptions that the torque is zero at a speed slightly less than synchronous speed, and it is zero again at zero speed. That is, the single-phase motor is not self-starting. It can be made self-starting by supplying the stator with an auxiliary winding for starting purposes, spacing this winding 90 electrical degrees from the main stator winding, and arranging that the current through it is as nearly in quadrature with the main current as possible. For starting purposes, the motor is thus converted temporarily into a two-phase motor. The most common method of obtaining the required phase difference is the split-phase method.

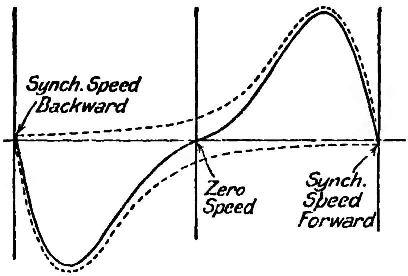


Fig. 325

Torque/Slip Curve for Single-phase Motor

A choker is connected in series with the auxiliary winding, thus giving it a greater reactance than the main winding. This auxiliary winding is only used for starting purposes, and it is cut out of circuit as soon as the motor attains full speed. In the Heyland single-phase motor, the auxiliary winding is placed in a few deep slots; it is thus situated farther from the surface of the stator than the main winding, and is more deeply embedded in the iron. This gives it the necessary reactance.

The presence of the backward travelling field in the single-phase induction motor is a disadvantage in several respects.

(*a*) It reduces the total torque since it acts in opposition to the forward field

(*b*) It renders the torque zero at starting so that an auxiliary winding has to be provided, thus increasing the size and cost of the motor

(*c*) It requires a wattless magnetizing current just as the forward field does, the total magnetizing current being twice what it would be if the forward rotating field could be produced alone. This large magnetizing current gives the motor a very poor power factor.

(*d*) Owing to the greater magnetizing current, and the fact that the rotor current is not zero at synchronism, the copper losses in the single-phase motor are greater than in the polyphase motor. The rotor core losses are also not negligible.

Because of the increased losses and the space taken up by the starting winding, it can be taken that on the average a frame which will give a certain horse-power at a given speed when used for a polyphase motor will only give about two-thirds that horse-power when wound as a single-phase motor.

EXAMPLES ON CHAPTER XX

(1) A 6-pole induction motor is supplied by an 8-pole alternator running at 750 r.p.m. If the slip of the motor is 3 per cent, what is its actual speed?

Ans.—970 r.p.m.

(2) A three-phase, 50 cycle induction motor, with its rotor star connected, gives 100 volts (root mean square) at standstill between slip rings on open circuit. Calculate the current in each phase of the rotor winding when joined to a star-connected circuit, each limb of which has a resistance of 10 ohms and a reactance of 10 ohms. The resistance per phase of the rotor winding is 0·2 ohm and its reactance at stand-still 10 ohms. Calculate also the current in each rotor phase when the slip rings are short-circuited and the motor is running with a slip of 4 per cent. (London Univ., 1922.)

Ans.—2·58 and 5·2 amps.

(3) Prove that, ignoring stator losses, the input to an induction motor working at constant virtual voltage and frequency, is approximately proportional to the torque. Give a method of changing the number of poles in an induction motor. (London Univ., 1921.)

(4) A 200-h.p. three-phase induction motor has at no load a speed of about 500 r.p.m. and a slip of 1·5 per cent at full load. Show by a diagram drawn to scale how you would expect the torque of this motor to vary as the speed is varied from 0 to 500 r.p.m. (London Univ., 1921.)

(5) Define the "slip" of a three-phase motor. Draw two curves connecting the torque of the motor with the slip for two different values of the resistance in the rotor circuits. State the connection between these two curves and give the theory of the curves. (London Univ., 1915.)

(6) What arrangement would you propose for altering the speed of a 600-h.p. three-phase haulage motor? The motor is required to start at twice full load torque, to come up to speed in 30 sec., to run for 5 min. at full speed, and then to stop. It is to repeat this operation every 12 min. (London Univ. Mining, 1920.)

(7) State the advantages of cascade control of induction motors for traction work Define the term "cascade synchronous speed." (London Univ., 1911.)

(8) Describe a method of obtaining, economically, speed regulation in the case of a large three-phase induction motor, the range over which speed regulation is desired being approximately synchronous speed to about 70 per cent below synchronous speed. Describe briefly the apparatus required for this purpose, and explain its working. (London Univ., 1921.)

(9) Show how to arrange two windings on the rotor of a three-phase induction motor to provide automatically large torque at starting and low rotor copper loss when running. How would you calculate the starting torque of such a motor? (C. and G., 1923.)

(10) A 4-pole, 50 cycle, three-phase, 200-volt induction motor is run on light load and takes 12·5 amp. per line at a power factor of 0·2. The same motor,

when the rotor is allowed to rotate very slowly, takes 100 amp. when a pressure of 120 volts between phases is applied, at a power factor of 0·25. Estimate the pull-out torque and maximum power factor at which the motor will work. (C. and G., 1918.) [Graphical.]

(11) The speed of a 400-h.p. electrically driven fan is reduced by (*a*) a shunt motor with field control, (*b*) an induction motor with rotor resistance control. Compare the economy obtainable per hour in the two cases when the speed is reduced 15 per cent. Assume that the torque of the fan varies as the square of the speed and the cost of energy is 1d. per B.O.T. unit. Ignore losses in the motors. (C. and G., 1921.)

Ans.—There is a saving of 2s. 8d. per hour by using a shunt motor.

(12) Show how the losses in a three-phase slip ring induction motor can be analsyed from no load and short-circuit tests taken when the power is supplied (*a*) to the stator and (*b*) to the rotor. (C. and G., 1922.)

(13) How can the speed of an induction motor be varied by altering the number of phases? Give any two methods of varying the speed of an induction motor by supplying the slip energy to machines possessing commutators. (C. and G., 1922.)

(14) An induction motor whose windings can be arranged for 12 or 24 poles can be run in cascade with a direct coupled induction motor which can be arranged for 2 or 4 poles. With such an arrangement the cascade control is never used for the 24 pole speed. Why is this? Calculate the approximate speeds which can be obtained from the set, the supply frequency being 50.

Ans.—500, 250, 428, 375 r.p.m.

(15) A 4-pole, 35-h.p., 50-cycle induction motor has a full load slip of 2 per cent, and a pull-out torque equal to 2½ times the normal full load torque. Draw to scale the torque/speed curve for all speeds between zero and full speed.

(16) Draw the circle diagram for a 6-pole, star-connected, three-phase induction motor having the given data, and find the full load power factor, stator current, and slip. Rated output 40 b.h.p., at 380 volts, 50 cycles. No load current 16 amps. Power absorbed at no load 1,550 watts. Short-circuit (standstill) current 310 amps at a power factor of 0·23. Resistance from terminal to terminal of stator winding (hot) 0·156 ohm.

Ans.—0·91, 60 amps., 6 per cent.

(17) From the circle diagram of the motor in question 16, plot the following characteristics against torque: Speed, power factor, efficiency, output, and stator current.

CHAPTER XXI

CONVERTING MACHINERY

1. **Methods of Conversion.** At the present time over 90 per cent of the total electrical energy generated is produced in A.C. stations. A large proportion of this is utilized as direct current energy, and it is therefore necessary to convert from alternating to direct current. The most important methods of effecting this conversion are—

(*a*) Motor-generator sets. The motors are three phase alternating current motors, either synchronous or asynchronous; and the generators are direct current, shunt, or compound.

(*b*) Rotary convertors.

(*c*) Motor convertors.

(*d*) Mercury arc rectifiers.

(*e*) Valve, and other rectifiers.

2. **The Rotary Convertor.** This is essentially a direct current machine having a direct current armature of ordinary design, from which tappings are taken to slip rings as well as the ordinary connections to the commutator. The field system is identical with that of an ordinary direct current machine; it may be shunt or compound excited, and is usually fitted with interpoles. In all heteropolar machines, that is, machines with alternate *N*. and *S*. poles, whether for direct or alternating current, the E.M.F.s induced in individual armature conductors are alternating. The voltage across pairs of slip rings will thus be alternating, while the voltage across the brushes at the commutator will be direct. The following are the rules for taking off the tappings to the slip rings. If the armature is wave-wound, it has only two electrical circuits through it, whatever the number of poles. Hence, there will be only one tapping taken to each slip ring. For single phase there will be two slip rings, and there will be one-half the total number of armature conductors in each path from one tapping point to another. For three phase, there will be three slip rings and three tappings, the armature being divided into three equal parts. And so on.

If the armature is lap-wound, it has as many paths through it from brush to brush as there are poles. Hence, for any given potential relative to that of either of the brushes, there will be as many points in the winding having that potential, as there are pairs of poles. The number of tappings taken to each slip ring is thus equal to the number of pairs of poles. The total number of

tappings is equal to the product of the number of slip rings and the number of pairs of poles. The arrangement of these tappings is shown in Fig. 326. The numbers of conductors in each path between two consecutive tappings must be equal. Hence, the winding must be such that it can be divided into the requisite number of identical paths. The existence of this condition shows that any direct current winding is not necessarily suitable for use on a rotary convertor armature.*

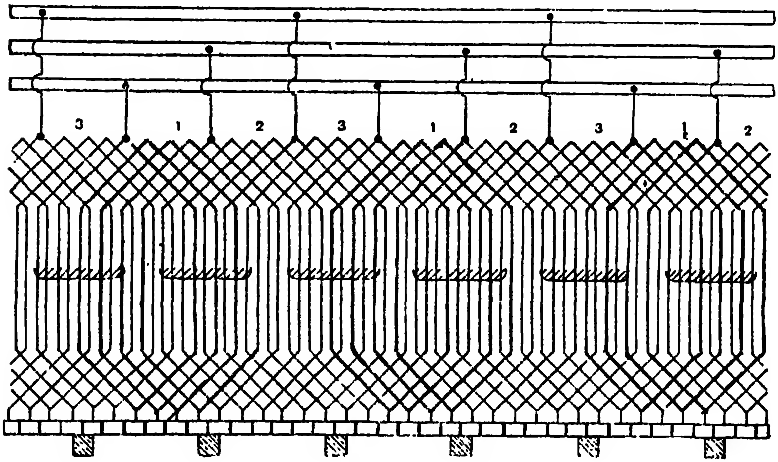


FIG. 326

ARRANGEMENT OF TAPPINGS ON ROTARY CONVERTOR ARMATURE

3. **Voltage and Current Ratios.** Fig. 327 represents an armature rotating in a two-pole field. The voltage at the direct current side, namely, the voltage across the brushes AB, is obviously the same as the maximum value of the single phase voltage, since in a single phase rotary the tappings are diametrically opposite.

$$\therefore \text{ Max. A.C. volts } = \text{D.C. volts} = E, \text{ say}$$

$$\therefore \text{ Effective A.C. volts } = E/\sqrt{2}$$

Consider now an "m" phase convertor. Two consecutive tappings such as C and D will be an angular distance of $2\pi/m$ radians apart. If the direct current voltage is represented by $2r$, the diameter AB of the circle, then the maximum value of the

* **The Transverter.** This is a modification of the rotary convertor; and consists of a group of phase-multiplying transformers, and a synchronously driven brush system, the units of which pass over fixed commutator segments.

alternating-current voltage between the points C and D will be represented by the length of the chord CD

$$\therefore\ E_{max} = 2r \sin\frac{\pi}{m}$$

$$= E \sin\frac{\pi}{m}$$

$$\therefore\ E_{eff} = E \times \frac{\sin\frac{\pi}{m}}{\sqrt{2}}$$

Hence, when there are m tapping points the voltage between adjacent points, and therefore between adjacent slip rings, bears the following definite ratio to the direct current voltage.

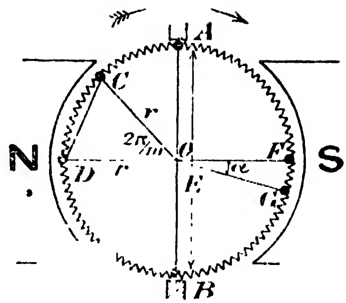


FIG. 327

CALCULATION OF VOLTAGE RATIOS

$$\frac{\text{A.C. volts}}{\text{D.C. volts}} = \frac{\sin\frac{\pi}{m}}{\sqrt{2}}$$

The numerical values of this ratio are given in the following table—

D.C.	1 phase.	2 phase.	3 phase.	4 phase.	6 phase.	12 phase.
1	$\sin\frac{\pi}{2}/\sqrt{2}$	$\sin\frac{\pi}{2}/\sqrt{2}$	$\sin\frac{\pi}{3}/\sqrt{2}$	$\sin\frac{\pi}{4}/\sqrt{2}$	$\sin\frac{\pi}{6}/\sqrt{2}$	$\sin\frac{\pi}{12}/\sqrt{2}$
	0·707	0·707	0·612	0·5	0·354	0·183

The value of m taken in calculating the above ratios is only equal to the number of tappings when the voltage between adjacent tappings is required. Thus for single phase there are two tappings and $m = 2$. For two phase there are four tappings, but since each pair of diametrically opposite tappings can be regarded as being the terminals of a separate single phase, $m = 2$ in this case also. In the four phase convertor, there are four equidistant tappings as in the two phase, but the phase voltage is now taken as the voltage between consecutive tappings, so that $m = 4$.

We have now to calculate the current ratios. If E = D.C. volts and I = D.C. line current, then

$$\text{Power on D.C. side} = EI$$

Let E_p = phase voltage and I_p = *phase* current, i.e. the alternating current flowing through the armature between two tapping points. Then if the power factor is $\cos\varphi$,

$$\text{Power on A.C. side} = mE_pI_p \cos\varphi$$

Now the A.C. line current I_l is the vector difference of the phase currents in two consecutive phases. Hence, from the vector diagram in Fig. 328, we have

$$I_l = 2I_p \cos\frac{AOB}{2}$$

$$= 2I_p \cos\left(\frac{\pi}{2} - \frac{\pi}{m}\right)$$

$$= 2I_p \sin\frac{\pi}{m}$$

$$\therefore\ I_p = \frac{I_l}{2\sin\frac{\pi}{m}}$$

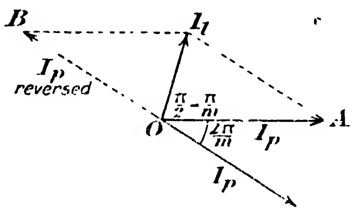


FIG. 328

CALCULATION OF CURRENT RATIOS

$$\text{Also } E_p = E \times \frac{\sin\frac{\pi}{m}}{\sqrt{2}}$$

$$\therefore\ \text{Power on A.C. side} = m \times \frac{E\sin\frac{\pi}{m}}{\sqrt{2}} \times \frac{I_l}{2\sin\frac{\pi}{m}} \times \cos\varphi$$

$$= \frac{m}{2\sqrt{2}} \,.\, EI_l \cos\varphi$$

Assuming 100 per cent efficiency and equating the expressions for the power on the D.C. and A.C. sides, we have

$$\frac{m}{2\sqrt{2}} \,.\, EI_l \cos\varphi = EI$$

$$\frac{I_l}{I} = \frac{2\sqrt{2}}{m} \,.\, \sec\varphi$$

For unity power factor, $\sec\varphi = 1$ and the current ratios are–

D.C.	1 phase.	2 phase.	3 phase.	4 phase.	6 phase.	12 phase.
	1·414	·707	·943	·707	·472	·236

4. **Wave Form of the Armature Current.** Suppose that a rotary convertor is converting from alternating to direct current. Then

relative to the alternating side it is motoring, while relative to the direct current side it is generating The total current in the armature is thus the difference between the alternating and the direct currents. Consider first of all a single phase convertor; let A and B be the tapping points (Fig. 327). For a point F mid-way between the tapping points, the alternating voltage has its crest value the instant F is opposite a pole centre; and if the current is in phase with the voltage, the current will go through its crest value at the same instant. The direct current is constant in magnitude and flows in one direction through the conductor at F while F is under the influence of the S. pole, and in the reverse direction while F is under the N. pole. It reverses each time the conductor passes under a brush. Hence, the direct current in one conductor can be regarded as an alternating current of rectangular wave form, which reverses each time the conductor passes under a brush. For the point F, mid-way between A and B, the alternating and rectangular waves are in phase, so long as the power factor is unity and the resulting wave form is as shown in Fig. 329 (I).

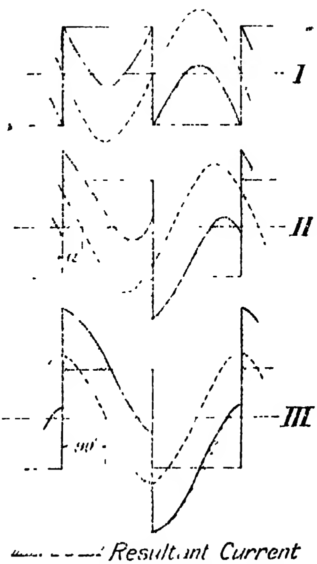


FIG. 329

WAVE FORM OF RESULTANT ARMATURE CURRENT

For a point G displaced any angle a from F, the alternating-current wave has the same form and is in phase with that through F, but G reaches a brush a degrees before F does, so that the rectangular wave for G reverses a degrees in front of the rectangular wave for F, thus modifying the resulting wave form as shown by curve II.

For a conductor at the tapping point the alternating and rectangular waves are 90° apart, the resulting wave form being as in curve III.

An examination of these curves shows that the effective value of the resultant current, and therefore, the armature I^2R loss, is the greatest at the tapping points and least at the mid points. Since the effective value increases as the phase angle between the sinusoidal and rectangular waves increases, we see that the greater the number of phases, the smaller will be the heating effect, because, as the number of phases increases, the maximum possible phase difference between the two waves decreases.

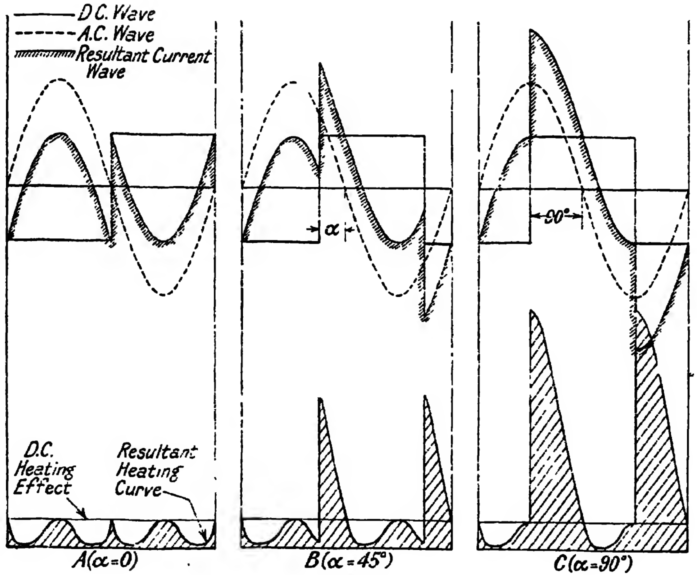


FIG. 330

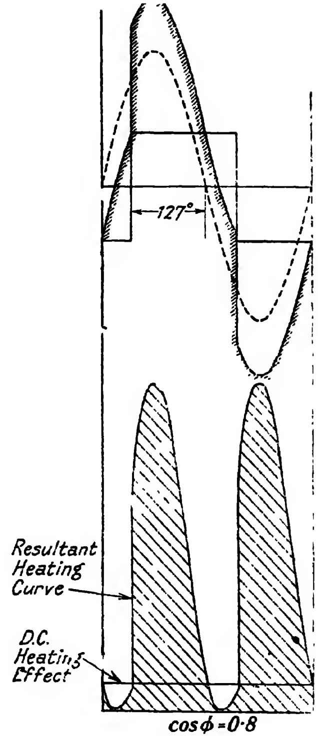


FIG. 331

HEATING EFFECT IN A ROTARY-CONVERTOR ARMATURE

The variation in the heating effect is best illustrated by curves of current squared, plotted against time for different values of the angle α. We will consider for simplicity a single-phase convertor, although the method can be extended to a machine with any number of phases. We require the relationship between I and I_p (not I_l) in order to draw the curves of resultant current. We have

$$mE_pI_p \cos \varphi = EI$$

$$\therefore \qquad I_p = \frac{1}{m} \cdot \frac{E}{E_p} \cdot \frac{I}{\cos \varphi}$$

For a single-phase machine $\frac{E}{E_p} = 1{\cdot}414$ and $m = 2$, hence, considering first of all a power factor of unity on the A.C. side, we have

$$I_p = \tfrac{1}{2} \times 1{\cdot}414 \times I$$
$$= I/\sqrt{2}$$

The direct current per conductor

$$I_a = I/2$$

$$\therefore \qquad I_p = \sqrt{2}I_a, \text{ and}$$

$$I_{p\cdot max} = 2I_a$$

The current curves are drawn to scale in Fig. 330, the cases being A for a conductor half-way between tapping points, B for a conductor displaced 45 electrical degrees from A, and C for a conductor at a tapping point. In the lower half of the figure the curves of current squared are plotted to the same scale, and it is at once apparent how much greater is the heating at a tapping point than at the mid-point.

Now suppose that the power factor is less than unity, say 0·8, then for the same value of I_a, $I_{p\cdot max}$ is increased in the ratio of sec φ to unity

$$\therefore \qquad I_{p\cdot max} = \frac{2I_a}{{\cdot}8} = 2{\cdot}5I_a$$

The angle φ is 37 degrees, very approximately, and as a result the maximum displacement between the direct- and alternating-current waves, which occurs at one of the tapping points can be $(90 + 37) = 127$ degrees. The corresponding curves for this case are drawn in Fig. 331, from which it is evident that the decrease in power factor produces a greater heating at one of the tapping points. It is true that the displacement at the other tapping point is now $(90 - 37) = 53$ degrees instead of 90 degrees, but this is offset by the fact that I_p is increased, so that on the whole the effect of a decrease in power factor is to produce a considerable increase in the heating.

A mathematical expression for the total I^2R loss in the armature is derived as follows—

We have seen that the alternating phase current

$$I_p = \frac{I_l}{2 \sin \frac{\pi}{m}}, \text{ and } I_l = \frac{2\sqrt{2}}{m} I \sec \varphi$$

$$\therefore \quad I_p = \frac{\sqrt{2}}{m} \operatorname{cosec} \frac{\pi}{m} \sec \varphi \times I$$

$$\therefore \quad (I_p)_{max} = \frac{2}{m} \operatorname{cosec} \frac{\pi}{m} \sec \varphi \times I$$

Considering the two-pole armature of Fig. 327, the direct current flowing through any conductor G is $\frac{I}{2}$, and the alternating current, being $(\alpha - \varphi)$ from its maximum value, is given by

$$i = (I_p)_{max} \cos (\alpha - \varphi)$$

Hence, total current in any conductor

$$= \frac{I}{2} - i$$

$$= \frac{I}{2} \left\{ 1 - \frac{4}{m} \operatorname{cosec} \frac{\pi}{m} \sec \varphi \cos (\alpha - \varphi) \right\}$$

To find the total copper loss it is necessary to calculate the R.M.S. value of the above expression for one particular conductor as the armature makes half a revolution. The conductor at F passes from one brush to the other in half a revolution, and if angles are reckoned from the mid-position F, the alternating E.M.F. will go through one-half cycle between the limits $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$. But we must consider any conductor. Let its distance from F be β. The limits for this conductor are obviously

$$-\left(\frac{\pi}{2} + \beta\right) \text{ and } \left(\frac{\pi}{2} - \beta\right)$$

Hence, the mean of the squares of the instantaneous values of the current, i.e. the square of the R.M.S. current in the particular conductor considered is

$$\frac{1}{\pi} \int_{-\left(\frac{\pi}{2} + \beta\right)}^{\frac{\pi}{2} - \beta} \left[\frac{I}{2} \left\{ 1 - \frac{4}{m} \operatorname{cosec} \frac{\pi}{m} \sec \varphi \cos (\alpha - \varphi) \right\} \right]^2 . d\alpha$$

$$= \frac{I^2}{4} \left\{ 1 - \frac{16}{\pi m} \operatorname{cosec} \frac{\pi}{m} \sec \varphi \cos (\beta + \varphi) + \frac{8}{m^2} \operatorname{cosec}^2 \frac{\pi}{m} \sec^2 \varphi \right\}$$

We have now to find the mean of this for the whole armature. This is obviously the same as the mean for one armature path between two consecutive tappings. Now two such tappings are $\frac{2\pi}{m}$ apart, and therefore, the limits of integration are $-\frac{\pi}{m}$ and $+\frac{\pi}{m}$. The expression for the square of the R.M.S. value can be written

$$\frac{I^2}{4}\{1 - A \cos(\beta + \varphi) + B\}$$

$$\therefore \text{ Mean value} = \frac{I^2}{4}(1 + B) - \frac{I^2A}{4} \times \frac{m}{2\pi}\int_{-\frac{\pi}{m}}^{+\frac{\pi}{m}} \cos(\beta + \varphi)d\beta$$

$$= \frac{I^2}{4}(1 + B) - \frac{I^2A}{4} \times \frac{m}{\pi} \cdot \sin\frac{\pi}{m}\cos\varphi$$

Finally substituting the values of A and B, we have mean value of the square of the current for the whole armature

$$= \frac{I^2}{4}\left\{1 - \frac{16}{\pi^2} + \frac{8}{m^2}\operatorname{cosec}^2\frac{\pi}{m}\sec^2\varphi\right\}$$

Hence, total I^2R loss in the armature

$$= 2 \times \frac{I^2R}{4}\left\{1 - \frac{16}{\pi^2} + \frac{8}{m^2}\operatorname{cosec}^2\frac{\pi}{m}\sec^2\varphi\right\}$$

since we are considering a two-circuit winding, R being the resistance of each circuit. If the armature were carrying a direct current only, its external value being I, as before, the I^2R loss would be

$$2 \times \left(\frac{I}{2}\right)^2 R$$

Hence

$$\frac{I^2R \text{ loss when acting as a rotary}}{I^2R \text{ loss as a plain D.C. machine}} = 1 - \frac{16}{\pi^2} + \frac{8}{m^2}\operatorname{cosec}^2\frac{\pi}{m}\sec^2\varphi$$

This shows that as the number of phases is increased, the I^2R loss decreases, and is always less than the loss in a plain D.C. machine, except in the case of a single phase rotary. For a unity power factor load, for which $\sec\varphi = 1$, the square roots of the above ratio are

D.C.	$m = 2$	$m = 3$	$m = 4$	$m = 6$	$m = 12$
1	1·18	·76	·62	·52	·455

If the output of the rotary armature were limited only by the armature copper loss, then the ratios of the maximum outputs

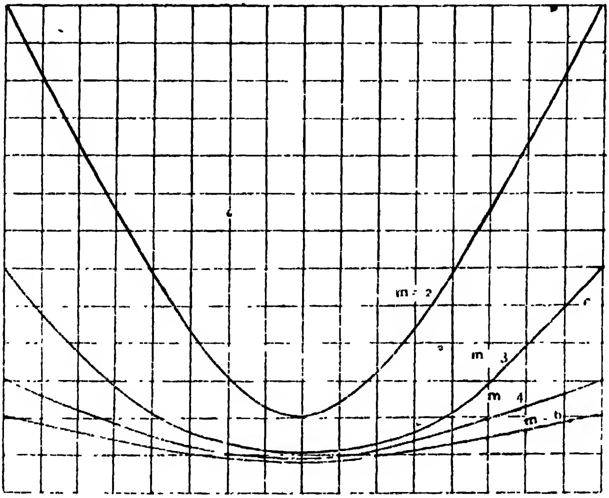


Fig. 332

Curve Showing Relative Heating Between Tapping Points of Convertor Armature Having m Slip Rings

possible to the output when working as an ordinary D.C. machine, would be the reciprocals of the ratios in the above table, namely

D.C.	$m = 2$	$m = 3$	$m = 4$	$m = 6$	$m = 12$
1·0	·85	1·33	1·63	1·92	2·20

This table shows how, in the case of all but a single phase convertor, the output from a given armature increases as the number of phases increases. In practice it is usual to work rotary converters six phase, since twelve phase working involves too much complication.

The heating effects in the armature, at different positions with respect to the tapping points, and also in terms of the number of phases, are illustrated graphically in Fig. 332.

The above comparisons are based on the assumption that the power factor on the alternating-current side is unity. If it is not unity, then since the expression for the heating effect contains a term $\sec^2 \varphi$, we see that as φ increases, the heating effect very considerably increases, the rotary convertor being no longer so very superior to the plain direct current machine from the point of view of armature heating.

5. Armature Reaction in Polyphase Convertors. From the consideration of armature reaction in direct-current generators, we have seen that the M.M.F. due to the direct current is directed along the brush axis, and is therefore fixed in space. On the other hand, the armature is rotating at synchronous speed, so that we can also regard the armature M.M.F. due to the direct component of the

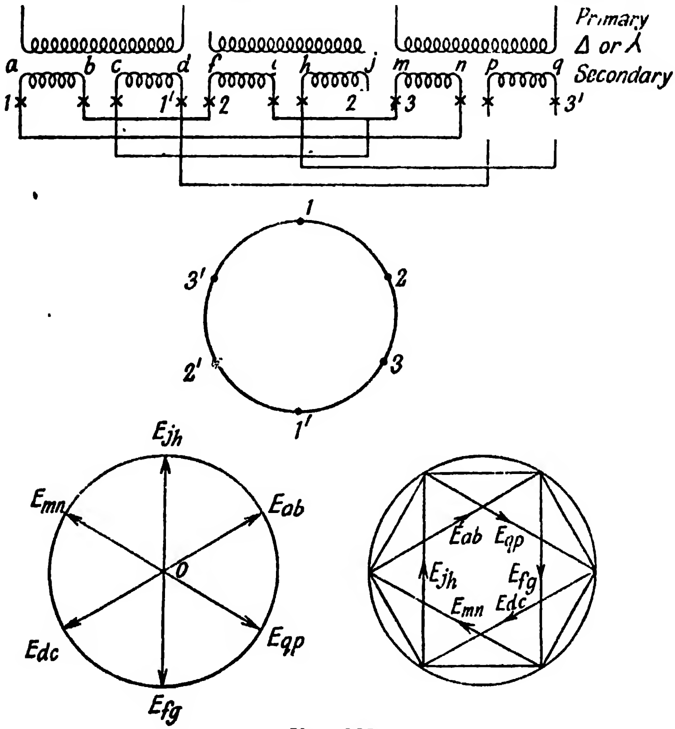


FIG. 333
DOUBLE DELTA SYSTEM OF CONNECTIONS

current as rotating at synchronous speed relative to the armature itself. Its direction is at right angles to the main field. The alternating component of the armature current also sets up an armature reaction which also rotates at synchronous speed with respect to the armature, and in consequence these two components of the armature reaction have no relative velocity. Also, the M.M.F. due to the alternating component acts at right angles to the main field, but in a direction opposite to that due to the direct component, these two thus tending to cancel one another out. Actually they do not quite neutralize one another, the direct M.M.F. being slightly the greater, but the difference is so small that to all intents and

purposes the rotary convertor can be regarded as having negligible armature reaction. As a result of this there is no distortion of the field form, and the machine can therefore withstand very heavy overloads without serious sparking at the commutator.. In fact, when rotary convertors are required for traction purposes, one of the standard works tests is actually to short-circuit the machine momentarily.

6. **Methods of Obtaining Six-phase Current.** Since there is a fixed voltage ratio between the D.C. and A.C. sides, it will be seen that to give a definite voltage at the commutator, a definite voltage must be applied to the slip rings, and this is almost sure to be different from the supply voltage. Transformers have therefore to be interposed between the alternating-current supply and the slip rings, the necessary six phase current being obtained by suitable arrangements of the secondary windings

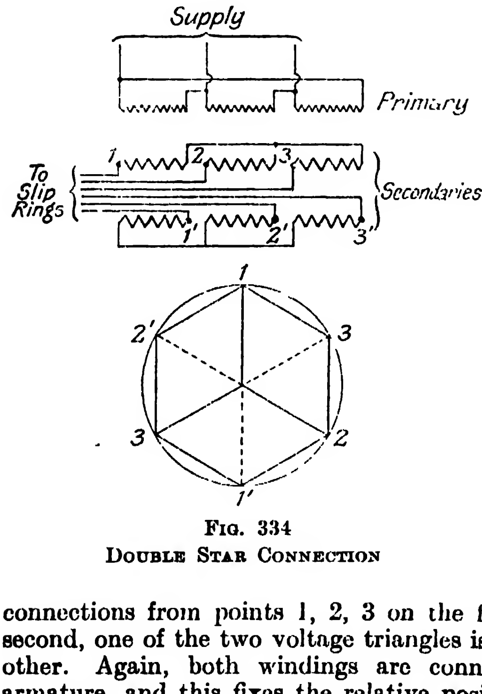


FIG. 334
DOUBLE STAR CONNECTION

(*a*) DOUBLE DELTA. Each phase of the secondary has two separate windings, which are arranged to form two independent mesh-connected secondaries, as shown in Fig. 333. By taking connections from points 1, 2, 3 on the first, and 1′, 2′, 3′ on the second, one of the two voltage triangles is reversed, relative to the other. Again, both windings are connected to a symmetrical armature, and this fixes the relative positions of the two voltage triangles, so that the lengths of the six vectors obtained by joining their corners are equal to one another. These vectors are the voltages applied to pairs of slip rings, and since there is a phase difference of 60° between two consecutive vectors, we see that a true six phase supply is obtained.

(*b*) DOUBLE STAR. As before, there are two separate secondaries, but in this case they are star-connected (Fig. 334) The first secondary gives the ordinary three-phase voltage star 1, 2, 3, while the second gives the reversed star 1′, 2′, 3′, since the connections from it are at the opposite ends of the windings to the connections,

from the first. Also, the two star points must coincide because of the symmetry of the armature winding, and therefore, as before, the sides of the hexagon give the voltages of the six-phase supply.

(c) DIAMETRAL. In the double star method, the two star points have the same potential and they could therefore be joined together if desired. If this were done, then each phase of the transformer secondary would have one winding only, the six ends being taken to the slip rings, and the middle points, connected together. But the potential of the middle point is fixed by the armature of the convertor, and therefore, the star point need not be made, thus leaving the three separate phases of the secondary winding connected to the slip rings, as in Fig. 335. In this case the voltage induced in each phase of the secondary is given by the diameter of the circle, whereas in the double star method it is given by the radius. This method is the most commonly used, because it is the simplest. If the rotary convertor is supplying a three-wire system on the D.C. side, then the star point is formed, and the middle wire connected to it. This fixes its potential midway between the potentials of the positive and negative outers, and no extra balancing plant is required.

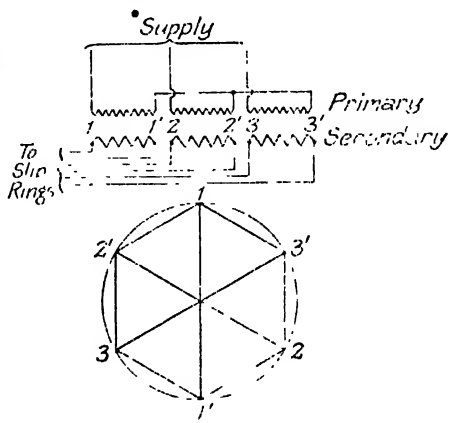


FIG. 335
DIAMETRAL CONNECTION

Example. A star-connected three-phase alternator, giving 1,500 volts per phase, supplies six-phase current by means of three mesh-connected transformers, each provided with two equal secondaries. The current is passed into a rotary convertor supplying direct current at 550 volts. Calculate the change ratio of the transformers and the current supplied by each of the secondaries, if 200 kW are taken from the convertor. Neglect magnetic leakage and loss in transformation. Show by diagrams how the six contact rings of the convertor are connected to the secondaries. (London Univ., 1903.)

The diagram of connections is given in Fig. 333. If E_a= R.M.S. voltage between adjacent slip rings, and I_a= current supplied to each slip ring, then

$$E_a = \frac{E}{\sqrt{2}} \cdot \sin\frac{\pi}{m} = \frac{550}{\sqrt{2}} \sin 30 = 194{\cdot}4 \text{ volts}$$

$$I = \frac{200{,}000}{550} = 364 \text{ amp.}$$

$$\therefore\ I_a = \frac{2\sqrt{2}}{m} \times I = \frac{2\sqrt{2}}{6} \times 364 = 171{\cdot}4 \text{ amp.}$$

The voltage induced in each secondary winding is given by the length of the side of an equilateral triangle drawn in the six-phase voltage hexagon. Each side of the hexagon represents a voltage of 194·4. Hence, voltage in each secondary

$$= \sqrt{3} \times 194{\cdot}4 = 337 \text{ volts}$$

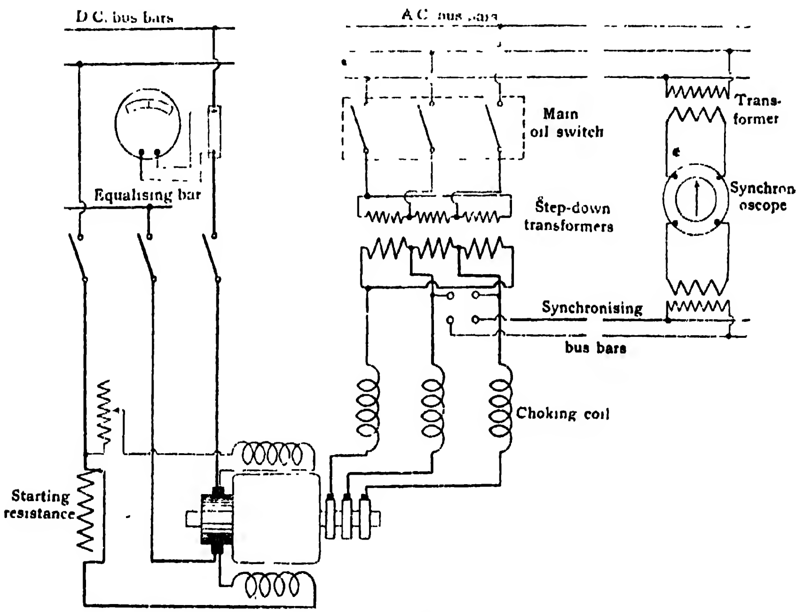


FIG. 336

CONNECTIONS FOR THREE-PHASE CONVERTOR TO BE STARTED FROM THE D.C. SIDE

∴ Transformation ratio of the transformer

$$K = \frac{337}{\sqrt{3} \times 1500} = {\cdot}1295$$

∴ No. of turns on each secondary = ·1295 × turns on each primary.

7. **Methods of Starting Rotary Convertors.** In the great majority of cases, rotary convertors are supplied with alternating current, and deliver direct current. It is therefore unlikely that there will be any direct current supply for starting, unless it is specially provided, so that the convertors in most cases are started up from the alternating-current side. If direct current is available, the

procedure is as follows: The direct current switch (Fig. 336) is closed, and the oil switch on the alternating-current side, left open. The rotary is then run up to speed like an ordinary shunt motor by means of the starting resistance. The shunt regulator is adjusted until synchronism is obtained, and the oil switch then closed. No voltage adjustment is necessary, as when synchronizing alternators, because the voltage ratio between the direct and alternating-current sides is fixed.

If the load on the direct current side is rapidly fluctuating while synchronizing, it may be very difficult to decide the correct moment of synchronism, because of the variations in speed produced by the variations in voltage. In such a case it is better to run the rotary up to a speed well above synchronism, open the D.C. switch, and thus allow the machine to slow down. As soon as synchronous speed is reached, as indicated by the synchroscope, close the oil switch. Alternating current will then flow through the armature and keep it running at synchronous speed. There is no danger of a heavy rush of current on closing the oil switch, because no power is being drawn from the direct current supply. At the worst, the armature will be violently pulled into step if the switch is not closed quite at the right moment. Finally the D.C. switch is closed.

There are three common methods of starting up from the alternating-current side—

(*a*) By Taps on the Main Transformer This is a very commonly used method, the arrangement of the control gear being as indicated in Fig. 337. It will be noticed that no synchronizing gear is required, but a moving coil zero-centre voltmeter *V* is connected across the brushes on the D.C. side, and the field winding is split up into a number of sections by means of a "field break-up switch." On the A.C. side, either auto-transformers are interposed between the slip rings and the main transformer, or the secondaries of the main transformer are used as auto-transformers, this second arrangement being shown in the figure. When starting, the D.C. switch and the field break-up switch are opened, and the brushes lifted off the commutator. The A.C. switch is closed, the auto-transformer being arranged so that the smallest voltage is applied to the brushes An alternating current flows through the armature and produces a rotating magnetic field, the lines of force of which sweep past the dampers on the pole faces, thereby inducing currents in them. Hence, the armature starts up like an inverted squirrel-cage induction motor. Now the rotating field produced by the armature rotates at synchronous speed *relative* to the armature, and therefore, as the speed of the armature increases, the actual speed, in space, of the rotating field becomes smaller and smaller, until eventually its poles move so slowly past the salient poles of the field system, that they magnetize the latter.

The two systems of poles then grip each other, the rotating field therefore becoming stationary in space. This means that the armature must now be running at synchronous speed, the rotary having what is called "jumped into step." It is obvious that it is possible for the polarity of the D.C. side to come up in the wrong direction, because this polarity depends upon the ultimate polarity of the main poles, and therefore, on the relative positions of the main poles and the rotating field at the instant the convertor jumps into step. If the polarity on the D.C. side is wrong it will

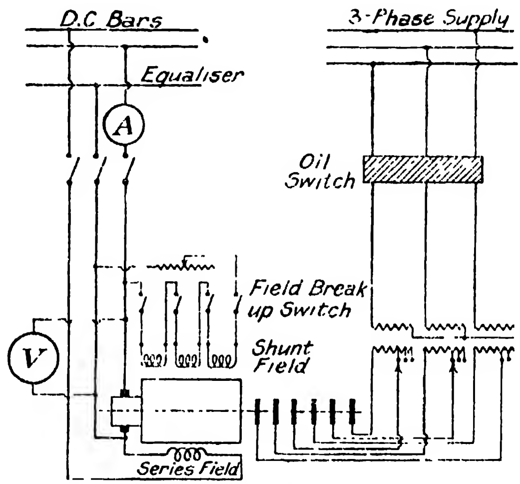


Fig. 337
Starting by Taps on the Main Transformer

be indicated by the zero-centre voltmeter V being deflected in the wrong direction. It can be corrected by what is called "slipping a pole." The field switch is momentarily reversed, thus causing the armature field to be repelled by the main field, and as a result each pole of the armature field moves on towards the next main pole. This is indicated by the needle of the zero-centre voltmeter swinging over to the right side, the field break-up switch being then closed in the right position. Finally the convertor is connected to the D.C. system by closing the D.C. switch.

The field is split up into sections by the field break-up switch, because, at the moment of starting, the actual velocity in space of the armature field is the synchronous speed, and consequently, owing to the very large number of field turns, a very high voltage would be induced in the field winding. When the armature is running at synchronous speed, the armature field is stationary in space, and consequently no voltage is induced in the field winding, the sections of which can therefore be all connected in series.

For the same reason, the brush gear on the D.C. side is lifted at starting, otherwise heavy currents would circulate across the brush faces and cause excessive sparking.

The full A.C. voltage is not applied to the slip rings at starting, because of the very heavy rush of current which would be produced on the A.C. side. Also, this current would be very nearly wattless, so that it would probably interfere with the voltage regulation of the line.

(*b*) By Separate Starting Motor. The convertor is started up by means of a small direct-coupled squirrel-cage induction motor. This motor has a smaller number of poles than the convertor, so that it is able to bring the convertor right up to synchronous speed. The convertor is started up by this motor, the field circuit being closed,

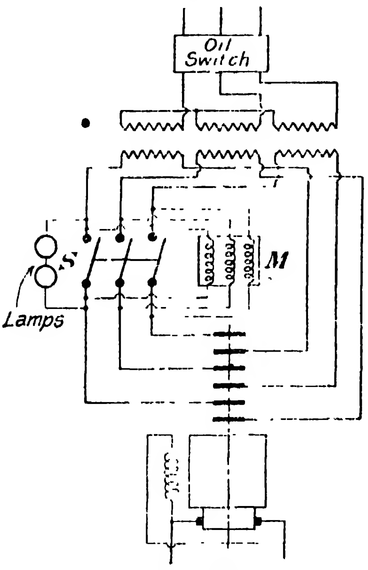


Fig. 338
Self-synchronizing Method

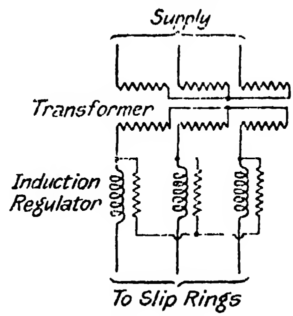


Fig. 339
Voltage Regulation by Induction Regulator

so that the voltage is built up in the right direction exactly as in the case of a shunt dynamo when brought up to speed. Synchronizing is necessary before connecting the slip rings to the alternating-current supply, the necessary speed regulation being obtained by means of an adjustable resistance in one stator phase of the starting motor. After synchronizing, the switch controlling the starting motor is opened and the convertor main switch on the D.C. side is closed.

At the present time it is usual to build the starting motor with a rotor consisting simply of a solid iron cylinder, since there is no possibility of trouble with such a rotor as it is only in operation a very short time.

(*c*) Self-synchronizing Method. This method is due to Rosenberg. The convertor is started up by a small starting motor

as in case (*b*), but the stator of this motor is in series with the convertor armature, as can be seen from Fig. 338. The main oil switch is first closed, the switch *S* across the starting motor being open. The shunt-field circuit is also closed, the regulator being set to the normal position. The starting motor *M*, having fewer poles than the convertor, brings the latter up to synchronous speed, and as soon as this speed is attained, the convertor jumps into step because its armature is also carrying the alternating current taken by the starting motor. There is very little possibility of reversed polarity, because the armature rotating field is produced by a very small current, and in consequence, it is not sufficiently strong to reverse the residual magnetism in the field. As soon as the convertor is in step the switch *S* is closed, thus short-circuiting the starting motor. Brush-lifting gear and field break-up switch are unnecessary, because the rotating field produced by the armature current during starting is not sufficiently strong to produce induced voltages of any importance.

8. **Voltage Regulation of Rotary Convertors.** There are four methods as follows—

(*a*) Hand Regulation. The main transformer is provided with tappings, so that the voltage applied to the slip rings can be varied within limits. This is not a very satisfactory method, first, because it is not automatic; second, because the voltage can only be varied in jumps; third, because of excessive wear and tear on tappings from which very heavy currents are taken.

(*b*) By Induction Regulator. The method of connecting a three phase induction regulator in the circuit is shown in Fig. 339. The regulator, which is very similar in construction to a three phase induction motor, acts like a booster, and raises or lowers the total voltage applied to the slip rings according to the position in which its rotor is placed relative to its stator. The regulator can be arranged for hand or automatic operation, and by means of it a variation of voltage on the D.C. side as high as 30 per cent or more can be obtained.

(*c*) By Choking Coils: Reactance Control. Choking coils are placed between the main transformer and the slip rings, as in Fig. 336, and the convertor is compound excited, having both shunt and series excitation. The voltage at the slip rings is obviously equal to the vector difference of the voltage E at the main transformer terminals, and the voltage E_c across the choker terminals, this latter voltage leading the current by 90°. On normal load the current I is in phase with the supply voltage OA (Fig. 340 (A)), the slip ring voltage being therefore equal to OC. If the load increases, the convertor becomes over-excited due to the series excitation, and the current vector I and choker voltage vector OB are shifted round in a counter-clockwise direction, thus increasing the slip ring voltage OC, as shown in Fig. 340 (B). This

method is very good so long as a voltage regulation of not more than 15 per cent is required.

(*d*) SYNCHRONOUS BOOSTER CONTROL. This method is often used where a wide range of voltage on the D.C. side is required, and where the fluctuations in load are very rapid as in traction work. The synchronous booster consists of an alternating current generator, whose armature is mounted on the convertor shaft and is connected electrically in series with the convertor armature, as shown diagrammatically in Fig. 341. The booster and convertor

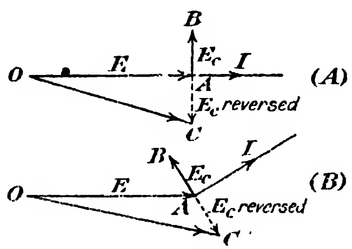


FIG. 340
REACTANCE CONTROL

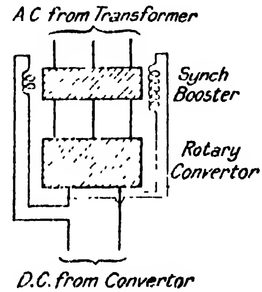


FIG. 341
VOLTAGE REGULATION BY SYNCHRONOUS BOOSTER

fields obviously have equal numbers of poles. If the booster field is supplied with series as well as separate excitation, the booster voltage will increase with the load, the voltage control thus being automatic.

The disadvantage of booster control is that it is liable to affect commutation if commutating poles are used. If the commutation is good when the voltage is about the middle of the range, then when the booster raises the voltage the commutating poles become magnetized by armature reaction, their effect then being too powerful. Conversely, when the convertor is giving the minimum voltage, the commutating poles are partially demagnetized and their effect is neutralized. This difficulty has been overcome by connecting diverters across the commutating poles when the convertor is delivering large D.C. voltages.

(*e*) SPLIT-POLE CONVERTORS. In the split-pole convertor the distribution of flux density under the poles is varied; this alters the wave form of the voltage across the slip rings and so alters the transformation ratio.

Let E_l = line voltage ; E_s = phase voltage ; E = D.C. voltage

$\therefore$ for a six-phase machine $(E_s)_{max} = \frac{E}{2}$ (= radius of the circle in Fig 327).

This can be put in the form

$$E = 2\frac{(E_s)_{max}}{(E_s)_{eff}} \times \frac{(E_s)_{eff}}{(E_l)_{eff}} \times (E_l)_{eff}$$
$$= 2K_1K_2 \times E_l$$

Thus $2K_1K_2$ is the transformation ratio.

Now for a sine wave $K_1 = \sqrt{2}$ and $K_2 = 1/\sqrt{3}$. As the wave form alters, K_2 remains practically constant, but K_1 varies very considerably, as shown below.

Sine Wave.	Rectangular Wave.	Triangular Wave.
$K_1 = 1{\cdot}41$	$K_1 = 1$	$K_1 = 1{\cdot}74$
$K_2 = 1/\sqrt{3}$	$K_2 = 1{\cdot}04/\sqrt{3}$	$K_2 = 1{\cdot}006/\sqrt{3}$

Hence, we can say that the limits of K_1K_2 are 1 to 1·74, so that by varying the wave form, a very large change of D.C. voltage can be produced. This change is effected by altering the distribution of flux. The pole consists of three limbs provided with main shunt, and auxiliary series windings as shown in Fig. 342. If the shunt winding produces, say, N. polarity, and the series windings tend to produce polarities of the order $N.,S.,N.$, then the D.C. voltage will decrease. If the series windings tend to produce polarities of the order $S.,N.,S.$, then the D.C. voltage will increase.

The objections to this method are: First, mechanical difficulties of construction; second, difficulty with commutation, which limits its application to low frequencies. It is therefore only suitable for small capacity, 25-cycle machines. To simplify construction the split-pole convertor is often made with a two-part, instead of three-part, pole.

9. **Inverted Rotary Convertors.** Usually, rotary convertors change the supply from A.C. to D.C., and if they are worked in the opposite sense they are styled "inverted." With a convertor working normally, the speed is fixed at synchronism, because it is operating like a synchronous motor on the A.C. side; and this also applies to an inverted convertor if it shares the load on the A.C. side with a synchronous generator. If it does not share the load in this way, there is nothing to fix the speed, and the speed will vary as the flux per pole varies. Thus, if the load is highly inductive, the lagging alternating current will produce a powerful demagnetizing effect, thus causing a serious increase in speed. To prevent this, inverted convertors are always separately excited from a direct-coupled exciter. The increase in excitation

due to a rise in speed will increase the flux per pole and so bring down the speed again.

10. **Parallel Operation of Rotary Convertors.** We have seen that two shunt generators work satisfactorily in parallel, because they have drooping voltage characteristics. Thus, if one machine takes too much load, its voltage drops, thereby automatically throwing the excess load on to the other machine. With convertors, the D.C. voltage is fixed because the A.C. voltage applied to the slip

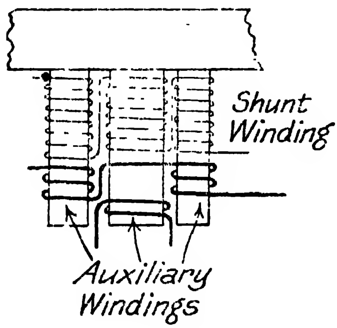


FIG. 342
SPLIT POLE CONVERTOR

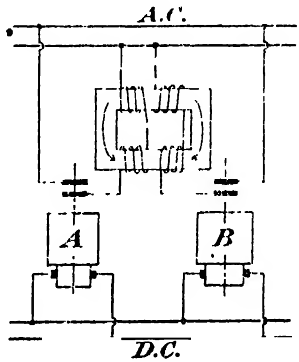


FIG. 343
ROTARIES IN PARALLEL

rings is fixed. Therefore, if two convertors which are working from the same A.C. supply are also parallel on their D.C. sides, trouble due to unequal distribution of the load will result. This does not hold if they are connected to the A.C. supply via transformers, because the drop in the transformers gives the necessary drooping voltage characteristic. Often, to save the cost of transformers, two rotary convertors are supplied direct from a low voltage alternator. In such a case, there are two methods of obtaining satisfactory parallel operation on the D.C. sides.

(*a*) The alternator can have two separate armature windings, one for each convertor.

(*b*) Balancing transformers can be used. The operation of these transformers will be understood by referring to Fig. 343, in which single phase convertors are represented for simplicity. A single core has two windings, connected in series with each convertor on its A.C. side, as shown. They are arranged so that they produce fluxes through the core in opposition, so that when the load is balanced, the core will carry zero flux. If convertor *A* carries a greater load than *B*, its transformer coil will produce a greater flux than that of *B*, so that the transformer will now carry a finite resultant flux. This flux induces in the balancing coil of

convertor B a current in the same direction as the ordinary load current, thus automatically equalizing the load.

These balancing transformers are quite small, and therefore cheap, since they have to deal, not with the total intake of the convertors, but only with temporary differences in load.

11. **The Motor Convertor.** This consists essentially of an induction motor and a D.C. generator mechanically coupled together. The rotor of the induction motor is wound for either 9 or 12 phases, according to the size, and tappings are taken to equidistant points on the D.C. armature. Three of the rotor phases are brought out to slip rings, as shown in Fig. 344, by means of which the set is started up exactly like an induction motor. When full speed is attained, the free ends of all the phases are short-circuited. Since the E.M.F. induced in the D.C. armature is injected into the rotor of the induction motor, the set will run at a speed less than the synchronous speed of the induction motor. The connections are so arranged that the direction of rotation of the rotating field set up by the alternating currents flowing through the D.C. armature, is opposite to the direction of motion. Hence, the speed of the set is such that this rotating field is stationary in space, because its poles are held by the salient poles of the D.C. generator field.

Let Ω = angular speed of the set in r.p.m.

Let ω_s = synchronous speed of induction motor, i.e. the speed of the rotating field produced by its stator.

Let ω = speed of induction motor rotor relative to this rotating field.

Then $\omega_s = \Omega + \omega$ . . . . (1)

Again let N_1 = No. of poles on induction motor,

N_2 = No. of poles on D.C. generator

Then, since the speed of the rotating field is inversely proportional to the number of poles, the speed of the rotating field of the D.C. armature *relative* to this armature is $\omega \times N_1/N_2$.

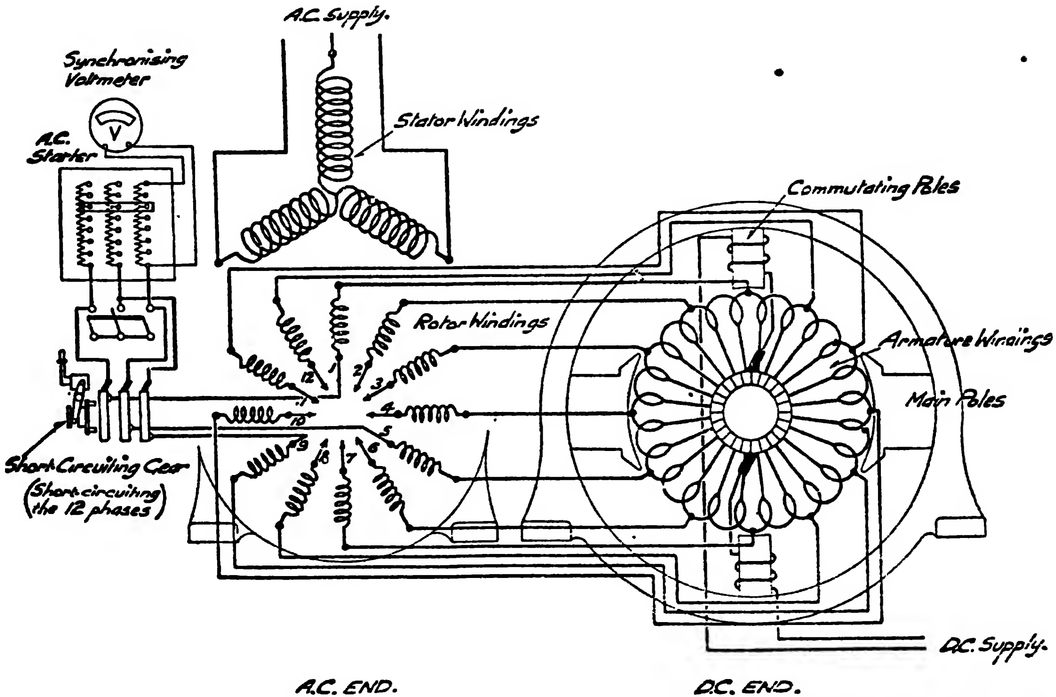
But this rotating field is stationary in space,

$$\therefore \quad \Omega - \omega \times \frac{N_1}{N_2} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\therefore \Omega = \omega \times \frac{N_1}{N_2} = (\omega_s - \Omega) \times \frac{N_1}{N_2} \text{ from equation (1)}$$

$$= \frac{N_1}{N_1 + N_2} \times \omega_s$$

Thus, if $N_1 = N_2$, then $\Omega = \frac{1}{2} \cdot \omega_s$; i.e. the set runs at one-half synchronous speed.



(By courtesy of Messrs. Bruce, Peebles and Co., Ltd.)

FIG. 344

MOTOR CONVERTOR CONNECTIONS

The torque on the induction motor rotor is equal to the retarding torque of the D.C. armature T, say. Power of the rotating field of the induction motor

$$= T \times \omega_s$$

Mechanical power conveyed along the shaft from the induction motor to the D.C. generator

$$= T \times \Omega$$

Hence, electrical power conveyed from the induction motor rotor to the D.C. armature

$$= T(\omega_s - \Omega)$$
$$= T \cdot \Omega, \text{ when } N_1 = N_2$$

Thus, when $N_1 = N_2$, the D.C. generator receives half of its power mechanically and half electrically.

The motor convertor compares very favourably with the rotary convertor, as is shown by the following points.

(*a*) The cascade convertor has exceptionally good commutating properties, because the frequency at the D.C. end is lower than the supply frequency.

(*b*) The cascade convertor has about the same efficiency as a rotary convertor, and is more efficient than a motor generator set. For an output of 500 kW, the full-load efficiency of a cascade convertor is about 92 per cent, the overall efficiency of a motor generator set, 89 per cent.

(*c*) No step-down transformer is required when working from a H.T. supply, unless this is greater than, say, 11,000 volts, because an induction motor stator can be wound for this voltage. This makes up for the greater floor space required by the two separate machines of the cascade set, a very important consideration in sub-stations.

(*d*) No synchronizing is required, the cascade convertor starting like an ordinary induction motor. These qualities are only obtained in the rotary at the expense of complications, such as starting motor and extra switch gear.

(*e*) The voltage regulation of the rotary convertor necessitates still further complication and expense, especially if a wide voltage range on a violently fluctuating load is required. The motor convertor can be arranged for a voltage variation as high as 40 per cent without additional apparatus.

(*f*) The power factor is better than that of a motor generator set with induction motor, as good as that of a motor generator set with synchronous motor, and may be better than that of a rotary convertor if the latter has to work over a wide range of voltage.

(*g*) The cascade convertor is not liable to reversal of polarity.

12. **The Mercury Arc Rectifier or Current Convertor.** The operation of this rectifier depends upon the "valve" action of an arc in

mercury vapour. The appearance of such an arc operating in a vacuum is shown in Fig. 345. The arc begins at the anode, which becomes heated up, but is not vaporized. Next comes a luminous column which, in the mercury-vapour lamp, is utilized as the source of light. Next, a dark gap, and finally a white-hot cathode spot. This spot is the basis of the arc, and it travels in an irregular manner at high speed over the surface of the cathode. With a mercury cathode a deep crater is formed under this spot, and also a pale negative flame rises from the cathode. In the rectifier this flame is an undesirable element. The mercury arc will only allow current to pass in one direction; hence, the valve action which is the fundamental principle of the rectifier. The theory of the arc is as follows: In gases, electricity travels in elementary charges called ions. The positive ions are bound to the chemical atom, or to matter, but the negative ions are chemically free. They are called electrons. Consider a point on the cathode brought to a white heat. At this temperature free electrons are liberated, and since there is a potential gradient maintained between the anode and cathode, they travel at a high speed towards the anode. They have frequent collisions with neutral mercury vapour molecules, and split up the bound charges of the latter, this process being called "ionization by shock." This results in the freeing of new positive ions and negative electrons, and also in raising the conductivity of the vapour path. The positive ions already present, and those newly created, move rapidly towards the cathode, the upper surface of which is brought to a white heat as a result of the continuous bombardment. The negative electrons also bombard the anode, but being unassociated with chemical atoms, they have no weighable mass and therefore do not raise the temperature of the anode very much.

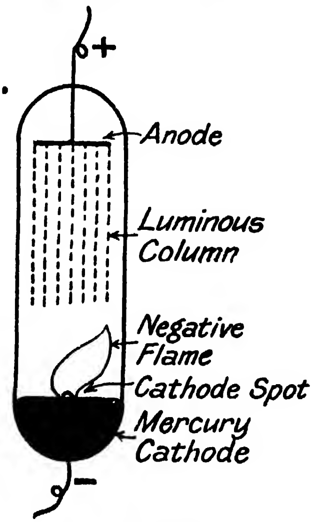


FIG. 345
THE MERCURY ARC

It will be seen that the arc will only form if some free electrons are present at first, and for this reason the arc has to be "ignited," like an ordinary arc between carbons, by bringing an ignition anode into contact with the mercury cathode and then withdrawing it.

There are certain losses in the arc, and they depend upon the pressure drop in it. This drop is made up of three separate drops,

that due to the anode, in the arc itself, and at the cathode. The drops at both electrodes are constant, being independent of the current density or the degree of vacuum. The drop at the iron anode is about 6·5 volts, and at the mercury cathode, about 5·5 volts, giving a total for the electrodes of 12 volts. The drop in the arc itself depends upon the degree of vacuum, to which it is directly proportional; and on the current density, to which it is inversely proportional. In large rectifiers it is about 0·1 volt per cm. Thus,

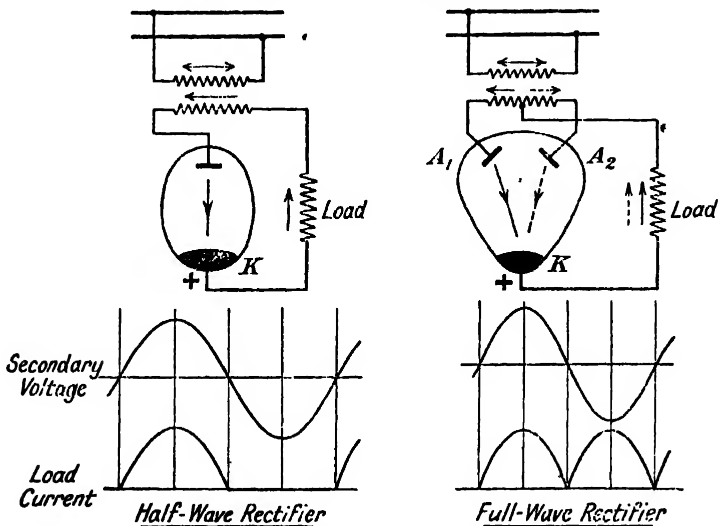


FIG. 346. HALF-WAVE AND FULL-WAVE SINGLE-PHASE RECTIFIERS

in an arc 80 cm. long, the drop in the arc itself is 8 volts, the total drop being therefore 20 volts. We thus see that since an increasing pressure does not increase the losses, it is preferable to work at as high a pressure as possible. Thus, at 1,500 volts the efficiency is 98·5 per cent.

The cathode drop is the result of the expenditure of energy in liberating electrons from the cathode and in evaporating mercury. The drop in the arc is the result of the energy expended in the ionization of the arc path, while the drop at the anode is the result of the energy expended in overcoming the electrostatic field in the vicinity of the anode.

The rectifier can be operated with any desired number of phases and Fig. 346 shows two methods of operation with single-phase supply. In the first figure there is only one anode, and, consequently, as current can only flow through the rectifier from anode to cathode, it follows that the external load will only carry current during those

half cycles which correspond to this direction of current flow. In other words, only alternate half waves are utilized, the rectifier therefore being called a "half-wave" rectifier.

In the second figure the load is taken to a centre tapping on the transformer secondary, with the result that each half wave is utilized, the current stream being from anode A_1 to cathode K when A_1 is negative with respect to the centre tapping, and from A_2 to K when A_2 is negative with respect to this tapping. In other

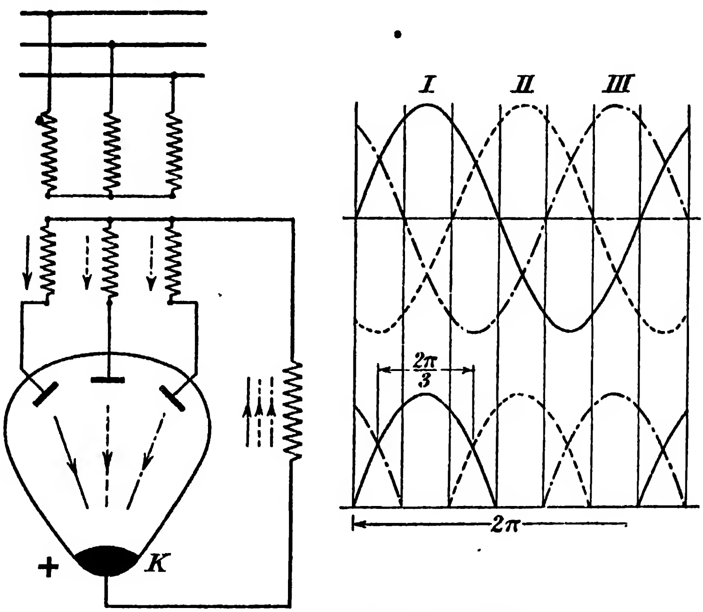


FIG. 347. THREE-PHASE RECTIFIER

words, the arc switches over from one anode to the other at the end of each half cycle. In this way both half waves are utilized and the rectifier becomes a "full-wave" rectifier.

In the three-phase rectifier there are three anodes, Fig. 347, and since none of the phases has a centre tapping, the rectifier can be regarded as three half-wave rectifiers combined, the three components of the total load current having the usual three-phase mutual displacement of $2\pi/3$ radians. Now the arc will only pass between an anode and the cathode when that particular anode is at a higher potential than the others: as soon as the next anode in the sequence attains a potential higher than the first, the arc is immediately transferred to this second anode. Now on examining the waves of secondary voltage, we see that intersections occur at intervals of

$2\pi/3$ radians, and this shows that each anode only carries current for $2\pi/3$, or one-third of a cycle, and not for one-half cycle as in the case of the single-phase rectifier.

If the anodes are arranged in a circle round the cathode, as is usual, then the rectifier can be regarded as a rotating switch which makes one complete revolution in each cycle and which maintains contact with each anode for one-third of a cycle per revolution. This is, in fact, the best way in which to regard the polyphase

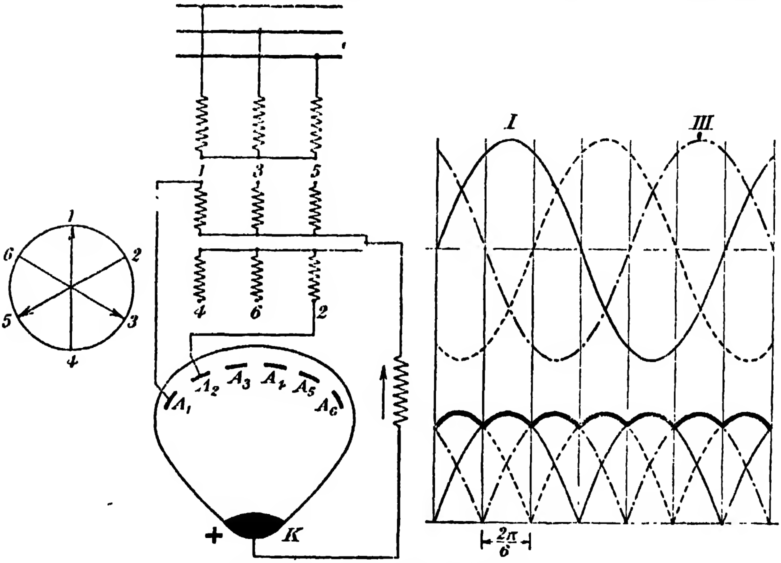


FIG. 348. SIX-PHASE RECTIFIER

rectifier, and it is important to note that the rectifier is not in any way the seat of the energy supplied to the load. This energy comes from the transformer secondary, and therefore originally from the A.C. supply. On comparing the curve of load current with either of those obtained from a single-phase rectifier, we see that an increase in the number of phases very appreciably diminishes the percentage variation.

Still further improvement is effected by a six-phase rectifier, as shown in Fig. 348. The supply to this rectifier is given as double star, which means that each phase is, in effect, centre tapped, and, as a result, we can regard the six-phase rectifier, supplied in this manner, as a combination of three full-wave rectifiers. To obtain the curve of load current, we therefore draw the waves for the three phases each with the negative half reversed. This gives intersections $2\pi/6$ apart, showing that the period of commutation, i.e. the time

during which the arc remains at any one anode, is now only one-sixth of the periodic time.

Now if the current ceases for an interval of time as short as 1×10^{-6} sec., it is sufficient to cool the cathode spot and thus stop the emission of electrons. Means must therefore be provided to prevent this occurring, the usual method being to have a separate "exciting" anode worked from a separate external source. Thus, besides the main anodes, there are two auxiliary anodes, the ignition anode and the excitation anode. Cooling of the cathode spot can also be prevented by placing a choker in the D.C. circuit. This has the effect of prolonging the half waves of current, with the result that the alternate half waves overlap, and the resulting direct current never falls to zero. This is shown in Fig. 349 for a single-phase full-wave rectifier.

Even without this choker the transfer of the electron stream from one anode to the next cannot be instantaneous because the current changes associated with the transfer, combined with the appreciable inductance of the circuits, result in an inductive drop which opposes the change. In addition, the ionized mercury vapour has a tendency to remain in the neighbourhood of an anode for an interval of time which, although very small, is comparable with the period of commutation. This effect sets an upper limit to the frequency at which this type of rectifier can be operated, this limit lying between 2,000 and 5,000 cycles per second.

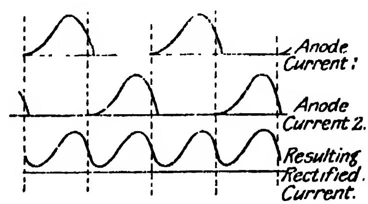


FIG. 349. WAVE FORMS OF RECTIFIED CURRENTS

The reasons for the choice of mercury for the cathode are as follows: during operation, the cathode is violently bombarded by massive positive ions with the result that, if solid, there is rapid deterioration. With a mercury pool as cathode there can be no such damage, and, when cool, all the condensed mercury vapour from the enclosure drains back into the pool. In addition, once the arc is started, the hot spot from which the electron stream is emitted is maintained at the temperature necessary for electron emission through the bombardment by the positive ions. Another important advantage of mercury is that, at a given operating temperature, a smaller voltage is required to release the electrons than for any other metal.

Large polyphase rectifiers are always worked with six or more phases, because this gives a smaller variation in the direct current than a three phase rectifier. The secondaries of the main transformer are arranged for diametral connection as with rotary convertors, and the middle point is connected to the negative pole on the D.C.

side. The connections of such a rectifier, including those to the ignition anodes, are shown in Fig. 350.* The ignition anode hangs down at the end of a steel wire about 10 mm. above the surface of the mercury cathode, but held up by a spring. It is pulled down by a solenoid actuated from an auxiliary D.C. supply when

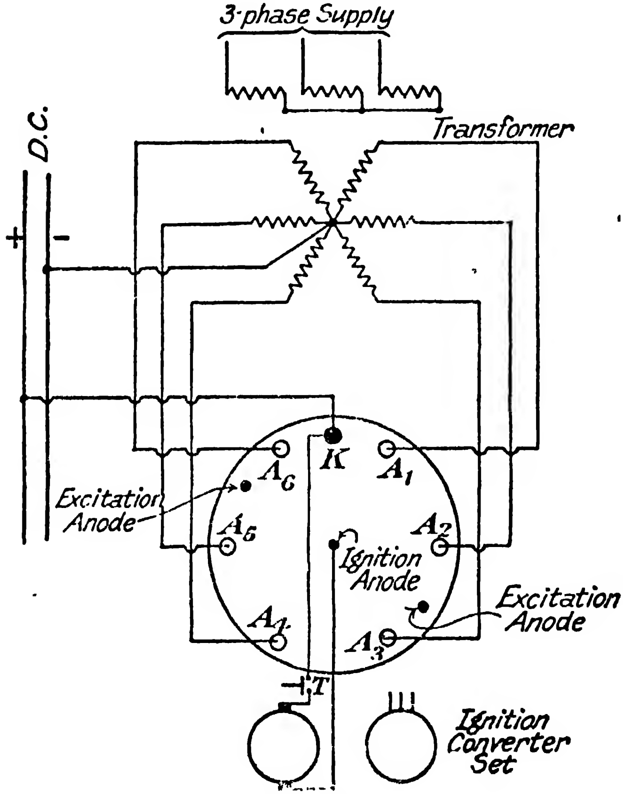


FIG. 350
POLYPHASE RECTIFIER

the push-button switch T is closed. It is then released, thus striking the arc and starting the rectifier.

In the polyphase rectifier, two excitation anodes are provided, these being operated by a small single-phase transformer in a manner similar to a single phase rectifier.

* Very complete information regarding systems of connection, as well as the general theory and mode of operation, is given in *Mercury Arc Current Convertors*, by H. Rissik. *Mercury Arcs*, by Teago and Gill, is also an admirable little book.

If a number of rectifiers have to work in parallel, it is desirable that they should have a drooping voltage characteristic. The drop in the rectifier is practically fixed, as we have seen, but the necessary droop in the characteristic can be obtained by connecting choking coils in the leads to the anodes.

The small rectifiers of up to about 500 kW capacity, as invented by Cooper-Hewitt, are made of glass; but for large power rectifiers this is obviously impracticable, a steel receptacle being used. Since the inlet glands for the electrodes are of large size, considerable difficulties were encountered in trying to make these tight, since the vacuum must be maintained at about 0·01 mm. of mercury. The final arrangement is a combination of asbestos packing with mercury sealing. In order to maintain the vacuum, the rectifier is supplied with a direct-coupled motor-driven air pump. A second constructional difficulty was that of cooling the anodes. If the anodes reach too high a temperature due, say, to continuous overloading, they will send off electrons just as the cathode does. As a result, the current will flow in both directions, thus destroying the valve action and constituting a short circuit. This is avoided by water-cooling the anodes. The greatest difficulty has been the elimination of the possibility of short circuits, the main causes of which are—

(*a*) Insufficient vacuum. This is looked after by the air pump.

(*b*) Continuous overloading; eliminated by water-cooling the anodes.

(*c*) Occluded gases in the anodes due to impure iron; eliminated by the use of chemically pure electrolytic iron for the anodes.

(*d*) Condensed mercury collecting on the anode surface. The cover is provided with a tall steel cylinder, and all the anodes, with the exception of the ignition anode, are arranged outside this.

(*e*) Effects of ultra-violet light from the luminous column of the arc.

(*f*) Contact between the anodes and the negative flame. The anodes are shielded from effects (*e*) and (*f*) by surrounding them by steel cylinders, the mouths of which are closed by a number of inclined steel slats.

Internal short-circuiting can be caused by back-firing or cross-firing which is the result of a failure of the rectifying action. From what has been said, it will be clear that this action can only take place if the arc continues to rotate round the ring of anodes, and to remain with each anode only for the correct length of time. If any of the anodes become overheated they may emit electrons themselves and thus try to act as cathodes. Irregular firing commonly takes place when starting up from cold, and this is due to small globules of mercury which have condensed on the anodes. These are converted to vapour when the anodes are warmed up so that the trouble is only temporary. The anodes are usually of iron, or

sometimes carbon, since neither substance is wetted by mercury, and, in addition, the number of electrons emitted, even when they are raised to high temperatures, is very small in comparison with the number emitted by mercury.

Back-firing and cross-firing are hindered by shielding the anodes

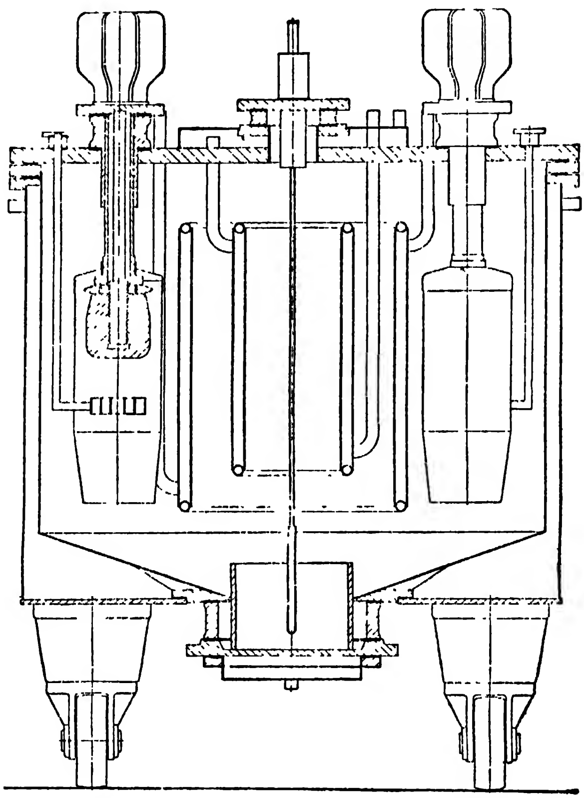


FIG. 351. 2,500-A. GRID-CONTROLLED STEEL-TANK RECTIFIER
English Electric Co., Ltd.

from one another, and from the cathode. With the glass bulb type this shielding is effected by housing the anodes at the ends of long extensions to the bulb, which are given one, or sometimes two right-angle bends. With the steel container type the anodes and the mercury pool are fitted with insulating shields.

The construction of a modern rectifier of large capacity is clearly indicated by the drawing of Fig. 351.

The efficiency of the rectifier is about 95 per cent, the maximum

value being at three-quarter load, and only falling off appreciably below one-quarter load. It is thus superior to any running machinery, even the rotary convertor. The fact that it is a static piece of apparatus very considerably lessens running costs and attendance charges. Also, it is started very easily, since it is only necessary to close momentarily the ignition circuit. With new rectifiers, it is usual to load up for the first time on an auxiliary load of 50 to 100 amp. in order to drive out all gases; also, when starting up after a shut-down, this precaution is observed for the first few months. If a rectifier has been stopped for some time, the anodes are brought into circuit separately, so as to warm them up, this precaution being due to the fact that the arc may otherwise start unevenly at the anodes and cause short-circuiting, as explained previously. Thus, with new rectifiers starting takes from 5 to 10 min.; but after a month or so when the above precautions are no longer necessary, it takes but a fraction of a second, since the ignition push-button is closed and almost immediately released.

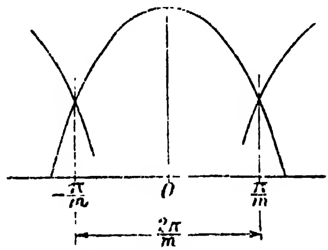


FIG. 352. TO ILLUSTRATE VOLTAGE RATIO

13. **Voltage and Current Relationships.** Let E_p = R.M.S. secondary voltage per phase. Then, with the origin at o, Fig. 352, we have

$$e = E_{max} \cos \theta$$
$$= \sqrt{2}\, E_p \cos \theta$$

$\therefore$ for the mean D.C. voltage, we have, for an m-phase rectifier,

$$E_{DO} = \sqrt{2}\, E_p \times \frac{m}{2\pi} \int_{-\frac{\pi}{m}}^{\frac{\pi}{m}} \cos \theta \,.\, d\theta$$

$$= \sqrt{2}\, E_p \times \frac{m}{2\pi} \times 2 \sin \frac{\pi}{m}$$

$$= \sqrt{2}\, E_p \,.\, \frac{m}{\pi} \,.\, \sin \frac{\pi}{m}$$

Actually the terminal voltage will be the above, less the sum of the drops in the rectifier and in the transformer.

Example. The full load terminal voltage on the output side of a six-phase rectifier is to be 1,500, and the sum of all the drops is 25 volts. Calculate the transformer secondary phase voltage.

$$V_{DC} = \sqrt{2}\, E_p \,.\, \frac{m}{\pi} \,.\, \sin\frac{\pi}{m} - \text{(sum of drops)}$$

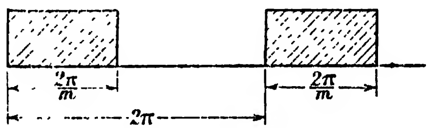
$$\therefore\ 1{,}500 = \sqrt{2}\, E_p \,.\, \frac{6}{\pi} \,.\, \sin 30 - 25$$

$$\sqrt{2}\, E_p \,.\, \frac{3}{\pi} = 1{,}525$$

$$E_p = \frac{1{,}525 \times \pi}{3\sqrt{2}}$$

$$= 1{,}140 \text{ volts.}$$

In calculating the current relationship, we have to note that the rectified current is supplied in turn by each secondary phase of the



transformer, and that this current persists in each phase only for $2\pi/m$ of a cycle per cycle. The secondary current is thus in the nature of a series of intermittent impulses, as shown in Fig. 353.

The mean current per phase is thus $I_{DC}\Big/m$, while the R.M.S. current is $I_{DC}\Big/\sqrt{m}$. Thus with a six-phase rectifier

$$I_{mean} = \frac{I_{DC}}{6}, \quad \text{while } I_{RMS} = \frac{I_{DC}}{2{\cdot}45}$$

14. **Grid Control of Mercury Arc Rectifiers.** Imagine that a perforated grid is interposed between cathode and anode in such a way that the electrons can only reach the anode via the grid. Then it is quite clear that the motion of the electrons will now depend on the potential of the grid relative to that of the cathode. If the grid is positive, the electrons will be accelerated in their path to the anode, as in Fig. 354 (*a*), whereas if the grid is negative, the electrons will be decelerated. Provided this negative grid potential is large enough, the electrons will be turned back by the repulsion of the grid and will not be able to reach the anode at all, as in Fig. 354 (*b*). In such a case, the flow of current will cease. Rissik defines the

function of grid control as follows:* "The function of any system of grid control is to permit the establishment of a current arc at the anodes of a current convertor at a predetermined instant in the anode voltage-cycle by the application of a positive potential to the individual control grids, and to prevent the re-establishing of the arc at some other instant by the subsequent application of a negative grid bias. The former potential is usually referred to as the *liberating* voltage, whilst the latter is generally known as the blocking potential." The liberating voltage varies between the limits 25 and 150, and the blocking potential between 100 and 300 volts.

It is clear that the grid control voltage will have to be synchronous with the voltage supply to the anodes, and therefore the most

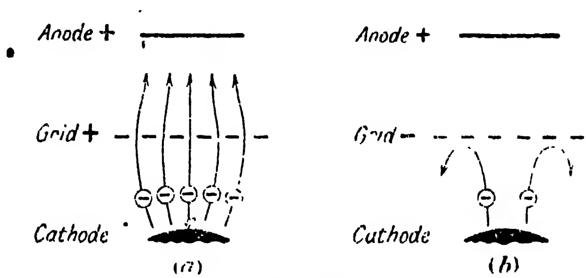


FIG. 354

obvious method of obtaining the control voltage is to use an alternating voltage of suitable value derived from the supply. The control is carried out by varying the phase of the grid voltage. Consider the first figure of Fig. 355, which is taken from *Mercury Arc Current Convertors*, by H. Rissik. The grid voltage v_g is in phase with the anode voltage. The horizontal ignition line of ordinate g represents the necessary voltage to be applied to the anode before ignition can take place, and therefore the point of intersection of this line with the grid voltage wave gives the moment at which the arc is struck. Now, once the arc is started, the grid has no further influence on the performance, the arc continuing at the anode as in the case of a rectifier without grids. In the figure, this is until the anode voltage falls to zero, and the anode delivers current for practically half a cycle. The angle a, by which ignition is delayed from the zero of the anode voltage, is called the ignition angle. In the second figure, the ignition angle a_2 is considerably greater than a_1, with the result that the intersection of v_g with the ignition line occurs considerably later in the cycle, and the time of delivery of current by the anode is considerably reduced. In the third figure, the angle a_3 is still greater, and we see that when v_g is nearly in phase opposition to e the time of delivery of current by the anode is very small

* *Loc. cit.*, Chap. IX.

indeed. We have seen that the average direct voltage is the average of the anode voltage curve reckoned over the period of ignition, and we therefore see that grid control gives a means of varying the output voltage. Furthermore, since the grid voltage is inoperative once the arc has been started, this voltage need not be sinusoidal

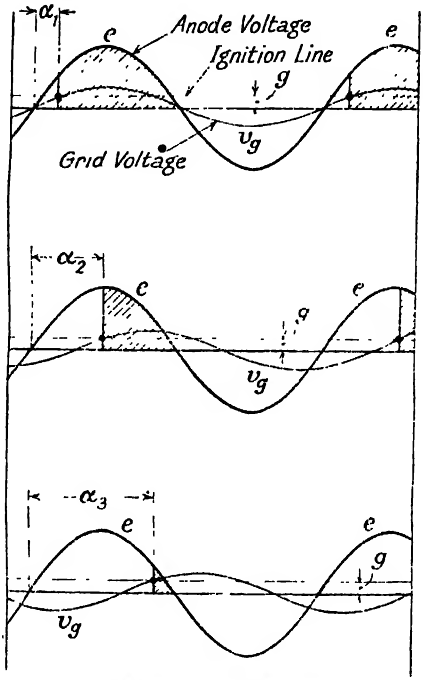


Fig. 355. Phase-shift Control

but can be in the form of almost instantaneous impulses, which can be obtained from a synchronously driven alternator which gives a very peaky wave form.

The application of this method of voltage control in the case of a six-phase rectifier is illustrated by the figures in Fig. 356. In the first figure, the grid impulses synchronize with the intersections of the various anode voltage curves so that the grids try to start the arc at the moment it would have started without their aid. The grids thus have no effect and the output voltage is that of a six-phase rectifier, without grids: this voltage is taken to be 100 per cent. In the second figure, the instant of application of the grid impulses has been advanced an angle $a = 30°$ with the result that the arc starts suddenly at an anode when the impulse is applied, and persists until the impulse has been applied to the next anode

in the sequence. With $\alpha = 30°$ the average output voltage, as represented by the average under the shaded area, is reduced to 86·6 per cent. With $\alpha = 60°$ this average is reduced to 50 per cent,

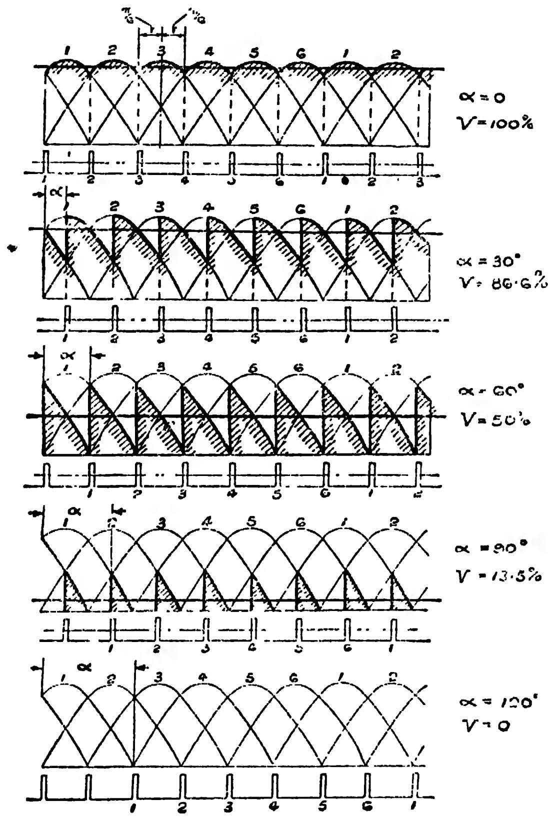


FIG. 356. VARIATION OF THE OUTPUT VOLTAGE WITH THE PHASE-ANGLE OF ARC COMMUTATION
(*English Electric Co.*)

and so on, until when $\alpha = 120°$ the output voltage is reduced to zero. The above curves of output voltage variation are theoretical curves corresponding to a pure resistance load, and, in practice, the irregularities are somewhat smoothed out by the inevitable inductance of the load circuit. If smoothing equipment is used, the output voltage can be made almost uniform.

The connection scheme for such a method of voltage control is illustrated diagrammatically in Fig. 357 in which the liberating and blocking voltages are shown as derived from batteries, and the grid

impulses applied via a synchronously driven distributor. In such a scheme the instant of ignition is controlled by varying the angular position of the distributor. There are other systems of voltage control, but standard works must be consulted for these.

The relationship between the mean output voltage with retarded ignition to the output voltage without any control of the ignition,

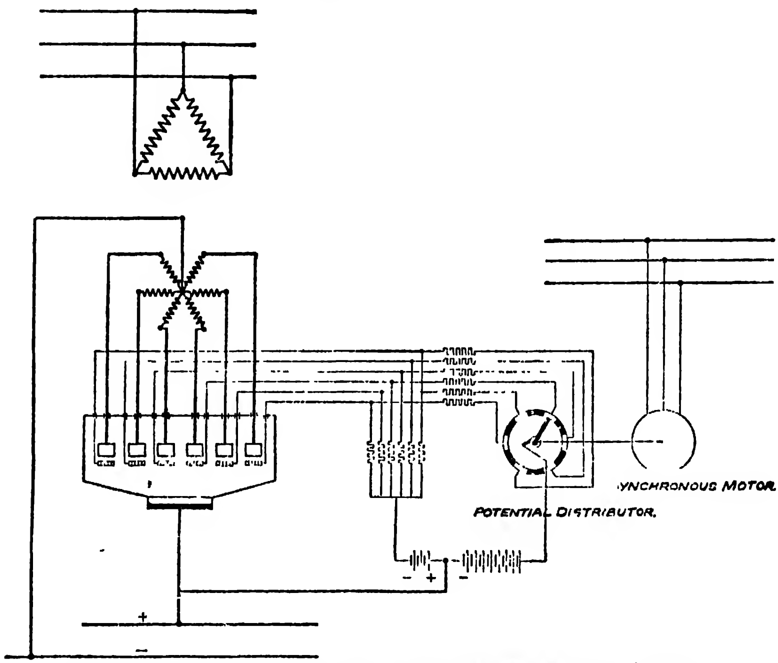


FIG. 357. DIRECT-CURRENT IMPULSE SYSTEM OF GRID CONTROL
(*English Electric Co.*)

is very easily deduced as follows, the assumption being that the voltage waves are not distorted. Fig. 358 shows two consecutive intersections of the anode voltage waves. For uncontrolled ignition the period of ignition will be that corresponding to the angular range P to Q, and, as before, we have

$$E_{DC} = \sqrt{2}E_p \times \frac{m}{2\pi} \int_{-\frac{\pi}{m}}^{\frac{\pi}{m}} \cos\theta \, . \, d\theta$$

$$= \sqrt{2}E_p \cdot \frac{m}{\pi} \cdot \sin\frac{\pi}{m}$$

If the ignition is retarded an angle a by means of grid control, then the arc commences at the instant corresponding to the point R and changes over to the next anode in the sequence, at the instant

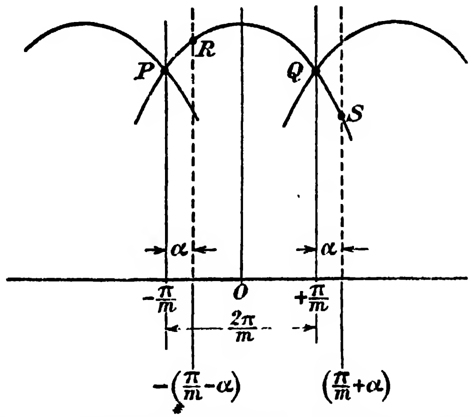


FIG. 358. TO ILLUSTRATE THE EFFECT OF THE ANGLE OF GRID EXCITATION

corresponding to point S. The total period is the same as before, viz. $2\pi/m$, but the limits are now $-\left(\frac{\pi}{m}-a\right)$ to $+\left(\frac{\pi}{m}+a\right)$

$$\therefore\ E'_{DC} = \sqrt{2}E_p \times \frac{m}{2\pi}\int_{-\left(\frac{\pi}{m}-a\right)}^{+\left(\frac{\pi}{m}+a\right)} \cos\theta\,.\,d\theta$$

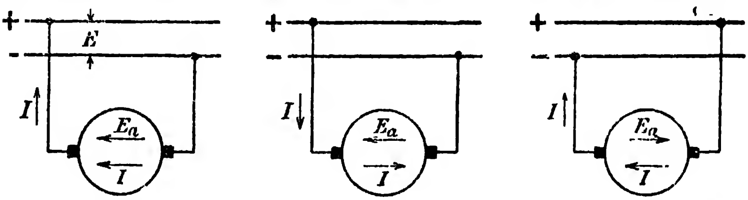
$$= \sqrt{2}E_p\,.\,\frac{m}{\pi}\,.\,\sin\frac{\pi}{m}\,.\,\cos a$$

$$\therefore\ \frac{E'_{DC}}{E_{DC}} = \cos a$$

In other words, the ratio of the reduced output voltage, obtained by grid control, to the output voltage given with no such control, is equal to the cosine of the angle of ignition retard.

15. **Inverted Operation.** The rotary convertor can operate either direct or inverted, but the rectifier, without grids, can only operate direct, because it is not possible to reverse the direction of current flow. With grid control, however, inverted operation is possible in spite of the fact that reversal of current in the arc is not possible. To understand this, consider, first, the case of a direct-current shunt machine which, as we know, will either run as a generator or a

motor. Fig. 359 shows three possible modes of operation. In (*a*) the armature voltage E_a is greater than the bus-bar voltage E, and the machine generates. In (*b*), E_a is less than E, and the machine motors, but neither voltage is reversed in sign and consequently the current has to reverse. In (*c*), E_a is again less than E, but the connections to the bus-bars have been reversed, and E_a has been reversed in direction, with the result that motoring action has been secured without the reversal of current flow. The inverted operation of a mercury arc rectifier is analogous to the above; the current flow cannot be reversed, and consequently the voltage must be reversed, and at the same time must be reduced. Now without grids only the positive portions of the anode voltage curves are utilized so that reversal is impossible, but with grid control the



(*a*) $E_a > E$, Generating. (*b*) $E_a < E$, Motoring with I reversed (*c*) $E_a < E$, and reversed Motoring with I in the original direction.

FIG. 359

impulses can be timed so as to utilize the negative portions of the voltage waves and also to give this reversed voltage the required reduced value. For inverted operation the leads to the load must also be changed over as in the case of the direct current machine of Fig. 359. Fig. 360 shows, in a diagrammatic way, the difference between normal and inverted running of the rectifier. The external load is shown as a direct-current machine which in the first case runs as a motor, and when the rectifier is functioning inverted, runs as a generator.

There are several practical applications of the mercury arc invertor, one of these being Ward-Leonard control of a direct current motor with two rectifiers taking the place of the motor-generator set, as shown diagrammatically in Fig. 361. The control of the speed is effected by grid control of the output voltage of a rectifier in place of excitation control of the generator voltage in the ordinary system. One of the rectifiers is used for forward running, while the other is used when the direction of rotation of the motor has to be reversed. For simplicity, a single-phase supply is shown in the diagram, the dotted connections referring to the rectifier used for reverse running. So long as the driven machine is operating only as a motor, then neither rectifier will be used as an invertor,

but if regenerative control is required, motor action will change over to generator action and one or other of the rectifiers will have to operate inverted, according to the direction of rotation.

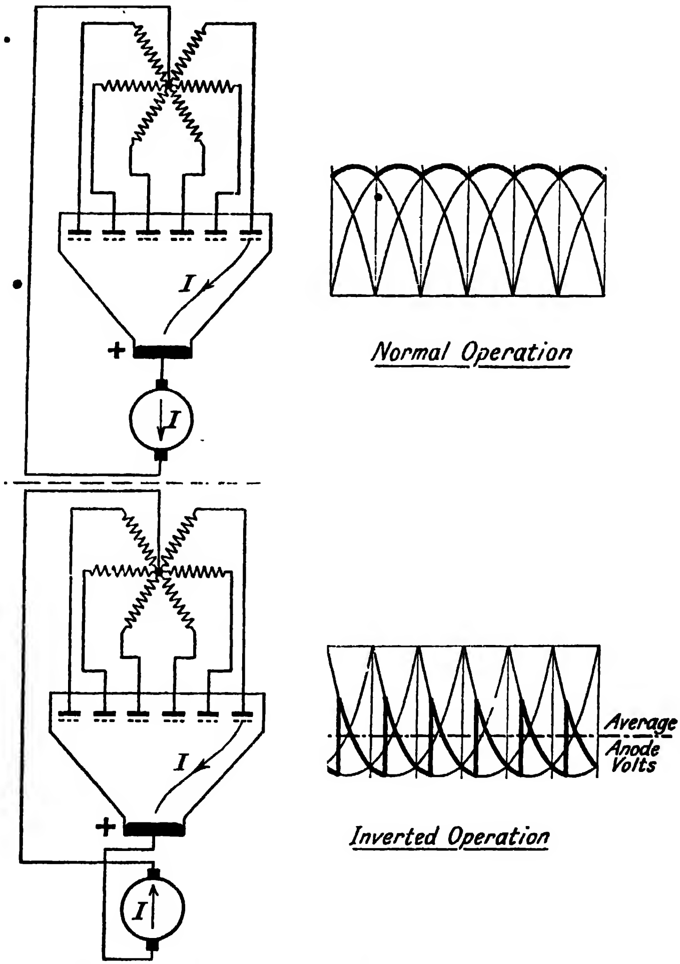


FIG. 360. INVERTED OPERATION OF MERCURY-ARC CONVERTOR

16. **Hot Cathode Rectifiers.*** The mercury arc rectifier previously described is a cold cathode rectifier, in that the heating of the cathode necessary for electron emission is localized at the small

* See Henney, *Electron Tubes in Industry*, and Guilliksen and Vedder, *Industrial Electronics*.

cathode spot and is derived from the bombardment of the cathode by the positive ions. In the hot cathode rectifiers, the cathode is

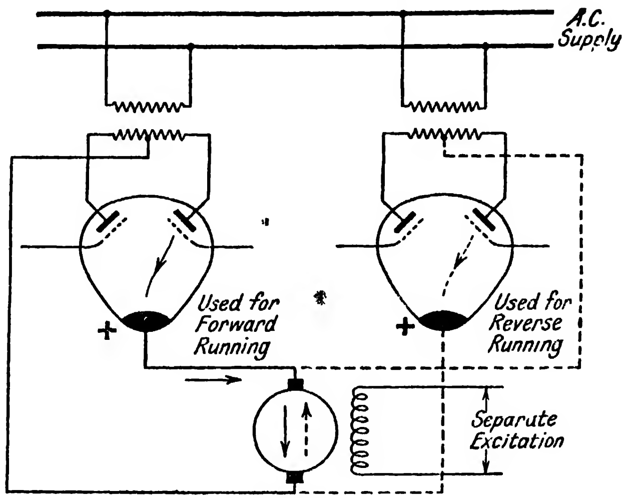


FIG. 361. THE APPLICATION OF MERCURY-ARC RECTIFIERS TO D.C. MOTOR CONTROL

heated as a whole from an external source. The enclosure is highly evacuated and a very small amount of mercury is enclosed, but this

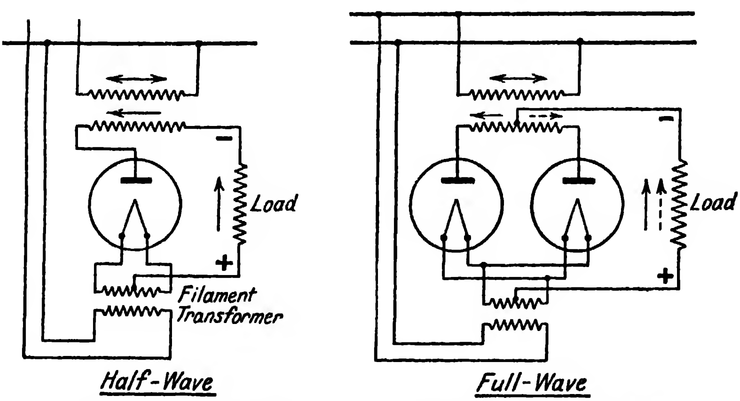


FIG. 362. HALF-WAVE AND FULL-WAVE HOT CATHODE RECTIFIERS

mercury does not form the cathode. Its function is to supply gas molecules to be converted into positively charged ions by collision, so that, for a given anode voltage, the current output is increased.

The operation is fundamentally the same as that of the mercury arc rectifier, the current passing through the tube as an arc, but only with a possibility of one direction of flow, viz. from anode to cathode. With a single tube, half-wave rectification is obtained, while with two tubes and a centre-tapped transformer, the rectification is full wave. Comparison of the connection schemes of Fig. 362 with those of Fig. 346 will show that the hot cathode rectifier and mercury arc rectifier are essentially similar, the chief difference being

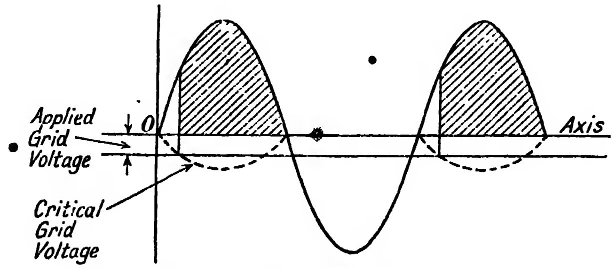


FIG. 363. PRINCIPLE OF THE THYRATRON

that the cathode of the former is a filament which is heated by the passage of current derived from a small filament transformer.

This type of rectifier has several advantages as compared with the mercury arc rectifier previously described, viz.

1. Smaller size for a given capacity.
2. Smaller arc length, leading to a lower arc drop and therefore a higher efficiency.
3. Lower drop at the cathode, this again increasing the efficiency.
4. Freedom from back-firing.

The cathode is usually a barium-coated filament, since this material is very efficient from the point of view of electron emission when its temperature is raised.

The tungar rectifier is similar to the above, except that it has a tungsten filament and the tube contains an atmosphere of argon, there being no mercury present. It is suited to low voltages and comparatively heavy currents, and is therefore commonly used for charging batteries and similar small power requirements. The kenotron is a highly evacuated rectifier with no heavy gas molecules, such as those of mercury vapour, present. It is therefore only suitable for very small currents, its application being to high-voltage rectification.

If the hot cathode rectifier is provided with a control grid, then the rectifier becomes a gas-filled relay or thyratron. Just as the two-element hot cathode rectifier is similar in operation to the mercury arc rectifier, so is the thyratron similar in operation to the mercury arc rectifier with grid control. Thus if an alternating

potential of supply frequency is applied to the grid, the output will be dependent on the phase of this grid potential. The grid voltage necessary to control the anode current is called the *critical grid voltage*: it is not a constant, but varies with the anode voltage. Hence, if the anode voltage is alternating, the critical grid voltage will also be cyclic (although not sinusoidal), as shown in Fig. 363.

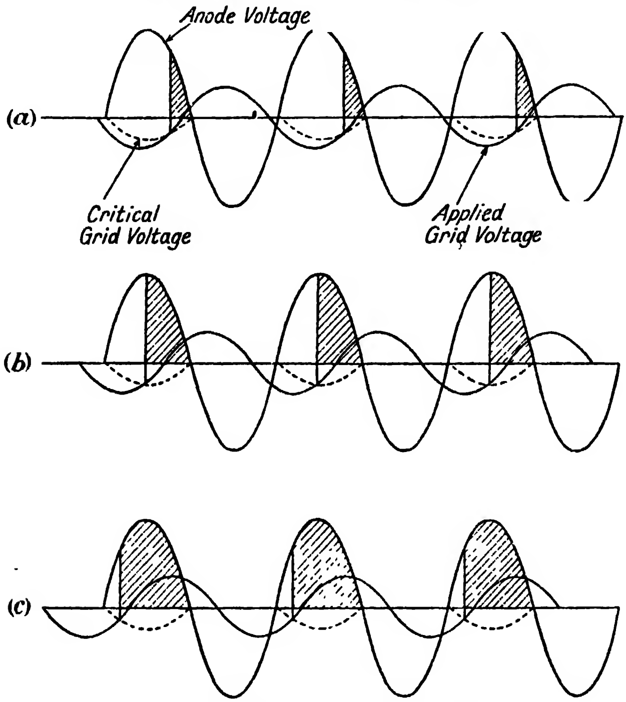


FIG. 364. PRINCIPLE OF THE THYRATRON

If the grid is given a steady bias, as represented by the horizontal line below the axis, the thyratron will start to pass current at the intersection of this line with the critical grid voltage curve. By altering the grid bias, the grid line can be shifted either up or down, and can be made to coincide with the axis. There is no current flow during the negative half cycle and the range of control by this method is from the utilization of the whole of the positive half cycle to the utilization of half of the positive half cycle. Thus the average current can only be varied from full to half value by this method.

By applying an alternating grid potential, and making use of phase control, then the average current can be varied from full value right down to zero, as illustrated by the curves of Fig. 364.

The usual circuit for illustrating phase control of the grid voltage is shown in Fig. 365, in which it will be seen that the grid is connected to the junction of a condenser C and a non-inductive resistance R

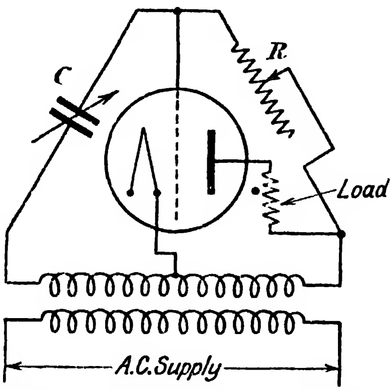


FIG. 365. PRINCIPLE OF THE THYRATRON

in series. By varying C and R, the phase of the grid voltage can be varied, and therefore the output of the thyratron controlled.

There are many industrial appliances of thyratrons, one of these being illustrated in Fig. 366. The thyratron is here operating as a

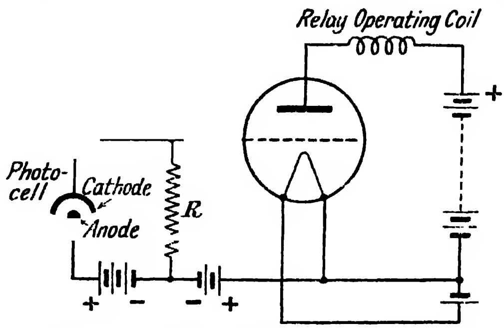


FIG. 366. AN APPLICATION OF THE THYRATRON

relay in conjunction with a photo-electric cell. When light is thrown on the cell, current flows through the cell and a potential difference is established across the ends of the high resistance R. This brings the grid of the thyratron within the ignition range and consequently an arc is struck and the thyratron anode current flows through the operating coil of the relay. In some cases the circuit is arranged so that the relay is operated when the light supply to the photo-cell fails. Typical examples of this application are the automatic lighting of street lamps with failing daylight, and the operation of

burglar alarms. In this second application the light beam consists of ultra-violet rays only and is therefore invisible.

17. **Solid Contact Rectifiers.*** The most important of the solid contact rectifiers is the copper disc rectifier, the action of which is based on the fact that a copper disc covered with a layer of cuprous oxide has a greater resistance to the current flow in one direction than in the reverse direction. Copper oxide rectifiers for small outputs consist of copper discs having on one side a layer of cuprous oxide obtained by the oxidation of pure copper blanks. The oxide layer has a thin annular coating of graphite, and contact with this is made by means of a lead washer: The requisite number of discs is mounted on an insulated steel rod with coppered steel space and

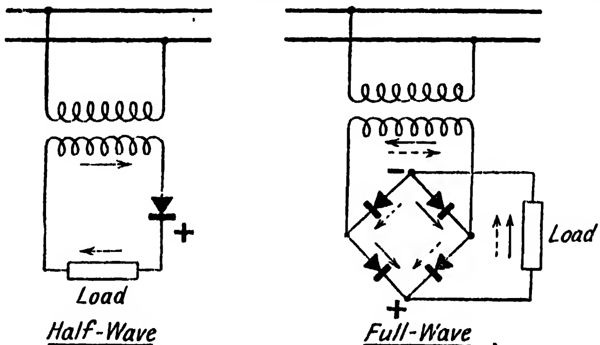


FIG. 367. HALF-WAVE AND FULL-WAVE RECTIFIERS

tinned cooling fins. The oxidation is carried out by heating in air for several minutes at about 1,020° C., the melting point being 1,040° C. The rectifying action is very poor, however, unless a special subsequent heat treatment is given, this consisting of annealing at about 600° C., followed by rapid quenching. The surface layer of black cupric oxide, CuO, formed during cooling is removed by cyanide solution.

The forward average current rating for three-quarter inch diameter discs is about one-third of an ampere, and the maximum safe reverse voltage about 9 volts. For large powers, of the order of several kilowatts, the large-plate type of element is used, consisting of a long oxidized copper strip about 2 in. wide. Series-parallel combinations involving standard elements can be arranged to meet any voltage and current requirements.

Another important dry-plate metal rectifier is the selenium type developed in Germany. With this rectifier, discs of steel are used

* It is not possible to describe the many forms of rectifier which have been devised for different purposes, e.g. the synchronously driven commutator, vibrating reed, electrolytic rectifier, and so on; very full particulars will be found in *Alternating Current Rectification*, by Jolley.

with a specially prepared surface on which is deposited a thin layer of selenium, a hard grey-coloured element. Contact is made by a ring of special alloy apparently deposited by spluttering. Selenium rectifiers are very efficient, and for the same output are more compact than the copper oxide type.

As with other types of rectifier, the copper oxide rectifier can be used either for half-wave or for full-wave operation, the latter requiring a bridge arrangement of the elements, as shown in Fig. 367. Alternatively, full-wave rectification can be secured by means of two rectifiers and a centre-tapped transformer exactly as with the other types of rectifier. Owing to its simplicity, and the fact

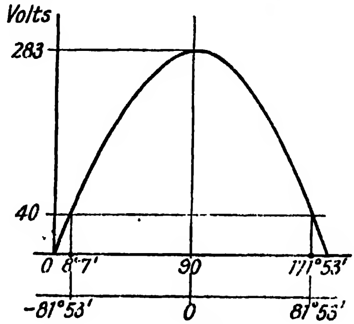


FIG. 368

that auxiliary batteries or filament supplies are not required, this type of rectifier is now very widely used. In very small sizes it is used in the mains units of wireless receiving sets (and in measuring instruments), while in larger sizes with large current output it is commonly used in the place of batteries if direct current is required where only an alternating-current supply is available.

Example. A half-wave rectifier is used to charge a 40-volt battery from an alternating-current supply of R.M.S. voltage 200. The resistance of the rectifier to current in one direction is 200 ohms, and in the other direction very high. If the battery resistance is negligible, calculate the ampere-hours put into the battery in 24 hours.

The battery E.M.F. of 40 volts is opposed to the supply voltage during the charging half cycle, so that for the resultant voltage we have

$$e = E_{max} \sin \theta - 40$$

But charging cannot commence until the supply voltage has risen to 40, and charging ceases just before the end of the half cycle at the instant the supply voltage has fallen to 40. The corresponding angles are

$$\text{arc sin} \frac{40}{E_{max}} = \text{arc sin} \frac{40}{283} = 8° 7'$$

and $\qquad 180 - 8° 7' = 171° 53'$, Fig. 368.

Taking the origin at the instant the applied voltage is a maximum, we have for the average charging E.M.F. over one half period

$$\frac{1}{\pi}\int_{-81°53'}^{+81°53'} 283 \cos\theta \, d\theta - 40$$

$$= \frac{2}{\pi} \times 283 \sin 81° \, 53' - 40$$

$$= \frac{2}{\pi} \times 283 \times \cdot 99 - 40$$

$$= 179 - 40$$
$$= 139 \text{ volts}$$

$\therefore$ Average charging current per half cycle

$$= \frac{139}{200} = \cdot 695 \text{ amp., or } \cdot 35 \text{ amp. per cycle.}$$

Hence ampere-hours per 24 hours

$$= \cdot 35 \times 24 = 8 \cdot 4.$$

EXAMPLES ON CHAPTER XXI

(1) A six-phase rotary convertor is supplied from a transformer with the secondaries connected in double delta. If the convertor delivers 500 amp. to the D.C. load, what current will flow in the secondary winding of each transformer ?

Ans.—136 amp.

(2) A rotary convertor with six slip rings is fed from the secondaries of a three phase transformer with star-connected primaries. Each primary coil has ten times as many turns as the secondary. A load of 200 amp. at 500 volts is taken from the continuous-current side. Draw carefully a diagram of the connections, and also, a vector diagram showing the magnitude and phase relationship of the voltages of the line, of the transformer coils, and of the slip rings. Calculate the approximate voltages on the mains and the currents in the primary coils (assume the efficiency to be 100 per cent, and the power factor, unity). (London Univ., 1914.)

Ans.—Assuming diametrical secondary connections, line voltage 6,100, current in primary coils 9·5 amp., voltage at slip rings 353.

(3) A six-phase rotary convertor is supplying 100 amp. to the D.C. side. Draw graphs of the current in the armature (*a*) for a conductor mid-way between two tapping points, (*b*) for a conductor at a tapping point.

(4) Explain the action of the rotary convertor, and explain why it is more economical to use a rotary convertor with a large number of phases. Show that in a six-phase rotary convertor working at unity power factor, the heating of a coil close to the tapping point is about twice as great as that of a coil midway between two tapping points. (London Univ., 1922.)

(5) Explain the switchboard connections that must be made in a sub-station containing self-synchronizing rotary convertors, and compare the

advantages of this type of convertor with that in which the rotary is started up without this special device. Give a diagram of connections of the self-synchronizing rotary convertor, and explain the method of starting up. (C. and G. Distribution.)

(6) Compare critically the following methods of starting rotary convertors: (*a*) starting electrically from the continuous current side; (*b*) starting electrically from the alternating current side; (*c*) starting mechanically by a small motor. Consider the matter under the following headings: (*d*) ease of operation; (*e*) time taken to come to synchronism; (*f*) reliability in times of stress when the generating voltage is steady. (C. and G., 1914.)

(7) Examine critically the suitability of the various methods of adjusting the voltage on the D.C. side of a rotary convertor when the range of voltage is a wide one, say, from 440 to 550 volts.

(8) Explain why, in a rotary convertor, the heating of the armature for a given load is reduced as the number of phases on the A.C. side of the machine is increased. Show generally how the distribution of temperature in the armature changes as the excitation of the rotary convertor is changed. (C. and G., 1918.)

(9) The speed of a rotary convertor, when used to convert direct current power into alternating current power, varies with the power factor of the load on the alternating current side of the machine. Explain this result, and describe an arrangement which may be used to reduce the speed variation. (London Univ., 1924.)

CHAPTER XXII

COMMUTATOR MOTORS

I. The Single-Phase Series Motor

1. If an alternating current is sent through an ordinary D.C. series motor, the field and armature currents reverse simultaneously, so that a unidirectional torque is produced and the armature rotates. The motor has to be modified very considerably for work with alternating currents. Thus, all the iron has to be thoroughly laminated both in armature and field.*

There are two E.M.F.s induced in an element of the armature winding. First, there is the dynamically induced E.M.F. caused by the rotation of the armature conductors in the magnetic field. Second, there is the statically induced E.M.F. produced by the alternations in the flux, independently of rotation. Whatever the brush position may be, there is always an E.M.F. induced in the short-circuited winding element. Thus, when the brushes are in the normal position, the short-circuited element has a statically induced E.M.F.; and when the brushes are perpendicular to the normal position, it has the dynamically induced E.M.F. Hence, the motor shows a much greater tendency to sparking than a D.C. motor, and the elimination of sparking has been one of the most difficult problems in the evolution of the motor. Both excessive sparking and poor power factor are prevented by "compensating" the motor, the aim of which is to neutralize the effect of self-induction. This can only be done for the armature, since to neutralize the self-induction of the field means that the field flux must be neutralized, which is obviously impossible. The armature can be compensated by employing coils carrying the armature current, and placed in such a way as to neutralize the armature flux. The Fig. 369 A-B-C shows how this can be done. In 369(A), the field flux is produced along the axis AB, and the armature produces its flux at right angles to this, namely, along the brush axis CD. Thus the compensating coil must produce its flux along CD and in a direction opposite to that of the armature flux at every instant. In the figure the field and compensating windings are shown as solenoids; in the actual motor they are distributed windings.

Since two fields at right angles produce a single resultant field, it is possible to combine the main field and compensating windings into a single winding producing its flux in the direction of this resultant, as shown in Fig. 369(B).

The compensating coil, instead of being connected in series with the main field coil, can be short-circuited as in Fig. 369(C). It then acts like the short-circuited secondary of a transformer, of which the primary is the armature winding. Since the fluxes produced

* For full particulars see Dover's *Electric Traction*.

by the primary and secondary of a transformer almost neutralize one another, we see that this method gives good compensation, in spite of the fact that, due to the air gap, magnetic leakage is greater than in an ordinary transformer.

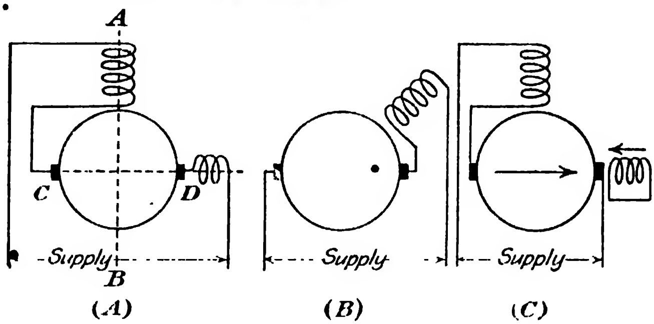


Fig. 369
Compensated Series Motor

In spite of compensation, there is often great difficulty in preventing sparking, and therefore, some manufacturers increase the resistance of the path taken by the current in a winding element as it is short-circuited at the brushes. This is done by using leads of high resistance between the windings and the commutator segments. It can be seen from Fig. 370 that these leads form a high resistance both for the short-circuit current and also for the main current flowing from brush to armature. Hence, the I^2R loss due to the circulating currents in the short-circuited elements is considerably reduced (since the currents are reduced) while the I^2R loss due to the main current is increased. In practice, the inserted resistance is so chosen that the sum of these two losses is a minimum.

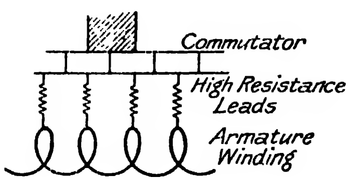


Fig. 370
Resistance Leads

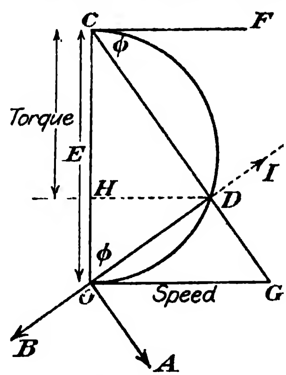


Fig. 371
Circle Diagram of Ideal Motor

2. **The Circle Diagram.** Consider a compensated motor having no losses of any kind, and connected to constant voltage mains.

Let OC (Fig. 371) represent the applied E.M.F., E, and OI represent the current, I, in phase. Then $\angle COI$ is the angle of lag, φ. The statically induced E.M.F. due to self-induction of the field is equal to $L\omega I$, and lags 90° behind I It is represented by vector OA on the diagram. The dynamically induced E.M.F. in the armature is obviously the motor back E.M.F., and it is therefore represented by OB, in phase opposition to the current. Also, OB must have a component in phase opposition to E, since the machine is a motor. The applied E.M.F. is equal and opposite to the resultant of OA and OB,

$$\therefore\ E = OC = OB \text{ reversed} + OA \text{ reversed.}$$

If we draw a semicircle on OC as diameter and join CD, we see that E is the resultant of OD and CD. Again, OD is in line with OB and DC is parallel to OA, the point D being the intersection of the current vector with the semicircle.

Hence, $OD = OB$, the dynamically induced E.M.F.,

and $CD = OA$, the statically induced E.M.F., $L\omega I$.

Now OC is constant, and therefore, the semicircle ODC is the locus of the intersection of these two vectors.

Again, $CD = L\omega I$, and since L and ω are constant, CD represents the current in magnitude. If now a perpendicular CF is drawn, then $\angle DCF = \angle COD = \varphi$, and therefore, referred to CF, the vector CD represents the current both in magnitude and phase. If we assume that the iron is not saturated, but is worked on the straight part of the characteristic, a condition which approximately holds in order that the iron losses may not be too high, the flux is proportional to the current. Hence, CD also represents the flux.

$$\text{The power factor} = \cos\varphi = \frac{OD}{OC}$$

is proportional to OD, since OC is constant.

Torque is proportional to the product of the current and flux.

$$\begin{aligned}\therefore\ T &\propto CD \times CD\\ &\propto CH \times OC\\ &\propto CH, \text{ since } OC \text{ is constant.}\\ \text{Intake} &= EI\cos\varphi\\ &\propto OC \times CD \times \cos\varphi\\ &\propto OC \times DH\\ &\propto DH\end{aligned}$$

Speed N is related to the dynamically induced E.M.F. and the flux as follows—

Dynamically induced E.M.F. $\propto$ flux $\times$ speed,

$$\therefore \quad N \propto \frac{OD}{CD}$$

$$\propto \cot \varphi$$

$$\propto \frac{OG}{OC}$$

$$\propto OG$$

If different positions of the point D are taken on the circle, and the above quantities, measured off, the characteristics can be

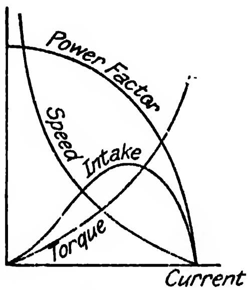


Fig. 372

Characteristics of Series Motor

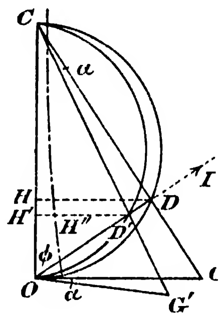


Fig. 373

Corrected Circle Diagram

drawn. These are of the form shown in Fig. 372. Since the current is a maximum at standstill, the speed becomes zero when the current reaches its maximum value. The speed tends to become infinite when the current is very small, so that the mechanical characteristics of the motor are very similar to those of a D.C. series motor. The motor is thus very suitable for traction use, in which sphere it is very largely used.

3. The Effect of Losses on the Diagram.

$$\text{Copper losses} = I^2R = I \times I\,.\,R$$

IR is the resistance drop, and is in phase with I. Hence, it can be represented by marking off a length DD' (Fig. 373) equal to IR.

$$\therefore \text{ Copper losses} = CD \times DD'$$

$$= 2\,\Delta\, CDD'$$

If we neglect for the moment the iron and friction losses, we have

$$\begin{aligned}\text{Output} &= 2\,\Delta\, OCD - 2\,\Delta\, CDD' \\ &= 2\,\Delta\, OCD' \\ &\propto D'H'\end{aligned}$$

Again, the ratio $DD'/CD = IR/I = R$

$= \text{a constant},$

and therefore the angle DCD' is a constant, $= a$, say; and the locus of D' is a circle whose centre is a little to the left of OC.

$$\text{Speed} \propto \frac{\text{dynamically induced E.M.F.}}{\text{flux}}$$

$$\propto \frac{OD'}{CD}, \frac{OD'}{CD'}$$

Draw OG' inclined at the fixed angle a to OG, meeting CD' produced in G'. Then

$$\text{Speed} \propto \frac{OD'}{CD'}, \frac{OG'}{OC}, OG'$$

The iron and friction losses cannot be represented geometrically, because the speed and flux are both variables. If they are determined by a separate core loss test, or calculated from design data, a curve of these losses can be drawn as shown dotted. The output, taking losses of all kinds into account, is then $D'H''$,

$$\therefore\ \text{Efficiency} = \frac{D'H''}{DH}$$

4. Conditions Necessary for a Good Power Factor. From the circle diagram,

$$\tan\varphi = \frac{CD}{OD} = \frac{\text{statically induced E.M.F. in the field}}{\text{dynamically induced E.M.F. in the armature}}$$

Let $f =$ supply frequency

$T_a =$ No. of turns on the armature

$T_f =$ No. of turns on the field

$\Phi =$ flux (max. value)

Hence, statically induced E.M.F. in the field

$$\propto \Phi T_f f$$

Now let the speed be such that the dynamically induced E.M.F. in the armature is of frequency f_1

$\therefore$ Dynamically induced E.M.F. $\propto \Phi T_a f_1$

$$\therefore \tan\varphi \propto \frac{\Phi T_f f}{\Phi T_a f_1}$$

$$= k \times \frac{f}{f_1} \times \frac{T_f}{T_a}, \text{ where } k \text{ is a constant}$$

$$\therefore \cos\varphi = \frac{1}{\sqrt{1+\tan^2\varphi}} = \frac{1}{\sqrt{1 + k^2 \times \frac{f^2}{f_1^2} \times \frac{T_f^2}{T_a^2}}}$$

Hence, for a good power factor the two ratios f/f_1 and T_f/T_a must be kept small. Now f is proportional to the synchronous speed ω_s, and f_1 is proportional to the actual speed ω. Thus, the ratio f/f_1 is kept small by having a low frequency supply and a high actual speed. On the Continent the frequency for single-phase traction work is as low as 15. Because of the low frequency, these motors have a larger number of poles than D.C. traction motors. As a rough rule, it can be taken that there is a pair of poles for each 100 to 150 h.p. They are built in sizes as large as 3,000 h.p.

To keep the ratio T_f/T_a small, the motors have a magnetically strong armature, this being possible because of the compensation of the armature.

The characteristics of A.C. series motors can thus be summed up as follows—

(*a*) They have a compensating winding located in the pole faces.

(*b*) A large number of poles.

(*c*) A large number of turns on the armature winding.

(*d*) A comparatively small number of turns on the field.

(*e*) A low frequency of supply. This, combined with the necessity for a low synchronous speed, is the reason for the large number of poles.

(*f*) Small air gap: necessary because of the weak field.

(*g*) Interpoles with shunt or compound excitation.

5. **Excitation of the Interpoles.** When a winding element undergoes commutation, a reactance E.M.F. is set up in it, due to the reversal of the current, exactly as in a D.C. machine. This is compensated by having series excitation of the interpoles. The interpoles also have to compensate for a voltage, called the "transformer voltage," which is induced in the winding elements by ordinary transformer action, independently of whether the armature is stationary or rotating. This transformer voltage, like all statically induced voltages, is in quadrature with the flux which produces it, and therefore it cannot be compensated for by series

excitation of the interpoles. On the other hand, shunt excitation gives a quadrature flux, and therefore both reactance and trans-

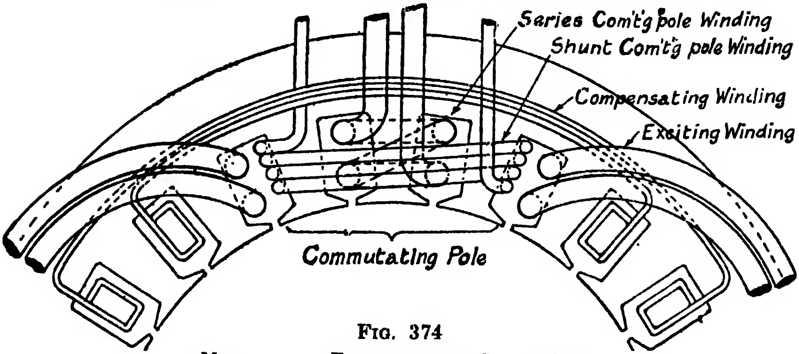


FIG. 374
METHOD OF EXCITING THE INTERPOLES

former voltages can be neutralized by having shunt, in addition to series, excitation of the interpoles. The actual arrangement of the windings is shown in Fig. 374.*

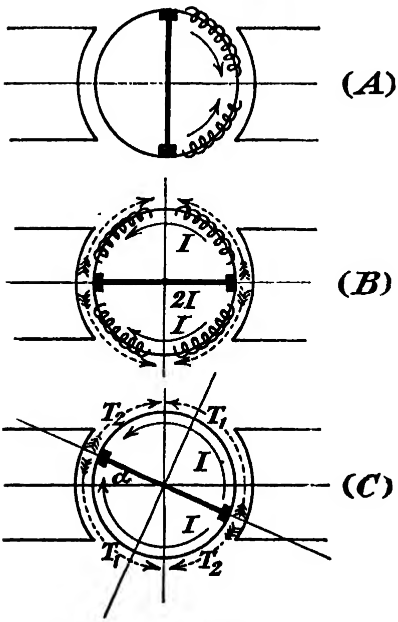


FIG. 375
ACTION OF REPULSION MOTOR WITH DIFFERENT BRUSH SETTINGS

II. THE REPULSION MOTOR

6. This motor differs from the A.C. series motor in that no current is led into the armature, the brushes being short-circuited as shown in Fig. 375.

Consider an ordinary D.C. armature with short-circuited brushes placed in a bi-polar field excited by alternating current.

(a) Let the brush axis be perpendicular to the main flux as in Fig. 375(A). Then the E.M.F.s induced in the two halves of the winding shown will neutralize one another, and the armature will neither carry any induced current nor produce any torque.

(b) Let the brush axis be along the direction of the main flux, as in Fig. 375(B).

* For further information, see Dover, *Electric Traction* and *Electric Motors and Control Systems*.

Then a current will be induced in the armature, the four quarters of which will produce four torques as indicated by the dotted arrows. These will neutralize one another, the total torque being again zero.

(c). Let the brush axis be inclined at any angle a to the direction of the main flux, as in Fig. 375(c). Then calling the torques T_1 and T_2 as shown,

$$\text{Resultant torque, } T = 2(T_1 - T_2)$$

The torque is also given by the expression

$T \propto$ (current between brushes) $\times$ (component of main flux perpendicular to brush axis)

Now, current between brushes

$$\propto \text{component of main flux along brush axis}$$
$$\propto \Phi \cos a$$
$$\therefore T \propto \Phi \sin a \times \Phi \cos a$$
$$\propto \sin a \cos a, \propto \sin 2a$$

Hence, T is a maximum when $2a = 90°$, i.e. when $a = 45°$.

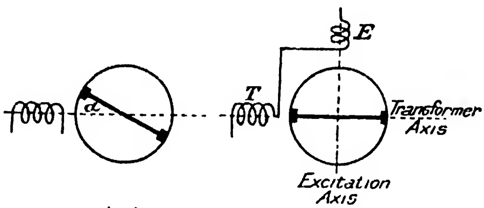


Fig. 376

The actual arrangement of Fig. 376(A) can be considered as replaced by that in Fig. 376(B). If the total number of turns on the actual field is T_f, then

Number of turns on winding $T = T_f \cos a$
Number of turns on winding $E = T_f \sin a$

The axis of the winding T, namely, the brush axis, is called the transformer axis of the motor. The winding E has no transformer action on the armature, except on those coils which undergo short circuit at the brushes, and therefore, it transmits no energy to the armature. Its function is to supply the excitation only, and the axis of E is called the excitation axis. When the motor is running, the armature develops mechanical energy, and this energy is imparted to it along the transformer axis, i.e. from the stator

winding T. Also the flux in the transformer winding T induces a static E.M.F. in T; and since the phase difference between a flux and the statically induced E.M.F. it sets up is 90°, it follows that the flux in winding T is in quadrature with the current, and therefore, with the flux along the excitation axis. But these two fluxes are 90° apart in space as well as in time, and therefore, the motor possesses

(*a*) A uniform rotating field, if the two fluxes are equal.
(*b*) An elliptical field, if they are different in magnitude.

In the armature there is a static E.M.F. along the energy axis set up by transformer action, and this is in quadrature with the flux producing it, i.e. with the flux through T. Again, the flux through T is in quadrature with the flux through E, and this latter flux produces the dynamically induced E.M.F. in the armature, which is, of course, in phase with the flux through E. Hence, the armature dynamically induced E.M.F. is in phase opposition, 180°, to the statically induced E.M.F. The difference between these two armature E.M.F.s is equal to the armature drop in volts RI. If we assume an ideal motor with zero resistance, then the two are equal and opposite.

Let Φ_t = max. flux produced by winding T
Φ_e = max. flux produced by winding E
T_a = No. of armature turns

Then dynamic E.M.F. $E_d \propto \Phi_e T_a \times Np$ . . . . (1)
Static E.M.F. $E_s \propto \Phi_t T_a f$ . . . . (2)

where N is the speed, and p, the number of poles.
But E_d is equal to E_s

$$\therefore \Phi_e Np = \Phi_t f$$

Again, if we call the synchronous speed r.p.m. N_s, then

$$f = \frac{N_s p}{120}, \propto N_s p$$

$$\therefore \Phi_e Np \propto \Phi_t N_s p$$

$$\therefore \frac{\Phi_t}{\Phi_e} \propto \frac{N}{N_s}$$

Hence, the two fields are equal at synchronous speed only, and at this speed the total field is a pure rotating one.

At speeds below N_s $\Phi_t < \Phi_e$

At speeds above N_s $\Phi_t > \Phi_e$

Under these conditions the resultant field is elliptical, the major axis at speeds below N_s becoming the minor axis at speeds above

N_s. At synchronous speed, the armature and the rotating field travel at the same rate, and therefore, there are no armature core losses. Also, there are no circulating currents in the coils undergoing commutation. At speeds above synchronism, the transformer flux increases rapidly with the speed, thereby resulting in large core losses and heavy circulating currents in the coils undergoing commutation. These conditions limit the operating speed of the motor to the neighbourhood of synchronous speed. If higher speeds are necessary, the transformer flux has to be weakened. On a low-frequency system the repulsion motor must therefore be a low-speed machine with relatively few poles, so that it will be considerably heavier than a series motor of the same power. This is because the fewer the poles, the greater must be the cross section of the magnetic paths for the same output. Thus, whereas series motors are suited to very low frequencies in the neighbourhood of 15, repulsion motors are better suited for frequencies of about 25.

7. **The Compensated Repulsion Motor.** In the compensated motor, the field windings E, which set up the excitation flux Φ_e along the excitation axis are dispensed with, the excitation flux being set up by the armature itself. To do this, the armature has another set of brushes which have their axis along the excitation axis. (*See* Fig. 377.) Thus, the armature itself takes the place of the field E in the simple repulsion motor, two important advantages being derived from this arrangement.

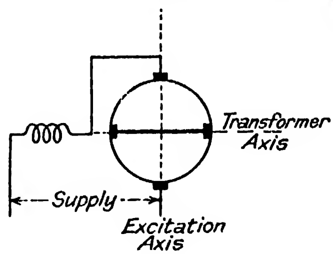


FIG. 377
COMPENSATED REPULSION MOTOR

(*a*) Since the main field is produced by the armature itself, the leakage which occurs between armature and field in all other motors is entirely eliminated.

(*b*) Practically complete compensation of the induced voltage in the excitation winding can be obtained at one particular speed, so that at this speed the motor operates at unity power factor. If there were no losses in the machine, this speed would be the synchronous speed, but because of losses it is slightly greater than this.

8. **The Doubly Fed or Series Repulsion Motor.** This is a modification of the repulsion motor to enable it to operate satisfactorily at speeds greater than synchronous speed. One scheme of connections is shown in Fig. 378. We have seen that at high speeds the transformer flux increases rapidly with the speed, thereby setting up heavy circulating currents in the coils undergoing commutation, and also causing heavy core losses. In the doubly fed

motor this is overcome by weakening the transformer flux. The diagram indicates that the point A can be moved along the secondary of the main transformer, thus altering the voltage applied to the stator winding. At starting, the brushes are often short-circuited and the motor is thus brought up to speed as an ordinary repulsion motor. Then, when the speed is near to synchronism, the connections are automatically changed over to

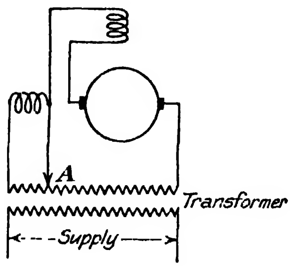


FIG. 378
DOUBLY FED MOTOR

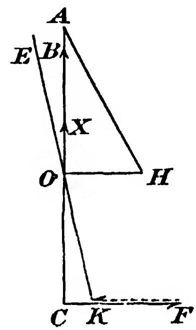


FIG. 379
VECTOR DIAGRAM OF COMPENSATED MOTOR

the doubly fed arrangement. Thus, the doubly fed motor combines some of the advantages of the series and the ordinary repulsion motor.

9. Vector Diagram for a Compensated Repulsion Motor. Consider for simplicity an ideal motor having no losses. In the armature there are (*a*) the transformer E.M.F., (*b*) the dynamic E.M.F. We have seen that in an armature with zero resistance these are equal and opposite. Now in the compensated motor the armature produces its own exciting field, as previously explained; and therefore, there are two more E.M.F.s to consider. These are (*c*) the statically induced E.M.F. set up by the pulsations in the exciting field; (*d*) the dynamically induced E.M.F. set up by rotation in the transformer field.

Let OA (Fig. 379) represent the stator current. Then a vector, such as OX, in phase with OA, will represent the exciting field, i.e. the field along the excitation axis. Let OB represent the dynamic E.M.F. (*b*) in phase with OX. Hence, OC, equal and opposite to OB, will represent the transformer E.M.F. (*a*). Now, OC is produced by the combined action of the stator and armature windings. The flux producing OC is perpendicular to it, namely, along OH; hence, OH also represents the transformer flux. This flux, as well as the flux OX, links with the armature, and therefore,

if OH is drawn proportional to the current producing this flux, the resultant of OH and OA, namely, HA, represents the total armature current.

The statically induced E.M.F. (c) lags behind the main field, as shown by the vector CF, and therefore the dynamically induced E.M.F. (d), which is in phase opposition to CF, will be represented by a vector such as FK. Now the supply E.M.F., E, is equal and opposite to the resultant of the E.M.F.s, OC, CF, and FK, namely, equal and opposite to OK. It is therefore represented by OE in the diagram.

III. The Polyphase Commutator Motor

10. **Action of the Commutator.** Imagine first of all an armature with a two-layer winding brought out to a commutator, a D.C. armature, in fact, and let this armature be rotated in a magnetic field. First of all let this field be stationary in space, then the E.M.F.s induced in individual armature conductors are alternating E.M.F.s, undergoing one complete cycle for an angular movement of two pole pitches. But the distribution of induced E.M.F.s in all the armature conductors is stationary in space, the E.M.F. at any particular point in space being proportional to the flux density at that particular point. Now, the action of the commutator is to connect the brushes which press on it to points in the winding which are fixed in space, and consequently if the magnetic field is stationary the distribution of potential round the commutator is also stationary. Again, since each armature coil is connected to adjacent commutator segments (assuming a simple lap winding, as is usual), the potential distribution round the commutator is the coil E.M.F. integrated right round the armature. There will be a position of minimum potential, which for convenience can be called zero potential, and a pole pitch away there will be a point of maximum potential. Also there will be one point of zero potential and one of maximum potential for each pair of poles on the machine. Obviously, if the machine is to be used as a D.C. machine, these points will give the positions of the negative and positive brush positions respectively, so that the potential distributions and brush positions for two- and four-pole machines will be as shown in Fig. 380. The magnitude of the E.M.F. between brushes of opposite polarity will be

$$E = \frac{\Phi Z N}{60} \times 10^{-8}$$

With a stationary magnetic field the external frequency is zero, i.e. the external circuit carries direct current, while the internal frequency is given by the speed N of the armature.

Now imagine that the magnetic field rotates in space, as would be the case in a machine consisting of a stator fed with polyphase currents after the manner of an induction motor, and having a

rotor carrying a D.C. winding. Then, since the distribution of E.M.F. in the armature conductors is, at any instant, dependent on the position in space at that instant of the magnetic field, it follows that as the field rotates in space, the E.M.F. distribution round the winding will also rotate in space at the same speed and in the same direction. But obviously the potential distribution round the commutator will move whenever the E.M.F. distribution

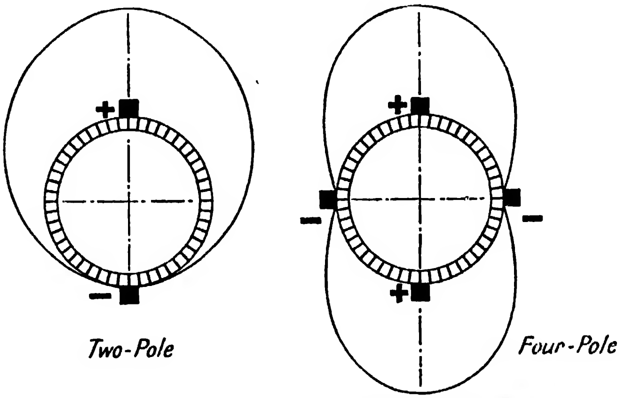


FIG. 380. POTENTIAL DISTRIBUTION ROUND COMMUTATOR

round the winding moves, so that we see, finally, that the potential distribution round the commutator rotates in space at the same speed, and in the same direction, as the magnetic field.

Thus it is obvious that the P.D. between two brushes pressing on the commutator will now be alternating, the frequency being determined by the speed of the magnetic field in space. But the internal frequency is determined by the relative speed of the armature to the stator rotating field, so that it follows that the commutator now acts as a frequency changer. Take, for example, the case of a 4-pole machine having its stator fed from a 50-cycle supply, and let the rotor travel at 1,000 r.p.m. in the same direction as the stator field.

$$\begin{aligned} \text{Speed of stator field} &= 120f/\text{poles} \\ &= 120 \times 50/4 \\ &= 1{,}500 \text{ r.p.m.} \\ \therefore \text{Relative speed of stator field to rotor} &= 1{,}500 - 1{,}000 \\ &= 500 \text{ r.p.m.} \end{aligned}$$

External frequency, i.e. frequency of the alternating P.D. at the brushes

= frequency corresponding to field speed
= 50 cycles/sec.

Internal frequency, i.e. frequency of currents in the rotor conductors

$$= \text{frequency corresponding to relative speed}$$
$$= 4 \times 500/120$$
$$= 16{\cdot}67 \text{ cycles/sec.}$$

It will be clear that (a) the P.D. between any two brushes will depend on their angular distance apart, and (b) that any number of phases of the external circuit can be arranged merely by providing the necessary number of brushes. Thus in the case of a 2-pole machine two diametrically opposite brushes will give single-phase, three equally spaced brushes will give three-phase, and so on. Thus the commutator can act, not only as a frequency changer, but as a phase convertor as well.

Suppose that the commutator is worked single-phase, then the brushes will be placed as in the case of a D.C. machine. The voltage given by the E.M.F. will be the maximum value, and therefore for the R.M.S. value (assuming a sinusoidal voltage) we have

$$E = \frac{\Phi Z N'}{60\sqrt{2}} \times 10^{-8}$$

where N' is the relative speed of stator field to rotor in r.p.m.

As in the case of the rotary convertor, we can regard the vector diagram for the armature as a complete circle and the voltage between two brushes of angular spacing a equal to the length of the chord which subtends an angle a at the centre. Thus, if the external circuit is three-phase, the angle a is 120° and we have to multiply the above value by

$$\sin \pi/m = \sin \pi/3 = 0{\cdot}866$$

Hence for three-phase external supply the brush voltage is given by

$$E = \frac{0{\cdot}866}{60\sqrt{2}} \times 10^{-8} \times \Phi Z N' \text{ R.M.S. volts}$$

11. **The Expedor.** Consider an armature with D.C. winding rotating inside a stator without any windings, the stator being present merely to complete the magnetic path of the flux. Then, since the stator can produce no flux, this must obviously be produced by currents flowing in the armature winding. For example, if the armature is connected via the commutator to the slip rings of an induction motor, as in Fig. 381, it will carry the current of the main motor and so set up a rotating field. The speed of this field in space will depend on the external frequency, as explained above, this external frequency being the rotor frequency of the main motor: the speed will be entirely independent of the speed of the armature.

First, suppose that the armature is stationary: then its conductors will be cut by the lines of force of its own rotating magnetic field,

and in consequence it will act like a choke in series with the rotor of the main motor. The result is that the power factor of the main motor will be reduced.

Now let the armature be rotated in the same direction as its own field in space; if its speed is less than the field speed, the relative velocity of field to armature will decrease, and consequently the induced E.M.F. in the armature will decrease. Hence, although still acting as a choke, the effective reactance of the armature will be reduced, and the overall power factor consequently improved.

Now suppose that the driving motor rotates the armature at exactly the same speed as the field speed, and in the same direction,

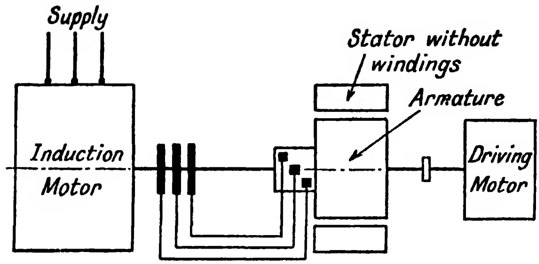


Fig. 381. The Expedor

then the relative velocity of field to armature will be zero and the induced E.M.F. will be zero. Consequently the armature will have no effect on the operation of the main motor beyond the slight increase in slip due to the additional I^2R loss in the rotor circuit caused by the resistance of the armature and leads.

Finally, let the armature speed be greater than the field speed, then the relative velocity of field to armature is reversed, the phase of the induced E.M.F. is reversed, and as a result the armature now acts as a condenser instead of a choke. It therefore follows that so long as the armature is driven at a speed greater than that of its own field, and in the same direction, it will improve the power factor of the main motor, the amount of the improvement increasing as the excess of armature speed above field speed is increased.

The machine is called a phase advancer, or sometimes an exciter. As it carries the full rotor current of the main motor it is also a series exciter, and to this class of machine Miles Walker has given the name Expedor.* Because it is a series machine the improvement in power factor effected by the expedor will be zero at no load, but will increase as the load current increases.

For the purpose of comparatively small industrial motors it is clearly desirable that there should be a single machine instead of a

* It is possible for the stator of the expedor to be provided with windings so that it can contribute to the total M.M.F., but the consideration of such machines is beyond the scope of this book.

combination of three machines. G.E.C. Ltd. developed a single machine in which an expedor is incorporated with an induction motor. In such a machine it will be clear that there can be no control over the speed of the exciter armature, or, as it is called, the compensating winding. Consequently the compensator field must be quite separate from the main field of the induction-motor part of the machine. This is accomplished by housing the compensating winding in entirely separate slots near the inner periphery

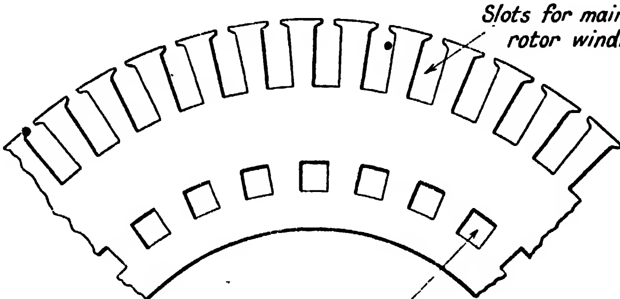


FIG. 382. CORE DISCS AND SLOTS IN THE EXPEDOR

of the rotor core. The core discs are therefore of the type shown in Fig. 382. The compensating winding is a lap winding brought out to a small commutator, and the electrical connections of main winding and compensating winding are the same as for Fig. 381.

The essential feature of this class of expedor is that the compensator field speed must be less than the actual speed of the compensator, and in the G.E.C. machine this is effected by arranging the compensator to have a different number of poles from the main field. A numerical example will make this clear.

Let f = supply frequency

p_1 = number of main motor poles

p_2 = number of compensator poles

σ = fractional slip of motor

$$\text{Then synchronous speed of motor} = \frac{120f}{p_1}$$

and

$$\text{actual speed of motor} = \frac{120f}{p_1}(1 - \sigma)$$

$$\text{Slip frequency of motor} = \sigma f$$

$$\text{Speed of compensator field} = \frac{120\sigma f}{p_2}$$

$$\therefore \qquad \frac{\text{Motor speed}}{\text{Compensator field speed}} = \frac{120f(1-\sigma)}{p_1} \times \frac{p_2}{120\sigma f}$$

$$= \frac{p_2(1-\sigma)}{p_1\sigma}$$

Thus if

$$p_1 = 6 \text{ poles}$$

$$p_2 = 2 \text{ poles}$$

$$\sigma = 5 \text{ per cent, i.e. } \tfrac{1}{20}$$

then

$$\frac{\text{motor speed}}{\text{compensator field speed}} = \frac{2(1-\frac{1}{20})}{\frac{6}{20}}$$

$$= 6{\cdot}33$$

Hence under these conditions the compensating winding is rotated 6·33 times as fast as the field produced by it, thus fulfilling the necessary condition.

12. **The Susceptor.** In the expedor type of machine the compensating winding is in series with the main rotor. This is a class of machine in which the compensator is in the nature of a shunt exciter, and to this class Miles Walker has given the name Susceptor. Consider again an induction motor with external exciter. The simplest susceptor is a small rotary convertor armature rotating in a stator without any windings. This armature is coupled to the main motor so that its speed cannot be varied independently of the main motor. Its slip rings are connected to the main supply, while its commutator is connected to the main rotor, the scheme being shown in Fig. 383.

The supply at line frequency to the susceptor will set up a rotating field at synchronous speed relative to the armature. Also since the slip rings are connected to fixed points in the armature winding, this relative speed will be maintained whatever the actual armature speed may be. The connections are such that the rotations of susceptor and its field are in opposite directions, and therefore, since the actual speed is the same as the main motor speed, the speed of the field *in space* is the slip speed. We have seen that the external frequency on the commutator side corresponds to the actual field speed in space, and consequently this frequency is the same as the slip frequency of the main motor. It is for this reason that the commutator can be in direct electrical connection with the main slip rings, as shown in Fig. 383.

Now, since the susceptor M.M.F. is produced by currents fed in at the slip-ring end, the position of the susceptor field in space at any instant will be entirely independent of the positions of the

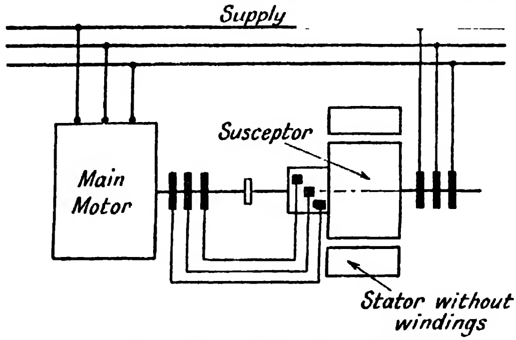


FIG. 383. THE SUSCEPTOR

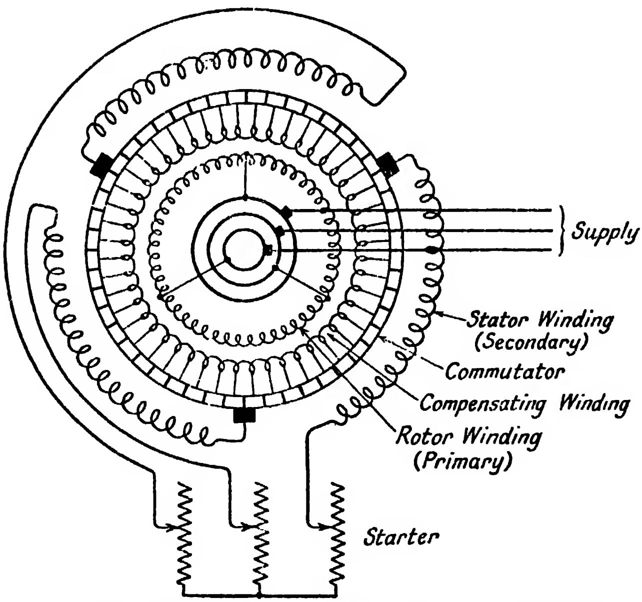


FIG. 384. THE COMPENSATED INDUCTION MOTOR—SUSCEPTOR-TYPE COMPENSATION

brushes on the commutator. But we saw in § 10 that the potential distribution round the commutator rotates in space with the field, and always keeps pace with it. Hence the E.M.F. at the brushes at any particular instant depends on the relative position of the brushes and the commutator potential distribution. In other words,

the phase of the E.M.F. at the commutator depends on the angular position of the brushes, and if the brush position is altered, the phase of this E.M.F. will be altered by a corresponding amount. It therefore follows that the phase of the E.M.F. injected into the main rotor circuit, and consequently the overall power factor of the equipment, can be controlled by rocking the brushes round the commutator.

Now, when considering the combination of induction motor and susceptor into a single machine, it will be realized at once that

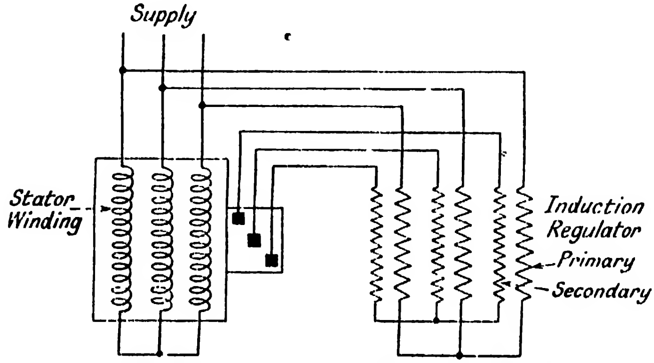


FIG. 385. VARIABLE-SPEED COMMUTATOR MOTOR

with an ordinary stator-fed induction motor this will be impossible, because the motor field speed is the synchronous speed, N_s, whereas the susceptor field speed is σN_s. The difficulty is overcome by using a rotor-fed motor, i.e. a motor whose rotor is used as the primary and stator as the secondary. In such a motor the field speed is the slip speed σN_s, with the result that one and the same field can act as both main field and susceptor field. This further simplifies the construction, since both main rotor (primary) winding and compensating winding can be housed in the same rotor slots. The scheme of connections is therefore somewhat different from that of Fig. 383 and is given diagrammatically in Fig. 384.*

13. **Speed Control.** If a D.C. motor is separately excited so as to have a field of constant strength, then the speed of the armature will be proportional to the voltage applied to it. Furthermore, for any given armature applied voltage the drop in speed with increased load will be small, the motor having a mechanical characteristic like that of a shunt motor. In fact, for a given motor, there will be a family of such characteristics, each corresponding to a definite value of the applied voltage. This principle can be applied to the induction motor. If the motor is stator-fed in the usual way, the field speed will be the synchronous speed, and consequently if

* This motor must not be confused with the Schrage motor described in § 14.

the rotor is a D.C. armature with commutator, the potential distribution round the commutator will rotate at synchronous speed. In other words, the frequency external to the armature will be the same as the stator frequency, showing that the commutator can be connected to a supply at stator frequency. Hence, in order that the speed may be varied, all that is necessary is that there shall be control over the voltage applied to the commutator. In the B.T.-H. variable-speed motor this is accomplished by means of an induction regulator, the essential connections being given in Fig. 385. It is to be noted that the commutator on this machine is used for the purpose of speed control, and not of power-factor improvement.

14. The Schrage Motor. This is a motor in which power-factor correction and speed control are both available in one and the

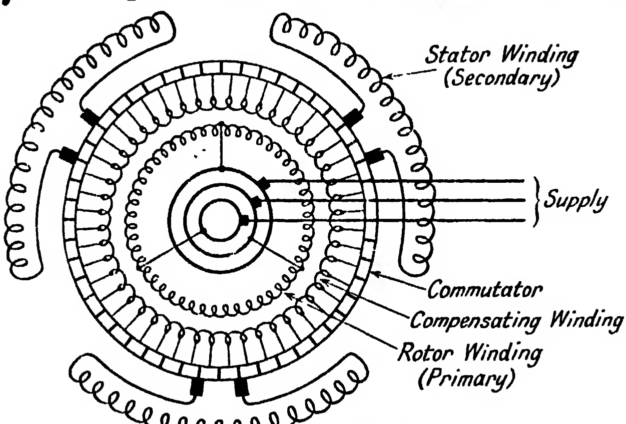


FIG. 386. CONNECTION SCHEME OF THE SCHRAGE MOTOR

same machine. It has been explained that the speed of an induction motor is dependent on the total E.M.F. acting in the secondary circuit. If the secondary is short-circuited in the usual way, this E.M.F is the induced E.M.F. and the motor runs at normal speed. If an E.M.F. of the correct frequency from some other source is injected into the secondary circuit, then if this injected E.M.F. has a component in direct opposition to the induced E.M.F., the speed will be below normal. On the other hand, if it has a component in phase with the induced E.M.F., the total E.M.F. acting in the secondary will be increased and the speed will be above normal. In other words, if the injected E.M.F. bucks the induced rotor E.M.F., the motor will run at subsynchronous speeds, whereas if it boosts the induced E.M.F., the motor will run at hypersynchronous speeds. In addition, if this injected E.M.F. has a quadrature component which leads the induced rotor E.M.F., the

power factor of the motor will be improved. The Schrage motor is a machine in which these adjustments are available. It is essentially an induction motor with integral compensator of the susceptor type, so that it is rotor-fed. It differs from the motor of Fig. 384 in that each brush set is duplicated, thus giving two separate brush sets, and these are arranged so that the brushes as a whole can be rocked round the commutator, or one set can be rocked relative to the other. Each phase of the stator (secondary) winding is connected to the brushes of one set according to the scheme of Fig. 386.

There are three possible modes of action illustrated in Figs. 387 (*a*), (*b*), and (*c*). In Fig. (*a*) the two brushes *a* and *b* of a set are in line with one another, so that at every instant they will be in direct electrical contact through a commutator bar. Hence

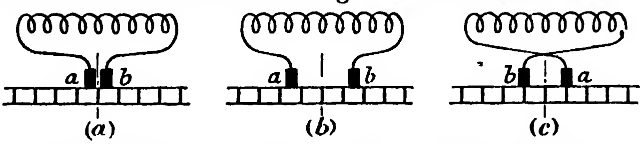


FIG. 387. METHODS OF OPERATION OF THE SCHRAGE MOTOR

each secondary phase is short-circuited and the motor will run like a plain induction motor. In Fig. (*b*) the brush sets are opened out and the connections are such that the commutator P.D. between the points *a* and *b* will have a component in direct opposition to the secondary induced E.M.F., so that the motor will run at subsynchronous speed. In Fig. (*c*) the position of the brushes *a* and *b* is reversed, so that the E.M.F. picked off at the commutator is in phase opposition to that of Fig. (*b*). Hence this E.M.F. will have a component in phase with the rotor induced E.M.F., and the motor will run at hypersynchronous speeds. For any given setting of the brushes the motor will have a shunt characteristic.

Although the magnitude of the E.M.F. between the brushes *a* and *b* is dependent on their relative angular positions, the phase of the E.M.F. depends upon the position of the centre line with respect to the centre of the rotor winding. Hence if the brushes as a whole are rocked round the commutator, the phase of the E.M.F. injected into the secondary circuit will be varied relative to the rotor induced E.M.F., and therefore the power factor of the motor will be varied. Hence the speed depends on the angular distance between the individual brush sets *a* and *b*, while the power factor depends upon the angular position of the brushes as a whole.

In the case of Fig. 387 (*c*) there will be a certain angular distance between brushes *a* and *b* for which the motor runs at exactly synchronous speed. The field speed will then be zero, or, in other words, the field will be stationary in space as in a D.C. machine. The compensating winding will therefore act like the armature of a D.C. generator, and it will deliver a direct current to the secondary winding.

CHAPTER XXIII

COMPLEX WAVE FORMS

1. Up to the present it has been assumed that alternating currents and voltages have been of simple sinusoidal form. Modern alternators are designed to give a terminal voltage which approaches very closely to a sine wave, but under certain conditions both current and voltage may be distorted very considerably. No matter what the wave form may be, the negative half-wave is an exact reproduction of the positive half, Fig. 388(A) being a possible wave form, but not Fig. 388(B). Since the wave form is repeated at regular intervals, it can be split up into fundamental and harmonic waves all of pure sinusoidal form. Also, because of the fact that the negative half is a reproduction of the positive half, there are no even harmonics. An alternating voltage of any wave form can therefore be represented by the general expression

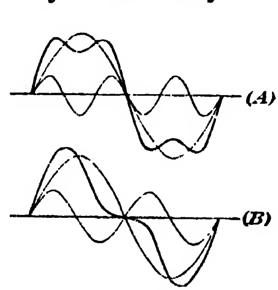


Fig. 388

$$e = E_1 \sin \omega t + E_3 \sin (3\omega t + \varphi_3) + E_5 \sin (5\omega t + \varphi_5) + \ldots$$

where E_1, E_3, E_5, etc., are the amplitudes of the fundamental and harmonic waves, $\omega = 2\pi f$, f being the fundamental frequency, and the angles φ_3, φ_5, etc., the phases of the harmonics with respect to the fundamental.

2. **Effective Value of a Complex Wave.** Consider the voltage wave represented by

$$e = E_1 \sin \omega t + E_3 \sin (3\omega t + \varphi_3) + E_5 \sin (5\omega t + \varphi_5) + \ldots$$

Then E_{eff} = R.M.S. value

$= \sqrt{\text{Average of } e^2}$

Now $e^2 = E_1^2 \sin^2 \omega t + E_3^2 \sin^2 (3\omega t + \varphi_3) + E_5^2 \sin^2(5\omega t + \psi_5) + \ldots$

$+ 2E_1E_3 \sin \omega t \,.\, \sin (3\omega t + \varphi_3) + 2E_1E_5 \sin \omega t \,.\, \sin (5\omega t + \psi_5) + \ldots$

The average value of $\sin^2\alpha$, where α is any angle, is $\frac{1}{2}$. The average value of $\sin \alpha \sin \beta$, where α and β are angles which correspond in different frequencies (as, for example, ωt and $3\omega t + \varphi_3$) is zero.

Hence, $E^2_{eff} = \frac{1}{2}E_1^2 + \frac{1}{2}E_3^2 + \frac{1}{2}E_5^2 + \ldots$

$$= (E_{1eff})^2 + (E_{3eff})^2 + (E_{5eff})^2 + \ldots,$$

$$\therefore\ E_{eff} = \sqrt{E_{1eff}^2 + E_{3eff}^2 + E_{5eff}^2 + \ldots}$$

Hence, the effective value of a complex wave is equal to the square root of the sum of the squares of the effective values of the components. Or, in terms of the maximum values of the components, we have

$$E_{eff} = 0{\cdot}707\sqrt{E_1^2 + E_3^2 + E_5^2 + \ldots}$$

3. Influence of the Nature of the Circuit on the Shape of the Current Wave. Consider first a circuit possessing resistance only. Then we have

$$I_1 = E_1/R\,;\ I_3 = E_3/R\,;\ I_5 = E_5/R, \text{ etc}$$

Also, the various currents are in phase with the corresponding voltages. As a result the current wave is an exact reproduction of the voltage wave, and is in phase with it, as shown in Fig. 389(A)

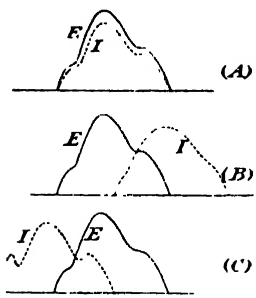


Fig. 389

Consider now a purely inductive circuit of inductance L, then

$$I_1 = E_1/L\omega\,;\ I_3 = E_3/3L\omega\,;$$
$$I_5 = E_5/5L\omega, \text{ etc.}$$

Thus, the harmonics in the current wave are diminished in proportion to their frequency numbers, with the result that the current wave is much smoother than the voltage wave, as shown in Fig. 389(B).

We have $E_{eff} = 0{\cdot}707\sqrt{E_1^2 + E_3^2 + E_5^2 + \ldots}$

$$I_{eff} = 0{\cdot}707\sqrt{I_1^2 + I_3^2 + I_5^2 + \ldots}$$

$$= \frac{0{\cdot}707}{L\omega}\sqrt{E_1^2 + \frac{E_3^2}{9} + \frac{E_5^2}{25} + \ldots}$$

Hence the effective reactance

$$= \frac{E_{eff}}{I_{eff}} = L\omega \times \sqrt{\frac{E_1^2 + E_3^2 + E_5^2 + \ldots}{E_1^2 + \frac{E_3^2}{9} + \frac{E_5^2}{25} + \ldots}}$$

Now consider a circuit containing capacity only. Then $I_1 = E_1C\omega\,;\ I_3 = 3E_3C\omega\,;\ I_5 = 5E_5C\omega$, etc., the effect of capacity thus being to magnify the harmonics in the current wave. This is shown in Fig. 389(C).

Again, $E_{eff} = 0{\cdot}707\sqrt{E_1^2 + E_3^2 + E_5^2 + \ldots}$

$$I_{eff} = 0{\cdot}707\sqrt{I_1^2 + I_3^2 + I_5^2 + \ldots}$$

$$= 0{\cdot}707\, C\omega \times \sqrt{E_1^2 + 9E_3^2 + 25E_5^2 + \ldots}$$

Hence, effective reactance

$$\frac{E_{eff}}{I_{eff}} = \frac{1}{C\omega} \times \sqrt{\frac{E_1^2 + E_3^2 + E_5^2 + \ldots}{E_1^2 + 9E_3^2 + 25E_5^2 + \ldots}}$$

Thus the effect of a complex wave form is to diminish the reactance of a condenser and to increase the reactance of an inductive circuit.*

4. **Selective Resonance.** If the frequency of the supply is suitable, resonance may take place, not with the fundamental, but with one of the harmonics. This is called "selective resonance." For resonance with the fundamental, we have the condition $L\omega = \frac{1}{C\omega}$. For resonance with the third harmonic, we have the condition $3L\omega = \frac{1}{3C\omega}$; and so on. The effect of selective resonance is to produce enormous distortion of the current wave, in which the fundamental may be actually of smaller amplitude than the particular harmonic which causes resonance, although in the voltage wave the amplitude of the harmonic is much less than that of the fundamental.

Example. A voltage wave having a fundamental of amplitude 500, and a third harmonic of amplitude 10, is applied to a circuit containing $R = 1$ ohm, $L = 1$ henry, and $C = \frac{1}{50}$ m.f. in series. Find the frequency for resonance with the third harmonic, and draw the voltage and current waves for this frequency.

For resonance with the third harmonic,

$$\omega^2 = \frac{1}{9} \times \frac{1}{LC};\ \therefore\ \omega = \frac{1}{3}\sqrt{\frac{1}{1/50 \times 10^{-6} \times 1}} = 2{,}360$$

$$f = \frac{2{,}360}{2\pi} = 375 \text{ cycles per sec.}$$

Impedance of the circuit to the fundamental,

$$Z_1 = \sqrt{R^2 + \left(L\omega - \frac{1}{C\omega}\right)^2} = \sqrt{1 + \left(2{,}360 - \frac{50 \times 10^6}{2{,}360}\right)^2}$$

$$= 19{,}000 \text{ approx.}$$

$$\therefore\ I_1 = \frac{E_1}{Z_1} = \frac{500}{19{,}000} = {\cdot}026 \text{ amp.}$$

* See also Golding's *Electrical Measurements and Measuring Instruments*, Chap. XV, and Kemp's *Alternating Current Wave Forms.*

The impedance of the circuit to the third harmonic is the same as the resistance since there is resonance with this harmonic.

$$\therefore\ I_3 = \frac{E_3}{R} = 10 \text{ amp.}$$

The voltage and current waves are shown in Fig. 390; and it will be seen that the harmonic is so greatly magnified in the case considered, that to all intents and purposes the current is a triple frequency current.

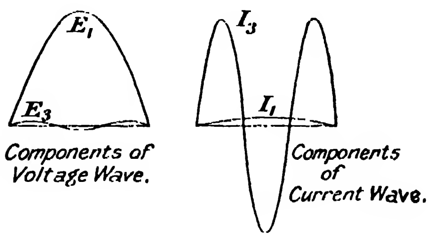


FIG. 390
TO ILLUSTRATE SELECTIVE RESONANCE

5. **Power Conveyed by Complex Waves.** An alternating current can contribute power to a circuit only when it is flowing under the influence of a voltage of the same frequency. Hence if the voltage and current waves are assumed to be resolved into their components, the total power is made up of the following terms—

E_1 acting on I_1 produces $E_1 I_1 \cos \varphi_1$ watts;

E_3 acting on I_3 produces $E_3 I_3 \cos \varphi_3$ watts; and so on.*

Hence, total power

$$W = E_1 I_1 \cos \varphi_1 + E_3 I_3 \cos \varphi_3 + E_5 I_5 \cos \varphi_5 + \ldots$$

the various power factors being given by

$$\text{Cos } \varphi_1 = \frac{R}{Z_1};\ \cos \varphi_3 = \frac{R}{Z_3};\ \cos \varphi_5 = \frac{R}{Z_5};\ \text{and so on.}$$

With complex waves the power factor obviously cannot be denoted by the cosine of a phase angle, because the various components have, in general, different phase angles. It can be defined by the expression

$$\text{Power factor} = \frac{\text{true power}}{\text{apparent power}}$$

Now the true power in a circuit not containing motors, and in which there are no iron losses, is equal to the copper loss, i.e. to RI_{eff}^2.

$$\therefore\ \text{Power factor} = \frac{RI^2_{eff}}{E_{eff} I_{eff}} = \frac{RI_{eff}}{E_{eff}}$$

* Note that E_1, E_3 . . . . , I_1, I_3 . . . . , now refer to R.M.S. valves and not to crest valves.

Again, $E_{eff} = 0{\cdot}707\sqrt{E_1^2 + E_3^2 + E_5^2 + \ldots}$

$I_{eff} = 0{\cdot}707\sqrt{I_1^2 + I_3^2 + I_5^2 + \ldots}$

But $I_1 = \frac{E_1}{Z_1} = \frac{E_1}{R} \cdot \frac{R}{Z_1} = \frac{E_1}{R}\cos\varphi_1$

Similarly, $I_3 = \frac{E_3}{R} \cdot \cos\varphi_3$; $I_5 = \frac{E_5}{R}\cos\varphi_5$; and so on.

$$\therefore\ I_{eff} = \frac{0{\cdot}707}{R}\sqrt{E_1^2\cos^2\varphi_1 + E_3^2\cos^2\varphi_3 + E_5^2\cos^2\varphi_5 + \ldots}$$

Hence, power factor

$$= \sqrt{\frac{E_1^2\cos^2\varphi_1 + E_3^2\cos^2\varphi_3 + E_5^2\cos^2\varphi_5 + \ldots}{E_1^2 + E_3^2 + E_5^2 + \ldots}}$$

This expression shows that with a complex wave the power factor can never have the same value that it would have with sinusoidal waves of the same effective value, except in the case of a pure resistance circuit. Also, resonance cannot bring the power factor to unity, as it does with sinusoidal waves, because if one of the angles φ_1, φ_3, φ_5, etc. is zero, then all the others will be finite.

6. Effect of Iron on the Shape of the Current Wave. Consider an iron circuit, say, an anchor ring, of cross section A square cm. Let it be wound with a coil of N turns, and a sinusoidal voltage, applied. The apparatus is then similar to a transformer on no load and a sinusoidal flux given by

$$E_{eff} = 4{\cdot}44\, B_{max} A N f \times 10^{-8}$$

$$\text{i.e.}\quad B_{max} = \frac{E_{eff} \times 10^8}{4{\cdot}44\, ANf}$$

will be produced. Also, this flux will lag 90° behind the applied voltage.

The magnetizing force H is proportional to the current, and therefore, if the B-H curve for the iron were a straight line through the origin, and there were no hysteresis, the current curve would be sinusoidal and in phase with the flux density curve. The current curve would thus be in quadrature (lagging) with the applied voltage, and the power would be zero. This is to be expected, since under these conditions the area of the B-H loop for a complete cycle would be zero, and we are not taking eddy current loss into account. Actually, the B-H loop has a definite area, and therefore, the current lags less than 90° behind the voltage in order that there may be some real power to supply the hysteresis loss. Also, since the B-H curve is not a straight line, but is a curve of the well-known shape, the current is distorted from the pure

sinusoidal form. In order to be able to draw the current curve it is necessary to calculate B_{max} and then draw the hysteresis loop corresponding to this value of B_{max}. (*See* Fig. 391.) Knowing B_{max}, and remembering that the flux density is sinusoidal, the curve of flux density can be very easily drawn as shown. Taking a series of points on this curve, projecting them on to the hysteresis loop, and reading off the corresponding value of the magnetizing current I, the current wave can be plotted. This wave contains a pronounced third harmonic whose amplitude, relative to the fundamental, increases as the saturation of the iron is increased. If

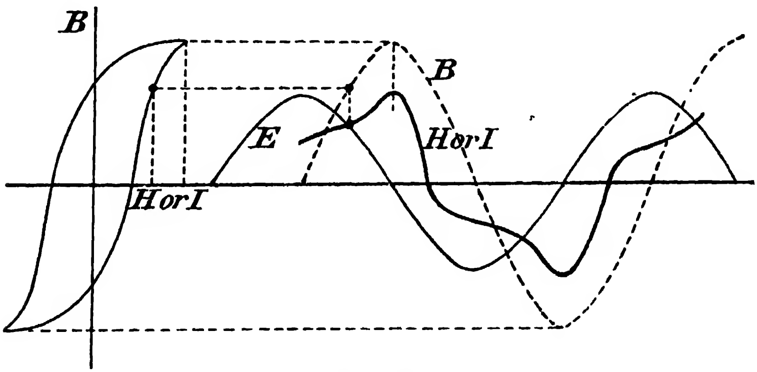


Fig. 391
Wave Form of Magnetizing Current

ordinates are drawn, the mean of the products of corresponding ordinates on the current and voltage curves over a complete cycle will give the true power. Also, the effective (R.M.S.) value of the current can be calculated. The equivalent sinusoidal current wave is that current of the same effective value, which, if acted on by the applied voltage, would produce the same power. If I_{eff} is the effective value, as determined from the ordinates, and φ is the angle of lag of the equivalent sinusoidal wave, then

$$\text{Average power} = E_{eff} I_{eff} \cos \varphi$$

$$\therefore \varphi = \text{arc.} \cos \left(\frac{\text{average power}}{E_{eff} I_{eff}} \right)$$

The curve of the equivalent current wave thus leads the curve of flux density by an angle ψ

$$\text{where } \psi = 90 - \varphi$$

The angle ψ is called the "Hysteric angle of lead."

It will be seen that the magnetizing current of a transformer will not be of pure sinusoidal form because of the hysteresis loss in the

core; hence, the assumption of a sinusoidal form as on page 263 is only approximate.

The distortion produced by magnetic saturation is such that the current wave possesses a very pronounced third harmonic. This effect can be made use of to produce a triple frequency current from a three-phase supply as follows. A transformer D (Fig. 392) has its primary winding in three separate parts, which are connected each in series with one of the chokers A, B, and C, to a three, phase supply. These chokers are designed to be very highly saturated, so that the primary currents each contain a large third harmonic. Now, while the mutual phase difference of the fundamentals is 120°, that of the harmonics is zero. As a result, there is no flux of supply frequency in the core of D, but there is a flux of three times normal frequency. This induces an E.M.F. of triple frequency in the secondary winding of D.

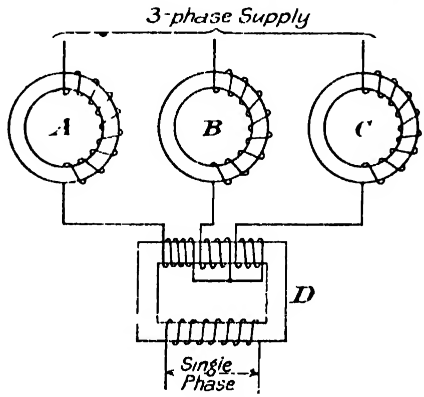


FIG. 392
TAYLOR'S METHOD OF TREBLING THE FREQUENCY

This method is sometimes used when a small amount of power for lighting is required from a three-phase low-frequency supply. A low frequency, although suitable for power work, is unsuitable for lighting, especially with metal filament lamps.

The above discussion of the effect of saturation of the iron applies to a circuit having a steady voltage of sinusoidal wave form applied to it. Now take the case of a coil with iron core, whose winding is connected in series with a circuit which is taking a sinusoidal current from the supply and whose impedance is so great that the introduction of the coil will not disturb this current. Then the curve of flux against time will no longer be sinusoidal, but will be flat-topped as is shown in Fig. 393. The flux wave is here obtained by projecting from the sinusoidal current wave on to the magnetization curve. Now the voltage induced in the coil, and therefore the voltage drop (neglecting the effect of resistance) is proportional to the rate of change of flux, the voltage drop curve thus being of the same form as the slope of the flux curve. This shows that the drop of volts is not sinusoidal, but is a peaked curve, the peaks being exceptionally pronounced when the iron is strongly saturated.

7. **Wave Form of an Alternator.** The terminal voltage is the

vector sum of the E.M.F.s in the individual conductors. Hence, to obtain a sinusoidal terminal voltage it is necessary that these individual E.M.F.s should be as nearly sinusoidal as possible. Each of these E.M.F.s has a wave form which is an exact reproduction of the distribution of flux round the air gap, so that the

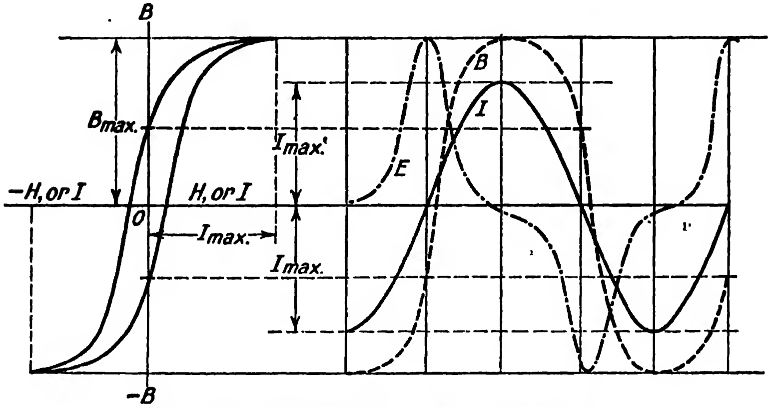
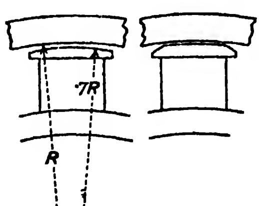


FIG. 393. WAVE FORM OF VOLTAGE DROP

first essential for a good wave form is that this flux distribution should be as nearly sinusoidal as possible. There are three methods of obtaining this—

(*a*) ROUNDING OR CHAMFERING THE POLE FACE (Fig. 394). This makes the air gap at the middle smaller than at the pole tips.



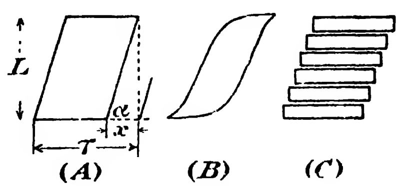


FIG. 394 FIG. 395

SHAPES OF ALTERNATOR POLE SHOES

Theoretically, the width of the air gap should be a sinusoidal function of the angular distance from the centre; but as this is not practicable, it is usual to make the radius of curvature of the pole face about 0·7 of the air gap radius.

(*b*) SKEWING THE POLE FACE. This causes the active length of conductor to vary from zero at the pole tip to a maximum at the centre. Theoretically, the axial length of the pole face should be a sinusoidal function of the angular distance from the pole centre,

as in Fig. 395(B). This is obviously impracticable, and therefore, the rhomboidal shape of Fig. 395(A) is adopted. Generally, the pole face is built up of laminations, and to enable stampings of the same shape to be used, the pole face is built up of separate packets of laminations, as in Fig. 395(C).

The pole pitch $\tau = \frac{\pi D}{p}$; where D = diameter.

Let β = ratio of pole span to pole pitch,

$$\therefore \quad \text{Distance } x = \frac{\pi D}{p}(1 - \beta)$$

and the inclination of the pole edge is given by

$$\tan a = \frac{L}{x}$$

An average value for the ratio β is 0·65 to 0·7

(c) USE OF A CYLINDRICAL ROTOR WITH DISTRIBUTED FIELD WINDING. This method is now invariably adopted with turbo-alternators. With a salient pole field system the flux distribution is approximately rectangular, as in Fig. 396(A), and it therefore possesses a very pronounced third harmonic.

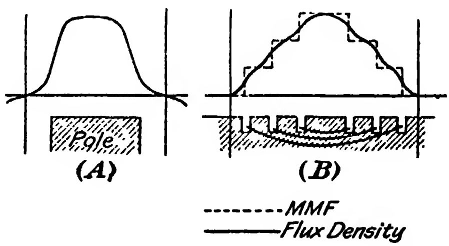


FIG. 396

COMPARISON OF SALIENT POLE AND CYLINDRICAL ROTORS

With a distributed field winding and a non-salient pole rotor, as in Fig. 396(B), the M.M.F. distribution is stepped as shown dotted. Because of fringing, the flux distribution rounds off the corners, thus giving a rough sine wave for the flux density.

8. **Elimination of Harmonics by Special Arrangements of the Stator Windings.** (a) USE OF SEVERAL SLOTS PER POLE PER PHASE. If a number of non-sinusoidal E.M.F.s having a small mutual phase difference act in series, their resultant will, as shown by Fig 397, be much more nearly sinusoidal than the components. Hence, if the winding is arranged so that the E.M.F.s in individual conductors in each phase have a slight phase difference, this will diminish the harmonics in the terminal voltage. This is done by adopting distributed instead of concentrated windings.

(b) THE USE OF FRACTIONAL PITCH WINDINGS. Generally, alternator windings are arranged with a whole number of slots per

pole per phase. If the number is fractional, e.g. $3\frac{1}{2}$ slots per pole per phase, the winding is called a "fractional pitch" winding. With an ordinary distributed winding, the E.M.F.s in the conductors of one group of coils are given a small phase difference, but the successive groups occupy identical positions relative to the poles. With fractional pitch windings, the various groups do not occupy identical positions with respect to the poles, with the result that phase differences are produced between groups of coils forming one phase, as well as between individual coil sides.

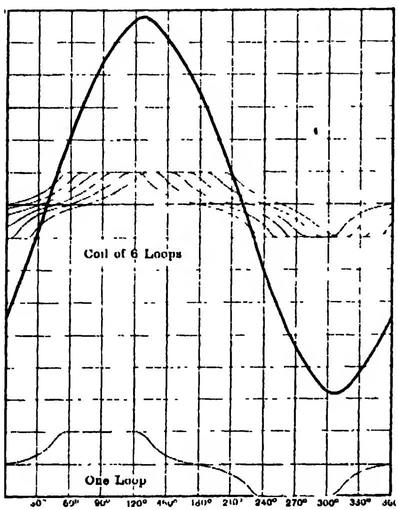


FIG. 397

TOTAL E.M.F. INDUCED IN A MULTI-TURN DISTRIBUTED COIL

(c) THE USE OF SHORT CHORD WINDINGS. A "short chord" winding is one in which the coil width is less than the pole pitch. There is thus a phase difference between the E.M.F.s in the two coil sides. By the use of such coils it is possible to eliminate one particular harmonic. Thus, if the distance x (Fig 398) is $\frac{1}{n}$th of the pole pitch, then the n^{th} harmonic will be eliminated, because the n^{th} harmonics in the E.M.F.s in the two coil sides will be in phase opposition.

9. **Suppression of Tooth Ripples.** As a pole moves past the

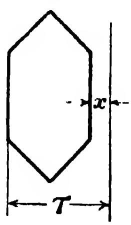


FIG. 398

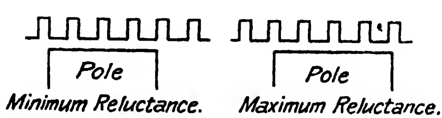


FIG. 399

VARIATION IN AIR GAP RELUCTANCE

stator teeth the configuration of the air gap is altered; its reluctance passing through a series of maximum and minimum values as shown in Fig. 399. These variations in reluctance set up a

stationary wave of magnetism, one cycle of which corresponds to one slot pitch.

Let a = number of slots per pole.

$\therefore\ 2a$ = number of slots per pair of poles.

Hence, one pair of poles corresponds to $2a$ cycles of this stationary wave, whereas it only corresponds to one cycle of the fundamental E.M.F. The frequency of the stationary wave is therefore $2af$.

Now we have seen that a stationary wave can be split up into two travelling components which rotate in opposite directions. Hence, their angular velocities relative to the field are—

$$(2a + 1)\omega \text{ and } (2a - 1)\omega,$$

ω being the synchronous angular velocity. As a result, they set up harmonics of frequencies $(2a + 1)f$ and $(2a - 1)f$.

EXAMPLE. If there are three slots per pole per phase $a = 9$, and the harmonics due to tooth ripples are the $(2 \times 9 + 1)$th and $(2 \times 9 - 1)$th, i.e. the 19th and 17th.

In order to suppress the tooth ripples it is necessary to reduce the variations of reluctance of the air gap to a minimum. This can be done—

(*a*) By making the pole span a whole number of slot pitches.

(*b*) By employing semi-enclosed or totally enclosed stator slots, as shown in Fig. 400. The placing of the windings in these slots is obviously more difficult than with open slots, and for this reason totally enclosed slots are rarely used except for low voltage bar windings, in which each slot contains one conductor only.

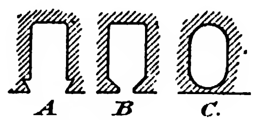


FIG. 400
SHAPES OF SLOTS

10. **Effect of Wave Form on Iron Losses.** (*a*) HYSTERESIS LOSS. Consider a transformer having a winding of N turns and a core of cross section A.

Then instantaneous applied voltage

$$e = N \cdot \frac{d\Phi}{dt}$$

$$= NA\frac{dB}{dt}$$

$$\therefore\ e \,.\, dt = NA \quad dB$$

Hence, the half area of the voltage wave (Fig. 401)

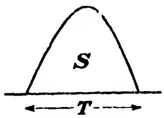


FIG. 401

$$\frac{S}{2} = \int_0^T e \, . \, dt$$

$$= NA \int^{B_{max}} dB = NAB_{max}$$

so that S is proportional to B_{max}. But the hysteresis loss

$$W_h \propto B_{max}^{1.6}$$

$$\therefore \; W_h \propto S^{1.6}$$

Now S is proportional to the average value of the applied voltage and

$$E_{av} = \frac{E_{eff}}{\text{Form factor}}$$

$$\therefore \; W_h \propto \left(\frac{E_{eff}}{\text{Form factor}}\right)^{1.6}$$

$$\propto \left(\frac{1}{\text{Form factor}}\right)^{1.6}$$

if E_{eff} is constant.

A peaked wave has a larger form factor than a sinusoidal wave, and consequently gives a smaller hysteresis loss for a given effective voltage. On the other hand, a flat-topped wave has a smaller form factor and therefore gives a greater hysteresis loss.

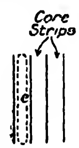


FIG. 402

(*b*) Eddy Current Loss. Let R = resistance of a local eddy current path (Fig. 402), and let e be the voltage acting round the path. Then power in the path = e^2/R. Now e is induced by the voltage E applied to the winding, and therefore, e_{eff} is proportional to E_{eff}.

Hence, eddy current loss is proportional to

$$\frac{E_{eff}^2}{R}; \text{ i.e. to } E_{eff}^2$$

Hence, so long as E_{eff} is constant, the eddy current loss is constant, and is independent of the wave form.

11. Effect of Wave Form on the Operation of a Synchronous Motor or on Two Alternators in Parallel. When two alternators

are paralleled on the same bus-bars, their E.M.F.s are in opposition with respect to the local path shown dotted in Fig 281, and therefore, one machine acts as a synchronous motor relative to the other. Suppose the two E.M.F.s are sinusoidal, equal in magnitude and in exact phase opposition. Then there will be zero current circulating between the two machines. Suppose now that one wave form is sinusoidal, while the other possesses a harmonic, say, the third. Then although the fundamentals will be in opposition, the harmonic will have no E.M.F. of the same frequency to oppose it, so that it will produce an idle current round the local path.

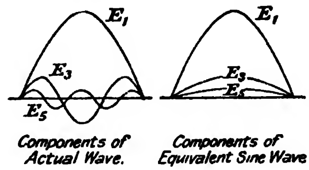


FIG 403

If both waves possess a harmonic and the wave forms are identical, then, if the fundamentals are in phase opposition, so also will be the harmonics. Thus, no circulating current, either of fundamental or harmonic frequency, will flow.

If both waves possess, say, a third harmonic, but the wave forms are not identical, then if the fundamentals are in phase opposition, the harmonics will not be in opposition. They will therefore combine to produce an idle circulating current.

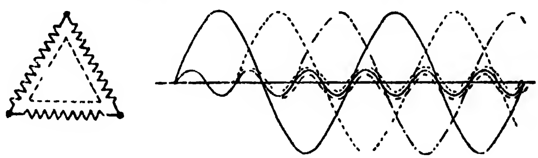


FIG. 404

Hence, for satisfactory parallel operation the wave forms of the different alternators should be as nearly identical as possible. It is for this reason that difficulty is sometimes experienced in running alternators of different makes in parallel, and that trouble is almost invariably experienced when it is attempted to parallel old slow-speed sets with modern turbine-driven sets.

12. **Effect of Wave Form on the Drop in a Transformer.** Let R and X be the effective resistance and reactance referred to the primary. Hence

$$\text{Drop} = I \times \sqrt{R^2 + X^2}$$

Split up the actual current wave into its various components and also split up the equivalent sine wave as shown in Fig. 403. In the latter the components will be of the same amplitude as in the former, but their frequencies will be that of the fundamental.

The drops produced by the various components can therefore be tabulated as follows—

Component.	*Drops in Volts.*	
	Actual Wave.	Equivalent Sine Wave.
Fundamental . .	$I_1 \times \sqrt{R^2 + S^2}$	$I_1 \times \sqrt{R^2 + S^2}$
3rd harmonic . .	$I_3 \times \sqrt{R^2 + 9S^2}$	$I_3 \times \sqrt{R^2 + S^2}$
5th harmonic . .	$I_5 \times \sqrt{R^2 + 25S^2}$	$I_5 \times \sqrt{R^2 + S^2}$
*n*th harmonic . .	$I_n \times \sqrt{R^2 + n^2S^2}$	$I_n \times \sqrt{R^2 + S^2}$

Hence, for a sine wave and a complex wave of the same effective value, the complex wave will produce the greater total drop in volts.

13. **Harmonics in a Mesh-connected Alternator.** Relative to the local circuit round the mesh, as shown dotted in Fig. 404, the fundamentals balance one another, their vector sum being zero at every instant. Hence, with pure sine waves there will be no current through the windings unless the machine is actually delivering load. If a third harmonic is present, or a harmonic whose number is a multiple of three, then round the local path these harmonics in the three phases of the winding are all in phase with respect to this path, as shown by the sine wave diagram. Hence, a parasitic current consisting of the harmonics which are multiples of three will circulate round the local path. These harmonics will therefore be absent from the current in the line. The terminal voltage wave obviously possesses all the harmonics produced in one phase of the winding.

14. **Harmonics in a Star-connected Alternator.** The line voltage is the vector difference of the voltages in the two phases joined to it. Now, there is a mutual phase difference of 120° between the fundamental voltages in the three phases, and therefore, the third harmonic voltages in the three-phase windings are all in phase, as shown in Fig. 405. The vector differences of these harmonics taken in pairs are therefore zero. Thus, the harmonics which are multiples of three disappear from the line voltage. If a three-phase four-wire system is used, or if two alternators working in parallel both have their star point earthed, then triple frequency currents will flow along the circuit between the star points. For this reason, when a number of star-connected alternators are working in parallel, it is usual to earth the star point of only one of them.

15. **Experimental Determination of Wave Form.** It is possible to determine the wave form of an alternator by a "step-by-step"

process, by means of a revolving contact connected to one point of the armature winding. This method is now obsolete except for instructional purposes, having been superseded by the oscillograph. The following description refers to the Duddell oscillograph. This instrument is essentially a permanent magnet galvanometer in which the moving coil is in the form of a phosphor bronze loop, as shown in the scheme of Fig. 406. The loop carries a very light mirror M, and tension is applied as shown. This loop has an exceptionally small natural period of oscillation of its own (about $\frac{1}{2000}$ to $\frac{1}{10000}$ sec.) and therefore, when an alternating current is

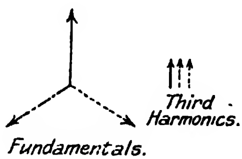


FIG. 405

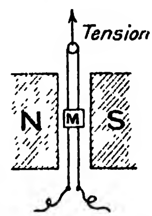


FIG. 406
PRINCIPLE OF DUDDELL OSCILLOGRAPH

passed through it, it can respond instantly to the changes in magnitude of the current. The deflection of the mirror at any instant is thus proportional to the instantaneous value of the current through the "vibrator" at that instant. The motion of the loop is damped and rendered aperiodic by immersing the loop and mirror in an oil bath It is obvious that if a cinematograph film is moved at a uniform speed in a direction at right angles to the motion of the beam of light reflected from the mirror, this beam being at the same time focused on the film, a photographic record of the wave form will be obtained. Very often it is necessary to visualize the wave form on a screen. To do this it is necessary to impart to the beam of light reflected from the mirror a motion at right angles, this motion being such that the deflection is proportional to the time. To accomplish this the beam is reflected from M on to a long rocking mirror N (Fig. 407), and is then focused by a cylindrical lens L on to a screen. The mirror N is rocked by means of a face cam C, driven by means of a small synchronous motor. It is this motor which gives the deflection proportional to the time, since the motor speed is constant and is an exact multiple of the speed of the alternator whose wave form is required. At the end of one revolution of the cam, the mirror N has to be returned very quickly to the starting point, and in order not to confuse the diagram, the light during this return motion is cut off by means of a vane V.

It is usual to provide the oscillograph with two vibrators side

by side, so that both current and voltage waves can be determined at the same instant. The voltage vibrator is connected in series with a high non-inductive resistance, while the current vibrator, with a certain amount of series resistance, is connected across a non-inductive shunt. It is necessary to be careful in making the connections to the vibrators, otherwise the line potential may be thrown across the two vibrators, with the consequent danger of their sparking across to one another and so being destroyed. Fig. 408 shows the correct and incorrect methods of making the connections.

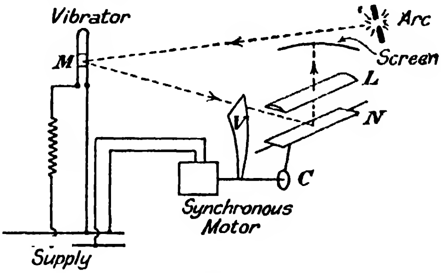


FIG. 407

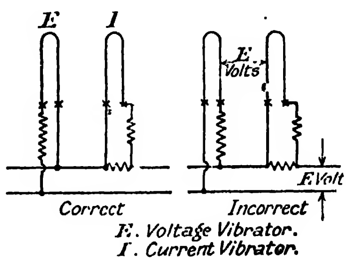


FIG. 408

16. The Cathode-Ray Oscillograph. When a high unidirectional P.D. is maintained between two electrodes in a highly evacuated tube, a stream of electrons passes from the cathode to the anode. As each of these has a negative charge, the stream of electrons is equivalent to an electric current, and consequently if a magnetic field is applied transversely to the stream, a deflection, whose direction is given by the left-hand rule, will take place. If the electron stream is brought to a focus at a fluorescent screen then the application of the magnetic field will produce a deflection of the light spot on the screen. Similarly, if the stream passes between a pair of plates between which a P.D. is maintained, a deflection will be obtained. In the first case the deflection is proportional to the strength of the magnetic field, while in the second case it is proportional to the strength of the electrostatic field.* If the magnetic field is produced by a pair of current-carrying coils placed on either side of the tube, then the deflection will be proportional to the current.

The application of alternating currents or P.D.s will result in the spot of light being drawn out in a line, and, owing to the very small inertia of the electrons, the deflection at any instant will be proportional to the current, or voltage, at that instant. In other words the appliance can be used as an oscillograph. One type of tube is illustrated in Fig. 409, from which it will be seen that a pear shape

* These phenomena are explained at length in physics textbooks; see, for example, *Electricity and Magnetism for Advanced Students*, by Starling.

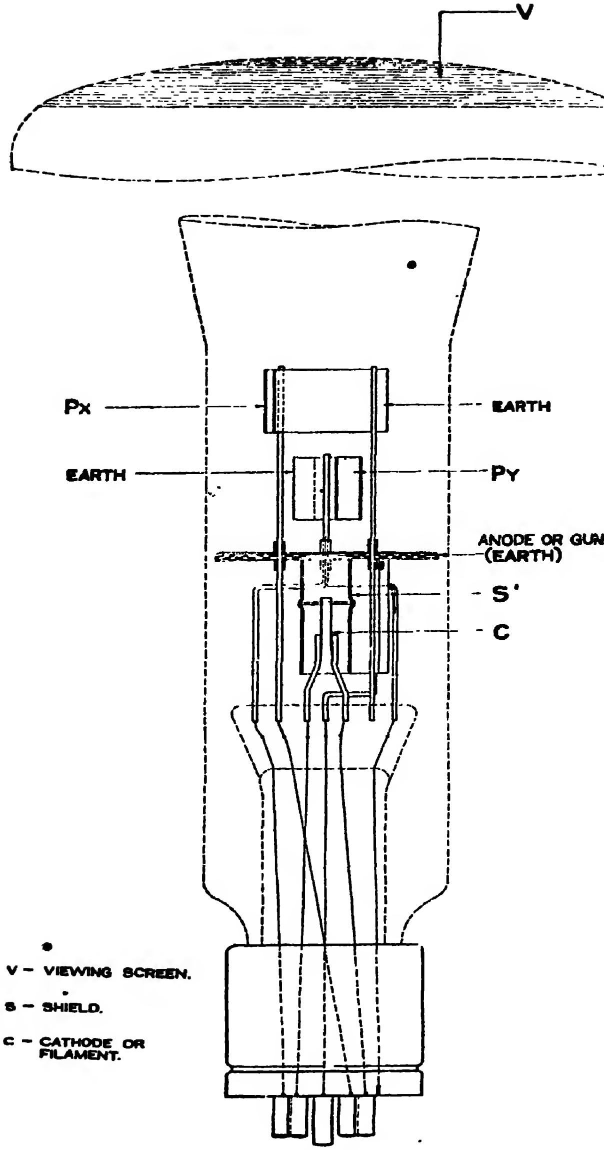

V
Px
EARTH
EARTH
PY
ANODE OR GUN
(EARTH)
S'
C
V – VIEWING SCREEN.
S – SHIELD.
C – CATHODE OR FILAMENT.

FIG. 409

is adopted, the wide end being coated on the inner surface with a fluorescent material.

The cathode is a heated filament C, and the electron stream from C first passes through a screen S, and then through an anode disc or "gun." It is the P.D. applied between the gun and the filament which produces the electron stream. The screen S protects the filament from bombardment by a stream of positive particles travelling in the opposite direction to the electron stream. The electrons issue from the anode as a fine pencil of rays, and on their way to the viewing screen they pass two pairs of deflecting plates P_x and P_y. The P_x plates are a little higher up the tube than the P_y plates and the planes of the P_x plates are perpendicular to those

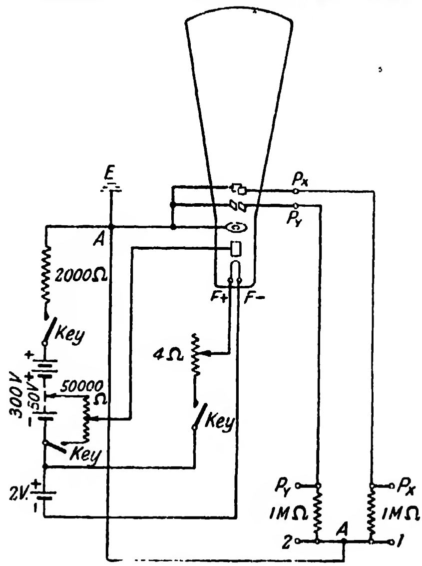


FIG. 410

of the P_y plates. The standard connection scheme for the type of tube just described is given in Fig. 410, and with it voltage deflections can be obtained by applying a P.D. to either the P_x or P_y pairs of plates. Current deflections, as explained previously, are produced by passing the current through a pair of coils placed one on either side of the tube at about the same level as the deflecting plates.*

In order to obtain a wave trace, as distinct from a straight line on the viewing screen, it is necessary to give the electron stream a deflection at right angles, exactly as with the Duddell oscillograph, but in this case it must be given electrically and not optically. For

* See also *Cathode-Ray Oscillography*, by MacGregor-Morris and Henley.

this purpose an auxiliary appliance, called the linear time base or sweep circuit, is necessary. Fig 411 shows such a circuit. P_e is a pentode and G a gas-filled triode (e.g. a thyratron). P_1 and P_2 are potential dividers, R_1 (about 50,000 ohms) is a resistance in the grid circuit of the triode, and R_2 (about 500 ohms) is a limiting resistance in its anode circuit. R_3 (about 2 megohms) and C_2 (about 2 microfarads) are for the purpose of isolating the D.C. potentials so that these do not produce a steady deflection of the spot.

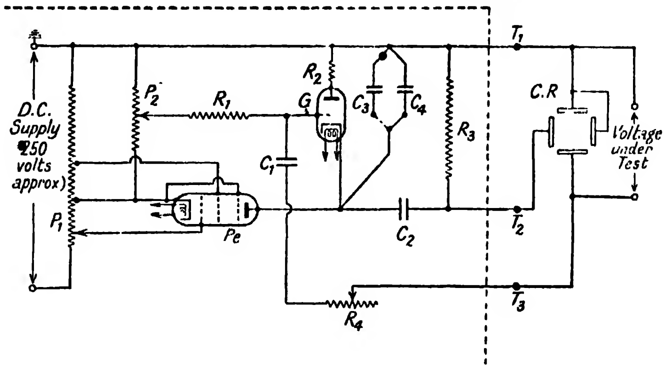


FIG. 411

The action of the circuit is as follows: when the D.C. supply is first switched on, the condenser C_4 (or C_3) acts momentarily as a short circuit, and the full D.C. voltage is applied to the pentode. C_4 charges (at almost constant current, due to the shape of the anode current-anode volts characteristic of the pentode) and the voltage across it rises until it is sufficient for the triode G to discharge. The voltage required for discharge depends upon the grid bias of G, and is therefore controlled by the potential divider P_2. Again, the charging current of C_4 depends upon the grid bias of the pentode and is controlled by the potential divider P_1. Now the rate of sweep on the oscillograph obviously depends upon the rate of increase of voltage across the time-base pair of deflecting plates, i.e. upon the rate of increase of the voltage across C_4. Let this rate of increase be dv/dt, then

$$\frac{dv}{dt} = \frac{d}{dt}\left(\frac{Q}{C_4}\right) = \frac{1}{C_4} \cdot \frac{dQ}{dt} = \frac{i}{C_4}$$

where Q is the quantity of electricity supplied to C_4 and i is the charging current. Since i is constant, the rate of sweep is constant and therefore the time base is linear.

The rate of sweep can be controlled by P_1 and the amplitude of the sweep by P_2. The condensers C_3 and C_4 are of the order of 1 microfarad and 0·1 microfarad respectively.

In order to enable the sweep circuit to remain in synchronism with the voltage or current under examination, a synchronizing branch circuit consisting of C_1 (about 0·1 microfarad) and R_4 (about 2 megohms) is added. This enables the sweep frequency to be locked to some sub-multiple of the test frequency, by applying to the grid circuit of the triode a small alternating voltage of this frequency.

17. **Effect of Harmonics on the Form of the Rotating Field Produced by a Three-phase Winding.** We have seen that the third harmonics neutralize one another, and we will therefore determine the effect of the fifth and seventh. The instantaneous E.M.F. in Phase I can be represented by the expression

$$e_1 = E_1 \sin \omega t + E_5 \sin (\omega t - \varphi_5) + E_7 \sin (\omega t - \varphi_7)$$

neglecting harmonics of orders higher than the seventh, and neglecting the third for the reason just stated. Hence, for the instantaneous E.M.F.s in Phases II and III, we have

$$\begin{aligned} e_2 &= E_1 \sin (\omega t - 120^\circ) + E_5 \sin 5(\omega t - \varphi_5 - 120^\circ) \\ &\qquad + E_7 \sin 7(\omega t - \varphi_7 - 120^\circ) \\ e_3 &= E_1 \sin (\omega t - 240^\circ) + E_5 \sin 5(\omega t - \varphi_5 - 240^\circ) \\ &\qquad + E_7 \sin 7(\omega t - \varphi_7 - 240^\circ) \end{aligned}$$

$$\begin{aligned} \text{Now } \sin 5(\omega t - \varphi_5 - 120^\circ) &= \sin \{5(\omega t - \varphi_5) - 240^\circ\} \\ \sin 7(\omega t - \varphi_7 - 120^\circ) &= \sin \{7(\omega t - \varphi_7) - 120^\circ\} \\ \sin 5(\omega t - \varphi_5 - 240^\circ) &= \sin \{5(\omega t - \varphi_5) - 120^\circ\} \\ \sin 7(\omega t - \varphi_7 - 240^\circ) &= \sin \{7(\omega t - \varphi_7) - 240^\circ\} \end{aligned}$$

$$\begin{aligned} \therefore \quad e_1 &= E_1 \sin \omega t + E_5 \sin (\omega t - \varphi_5) + E_7 \sin (\omega t - \varphi_7) \\ e_2 &= E_1 \sin (\omega t - 120) + E_5 \sin \{5(\omega t - \varphi_5) - 240^\circ\} \\ &\qquad + E_7 \sin \{7(\omega t - \varphi_7) - 120^\circ\} \\ e_3 &= E_1 \sin (\omega t - 240) + E_5 \sin \{5(\omega t - \varphi_5) - 120^\circ\} \\ &\qquad + E_7 \sin \{7(\omega t - \varphi_7) - 240^\circ\} \end{aligned}$$

The sequence of the phase angles for the fundamental voltages in the three phases is 0°, 120°, and 240°. The sequence of the seventh harmonics is also 0°, 120°, and 240°, showing that the phase rotation of the seventh harmonics is the same as that of the fundamentals. On the other hand, the sequence of the phase angles for the fifth harmonic voltages in the three phases is 0°, 240°, and 120°. Hence, the phase rotation of this harmonic is opposite to that of the fundamental, and the seventh harmonic. The effect of this from the practical point of view has already been considered on page 347 in connection with the starting of squirrel-cage induction motors.

EXAMPLES ON CHAPTER XXIII.

(1) An alternating voltage given by 100 sin 314t + 10 sin 1,570t is applied to the terminals of a condenser having a capacity of 1 microfarad. Find the expression for the current flowing through the condenser. (C. and G.)

Ans.—·0314 cos 314t + ·0157 cos 1,570t.

(2) Calculate the form factor of a triangular wave. If a sinusoidal voltage and then a triangular voltage wave are applied in turn to a transformer, the effective values and frequencies being the same, calculate the ratio of the hysteresis losses.

Ans.—Form factor = 1·15. Hysteresis loss with triangular wave is 0·94 of loss with sinusoidal wave.

(3) An electromotive force, $e = 2{,}000 \sin \omega t + 400 \sin 3\omega t + 100 \sin 5\omega t$, is applied to a circuit consisting of a resistance of 10 ohms, a variable inductance, and a capacity of 30 microfarads arranged in series with a hot wire ammeter. Find the value of the inductance which will give resonance with the triple frequency component of the pressure; and estimate the readings on the ammeter and on a hot wire voltmeter connected across the supply when resonant conditions exist. $\omega = 300$. (London Univ., 1922.)

Ans.—·041 henry; 1,440 volts; 31·6 amp.

(4) A current of 50 frequency, containing first, third, and fifth harmonics of crest values 100, 15, and 12 amp. respectively, is sent through an ammeter and an inductive coil of negligibly small losses. A voltmeter connected to the terminals shows 75 volts. What will be the current indicated on the ammeter, and what is the exact value of the inductance of the coil expressed in henrys? (C. and G., 1918.) *Ans.*—·0027 henry; 72 amp.

(5) A condenser of 1·5 microfarads capacity is supplied with a voltage having a wave form $e = 1000 \sin pt + 350 \sin 3pt + 270 \sin 5pt$, the frequency being 80 per second. Calculate the current taken as measured on an ammeter. If energy is being taken from the supply circuit at the same time, how do the harmonics affect the power factor? (London Univ., 1911.)

Ans.—1·06 amp.

(6) The wave-form of an alternator is found to differ from that of a pure sine function. Assuming this to remain unchanged, show how to calculate the wave-form of the current which the alternator would send (*a*) into a condenser, (*b*) through a choking coil. (London Univ., 1908.)

(7) A voltage represented by $e = 500 \sin \omega t + 10 \sin 5\omega t$, is applied to a circuit containing 1 ohm, 1 henry, and $\frac{1}{25}$ microfarad in series. Find the frequency for resonance with the 5th harmonic, and plot the voltage and current waves. *Ans.*—159.

(8) The connection between the magnetizing current and the flux for a particular alternating-current electromagnet is shown by the following table, which gives values for one-half of the magnetization loop, the complete loop showing the usual symmetry with respect to the axes—

Flux (kilolines) . .	0	30	75	120	150	179	176	165	138	102	60	0
Magnetizing current (amp.)	5·5	6·25	8	10·5	14·25	20·5	13·5	6·8	0	−3	−4·5	−5·5

Plot the loop on squared paper, and by its aid deduce and plot the wave-form of the magnetizing current when the flux follows a sine law, the amplitude of the flux being equal to the maximum value of the flux in the above magnetization loop. (London Univ., 1921.)

(9) Estimate the R.M.S. value of an alternating current of irregular wave shape in terms of the magnitudes of the fundamental component and the harmonics.

An alternating pressure is represented by $v = 1{,}000 \sin \omega t + 250 \sin 3\omega t + 200 \sin 5\omega t$. Estimate the reading which will be given by an electrostatic voltmeter connected to the circuit. (London Univ., 1924.)

Ans.—742 volts.

CHAPTER XXIV

PHASE ADVANCING

1. The power factors of the great majority of commercial loads are considerably less than unity. This is mainly because the induction motor is the most widely used of the various alternating-current motors, and this motor works at less than unity power factor, because of the wattless magnetizing current it draws from the line. Similarly, all transformers take a wattless magnetizing current. Electric lamps work practically at unity power factor, and therefore a system supplying a lighting load works at a much better power factor than one supplying power.

Rough average values for power factors under different conditions are as follows—

(a) System supplying lighting load only	0·9–0·95
(b) System supplying lighting and power, with the former predominating	0·8–0·85
(c) System supplying lighting and power, with the latter predominating	0·75
(d) System supplying power only	0·65–0·7
(e) Single-phase system supplying power only . . .	0·4–0·5

2. The Disadvantages of a Poor Power Factor.

(a) The amount of true power is less than the kVA capacity of the central station, with the result that, although the generators may be fully loaded from the point of view of current output, and therefore, of temperature rise, they will not be delivering their full load of true power. The same applies to the cables. Therefore, if the power factor can be increased, the earning capacity of the system will be increased, without installing any new plant. Thus, if the power factor is raised from 0·6 to 0·9, the earning capacity will be increased 50 per cent.

(b) In addition to the earning capacity of the system being lowered by their presence, wattless kVA actually cost something to produce. From the examination of the data of running costs for a large number of Italian stations, Professor Arno has deduced the following expression for the running costs of a station—

$$\text{Cost} = \text{constant} \times \int_0^t \left(\frac{2}{3}\,\text{kW} + \frac{1}{3}\,\text{kVA}\right) dt$$

where t is the time. It will be seen that the cost has two components. One, the cost of the true energy (kilowatt-hours) delivered, and the other, the cost of the kilovolt-ampere-hours.

(*c*) A poor power factor causes a large drop of volts in the alternators, partly because of the armature synchronous reactance, and partly because of the demagnetizing effect of armature reaction. As a result the excitation on low power factors has to be much greater than on high power factors, in order that the terminal voltage may be maintained at the proper value. This necessitates larger exciters. In some cases when the power factor of the system has been lower than anticipated when ordering the alternators, the necessary overloading of the exciters and consequent overheating of the alternator field have caused serious trouble, and this in spite of the fact that the alternators were not delivering their maximum possible load of true power.

(*d*) The voltage regulation of the transmission line is also very bad, with the result that expensive appliances may have to be used in order to keep up the voltage at the far end of the line. This is specially important on circuits supplying induction motors, since the torque of these motors is proportional to the square of the applied voltage, and therefore, if there is a large sudden fall in voltage, the motors may fall out of step.

3. **Methods of Obtaining a Good Power Factor.** The obvious method is to use, whenever possible, apparatus which has a good power factor. This is only possible with synchronous motors and rotary convertors in which any desired power factor can be obtained by adjustment of the excitation, or with compensated induction motors of the types described in Chapter XXII.

A second method is to attempt to raise the power factor of the system as a whole by installing phase-advancing apparatus either at the power station or across the transmission line near the load centre. The latter is obviously the better place, since it relieves both generators and transmission line, whereas in the former only the generators are relieved. In such a case it is usual to use an over-excited synchronous motor for the phase advancer, in which capacity it is generally called a "synchronous condenser." It should be noted that the cost of these machines is so high that, as a rule, it is not possible to relieve a great amount of wattless kVA by this method, the main object of installing the motor being to improve the voltage regulation of the system. The motor can, of course, be used to do a certain amount of mechanical work, such as driving a D.C. generator in a sub-station, but in such a case it cannot be worked at such a low leading power factor as when it is running on no load. Hence, if the synchronous motor is to be of the greatest amount of benefit to the system, from the point of view of improving the power factor and voltage regulation, it must be run idle.

A method of keeping the voltage constant at the receiving end of a transmission line is to connect an idle-running synchronous motor across the line at that end, and to have automatic regulation

of the excitation in such a way that with full load in the line the excitation is a maximum, and with the line unloaded, the excitation is a minimum. Hence, with the line fully loaded, the motor delivers the maximum kVA of leading current and thereby compensates for the lagging current taken by the load, this current being the main cause of the drop of voltage in an A.C. transmission line. When the line is unloaded the motor draws wattless lagging kVA from it, thus preventing the line voltage from rising.

Synchronous motors used for power factor correction have a much smaller air gap than A.C. generators, in order to give the maximum kVA rating for a given-sized frame. These motors are not a commercial proposition for a small amount of kVA correction, so that they are only used to improve the power factor of a system as a whole, or of a large capacity industrial installation They are thus never used to correct the power factor of individual induction motors.

A third method is to correct the power factor of individual motors. In such a case, large motors are usually supplied with their own phase advancer, whereas a group of small motors can be looked after by installing a battery of static condensers across the supply terminals. Modern condensers used for this service are very efficient, the dielectric losses in them being less than 0·5 per cent of the kVA capacity of the condenser. Very little error is therefore made by assuming that they take a current which leads the applied voltage by exactly 90°.

4. Use of Condensers. The wattless current of the motor per phase is $I \sin \varphi$. Hence, if a bank of condensers is connected across the motor terminals, the capacities being such that they take from the line a current of $I \sin \varphi$ per phase, the line will work at unity power factor when the motor current per phase is I. For motor currents less than I, the line will be over-compensated.

5. Advantage of Phase Advancers. In the plain induction motor the magnetizing current is taken in at the stator, and it is therefore of line frequency. Suppose the magnetizing current could be supplied through the rotor; then it would be a slip frequency instead of a line frequency current, and the necessary kVA would be very considerably reduced.

For example, if a motor takes 20 kVA of wattless current when operating in the ordinary way, it will, with a slip of, say, 2 per cent, take only 0·4 kVA of wattless current when the magnetizing current is supplied through the rotor. The voltage required is also very small because of the low frequency.*

6. The Kapp Vibrator. This consists essentially of three small motors whose fields are separately excited from an auxiliary D.C. supply, and whose armatures are connected in the rotor circuit of

* Figs. 369 and 371 refer to induction motors provided with external phase advancers of the expedor and susceptor type respectively.

the induction motor. The armatures thus carry the low-frequency currents induced in the rotor circuit, with the result that instead of rotating continuously they oscillate backwards and forwards with slip frequency. The rotor currents of large induction motors are very large and therefore the vibrator commutators are very large compared with the size of the armatures. The current per commutator, and therefore the size of the commutator, is made as small as possible by mesh-connecting the armatures, since with

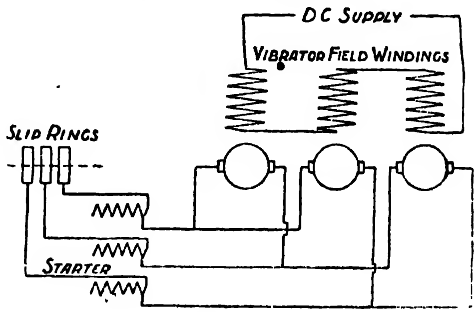


Fig. 412
Diagram of Connection for Kapp Vibrator

this connection the armature current is $1/\sqrt{3}$ of the current taken from each slip ring of the rotor. The scheme of connections is shown in Fig. 412. The vibrator is permanently in the rotor circuit, the motor being started up in the ordinary way by the three phase starter.

Since the flux is constant and not alternating, the torque on a vibrator armature is proportional to the armature current and in phase with the current. The torque is thus an alternating one and, because of the inertia of the armature, the angular speed lags 90° behind the torque, and therefore, 90° behind the current. The E.M.F. induced in the armature is also an alternating one, is zero when the speed is zero, and a maximum when the speed is a maximum. Actually, it is in phase opposition to the speed, that is, in quadrature, leading to the current. The reason is as follows. When the armature speed is increasing, power must be received by the vibrator in order to overcome the inertia. The vibrator can thus be regarded as motoring during this period, so that the E.M.F. induced in it must be a back E.M.F., i.e. opposite to the current. This condition is only fulfilled when the induced E.M.F. wave leads the current wave, as can easily be seen by reference to Fig. 413. The vibrator thus injects a leading E.M.F. into the rotor circuit, thus providing the necessary power factor correction. It will be seen that no mechanical power is supplied to the vibrator, and therefore, when the armatures are accelerating, electrical energy

is taken from the induction motor rotor, whereas when they are decelerating, electrical energy is returned to the rotor. The power curve is a sine function of twice the frequency of the current or E M.F., its average value being zero, exactly as in any electric circuit in which the current and E.M.F. are in quadrature. This shows that power is alternately taken from and returned to the rotor by the armatures of the vibrator.

When a phase advancer is connected in the rotor circuit of an induction motor, there are slight modifications made in the performance characteristics of the motor. Thus, the rotor copper loss

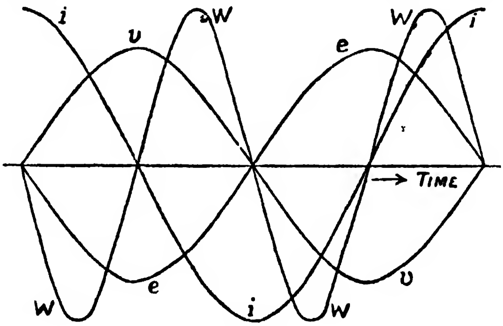


Fig. 413

Phase Relations Obtaining with Kapp Vibrator

i = *Current.* *v* = *Speed.* *e* = *E.M.F.* *w* = *Power output.*

is increased, but on the other hand the stator copper loss is reduced, because the stator no longer carries the magnetizing current There are also small losses in the phase advancer itself, the net result being a diminution in overall efficiency of perhaps 0·5 or 1·0 per cent. An important advantage is that the pull-out torque is increased. The effect on the slip depends upon the type of advancer used. The vibrator type increases the slip somewhat,

Load	*Power Factor.*	
	Without Vibrator.	With Vibrator.
¼	0·6	0·95
½	0·77	1·0
¾	0·85	1·0
Full	0·88	1·0
1¼	0·85	1·0

but as the slip of a large motor is exceptionally small, this is not very important. Certain advancers with rotating armatures can be arranged actually to reduce the slip.

The improvement in power factor effected by phase advancing is illustrated by the data on previous page for a 330 h.p. motor equipped with a Kapp vibrator.*

7. **Economical Limit of Power Factor Correction.** Although it is desirable to increase the power factor of a large motor or of the whole system to unity, from the point of view of the electrical operation of the system, it is not always a commercial proposition to do so. The maximum value to which the power factor can be economically raised depends upon the relative costs of generating and advancing plant.

Consider a system supplying a current per phase of I; let the original angle of lag be φ_1 and let this be reduced by the use of phase-advancing apparatus to φ_2. The vector diagram is shown in Fig. 414. Let the plant capacity in kVA be P. Then by power factor correction the true power of the system is raised from $P\cos\varphi_1$ to $P\cos\varphi_2$.

$\therefore$ Gain in true power $= P(\cos\varphi_2 - \cos\varphi_1)$ kW.

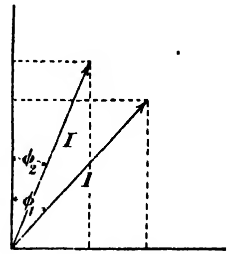


FIG. 414

The wattless kVA are reduced from $P\sin\varphi_1$ to $P\sin\varphi_2$, so that the kVA capacity of the phase-advancing plant must be equal to $P(\sin\varphi_1 - \sin\varphi_2)$. If the true power of the system were increased the same amount by installing extra generating plant, cables, etc., at a total cost of £a per kVA, the necessary expenditure would be

$$£\{aP(\cos\varphi_2 - \cos\varphi_1)\} \div \cos\varphi_1 \dagger$$

If the true power were increased by installing phase-advancing apparatus at £b per kVA, the necessary expenditure would be

$$£\{bP(\sin\varphi_1 - \sin\varphi_2)\} \ddagger$$

Hence, for economical correction, the value of the new angle of lag φ_2 is given by the expression**

$$£\{bP(\sin\varphi_1 - \sin\varphi_2)\} < £\{aP(\cos\varphi_2 - \cos\varphi_1)\} \div \cos\varphi_1$$

$$\therefore \frac{b}{a} < \frac{\cos\varphi_2 - \cos\varphi_1}{(\sin\varphi_1 - \sin\varphi_2)\cos\varphi_1}$$

* For other types of phase advancer, see Walker, *loc. cit.*, or Behrend, *loc. cit.*

† Assuming that the extra plant would work at the original power factor of $\cos\varphi_1$.

‡ It is here assumed for simplicity that the whole of the kVA intake of the phase advancer is wattless leading, as for example in the case of condensers.

** See also Bolton, *Electrical Engineering Economics*.

8. In order to illustrate the method of calculating the necessary advancer capacity to bring about a given amount of power factor correction, or, conversely, the change in power factor and true power effected by installing a given advancer capacity, it is advisable to take a numerical example.

Example. The load on the mains of a single-phase alternate-current supply system is 100 kW, at a power factor of 0·71, the current lagging behind the voltage. If phase-advancing apparatus is available for parallel connection, taking leading current at a power factor of 0·1, what must be its load in kVA if the power factor of the whole system is to be raised to (*a*) 0·8, (*b*) 0·9 and (*c*) 0·95? (London Univ., 1915.)

Since the methods of calculation for all three parts are the same, consider only part (*a*) The vector diagram is shown in Fig. 415. Assume a line voltage, say, 1,000 for convenience. The original current I_1 supplied to the load is represented by OA, its magnitude being

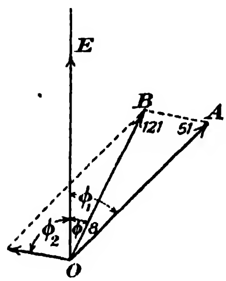


FIG. 415

$$I_1 = \frac{100{,}000}{1{,}000 \cos \varphi_1} = 141 \text{ amp.}$$

The angle φ_1 is very nearly 45°. The phase advancer will take a current OC leading E by an angle φ_2, where $\cos \varphi_2 = 0{\cdot}1$, i.e. $\varphi_2 = 84°$ approx. The resultant of OA and OC, namely, OB, gives the total line current after installing phase-advancing apparatus, and we know that the angle φ is given by $\cos \varphi = 0{\cdot}8$. Hence, $\varphi = 37°$ approx. We thus have for the angles at O, A, and B of the triangle OAB, 8°, 51°, and 121°; and we have

$$\frac{AB}{OA} = \frac{\sin 8°}{\sin 121°} = {\cdot}162$$

∴ Advancer current, $OC = AB = 141 \times {\cdot}162 = 23$ amp. approx. As we have assumed a line voltage of 1,000, the K.V.A. capacity of the advancer will be

$$\frac{1{,}000 \times 23}{1{,}000} = 23$$

Similarly for parts (*b*) and (*c*).

9. In paragraph 7 the capacity of the phase-advancing plant was deduced on the basis of constant current in the line, this method being called the constant kVA method. If condensers are used and no additional load is put on the system, the line current I_2 after correction will be less than the line current I_1 before correction. But as both currents have the same component in phase with E the power is the same both before and after correction, the method thus

being called the constant power method. Referring to Fig. 416, the wattless current before correction is

$$AD = I_1 \sin \varphi_1$$

and after correction is

$$BD = I_2 \sin \varphi_2$$

$\therefore$ Necessary wattless leading current to increase the power factor from $\cos \varphi_1$ to $\cos \varphi_2$, at constant power

$$I_c = AB = (I_1 \sin \varphi_1 - I_2 \sin \varphi_2)$$

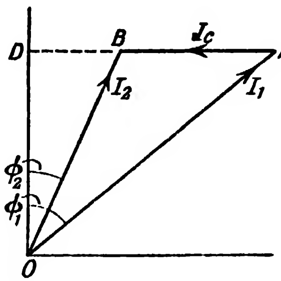


Fig. 416

Again, $$I_2 \cos \varphi_2 = I_1 \cos \varphi_1$$

$$I_2 = I_1 \frac{\cos \varphi_1}{\cos \varphi_2}$$

$$\therefore \; I_c = (I_1 \sin \varphi_1 - I_1 \cos \varphi_1 \tan \varphi_2)$$

The kVA of power-factor correcting plant is therefore given in this case by

$$P(\sin \varphi_1 - \cos \varphi_1 \tan \varphi_2)$$

Examples on Chapter XXIV.

(1) What are the disadvantages of a poor power factor in an alternating-current system? A single phase motor takes 20 amp. at 200 volts at a power factor of 0·8. Calculate the capacity of the condenser which, when shunted across the motor terminals, will bring the line current into phase with the voltage. Frequency of supply, 50.

Ans.—192 mf.

(2) Explain clearly the action of the reactive component in a synchronous machine. A slow-speed alternator and a turbo-alternator working in parallel supply 2,500 kW at a power factor of 0·8. If the turbo-alternator gives 1,000 kW at unity power factor, at what power factor must the slow-speed machine be working? (C. and G., 1921.)

Ans.—0·625.

(3) Explain, by means of vectors, how an over-excited rotary convertor can compensate for a lagging power factor. A generating station supplies power for the following: Lighting 100 kW, induction motor of 400 h.p. having a power factor of 0·8 and efficiency 0·93, a rotary convertor giving

1,000 amp. at 500 volts at an efficiency of 0·94. What must be the power factor of the rotary convertor in order that the power factor at the supply station may be unity? (C. and G., 1921.)

Ans.—0·91 leading

(4) Define the term "power factor" as applied to alternating-current circuits. What power factor does one usually find for the following: (*a*) A water resistance, (*b*) the primary current of an alternating-current transformer when the secondary is unloaded, (*c*) a hand-feed alternating-current arc? Illustrate each case by means of a rectangular co-ordinate diagram of wave forms. Assume the wave of the alternator electromotive force to be sinusoidal.

(5) A central station of 1,000 kVA capacity supplies a load whose power factor is 0·8. How many wattless leading kVA must be supplied by phase-advancing apparatus in order to raise the power factor of the station to 0·95: (*a*) the true power supplied by the station remaining constant, (*b*) the kVA supplied by the station remaining constant?

Ans.—(*a*) 336; (*b*) 400

(6) Describe a type of condenser used to compensate for lagging currents in a power supply system. A 2,000 volt, 50 cycle motor installation has a maximum load of 300 kVA at a power factor of 0·6. Calculate the capacity of a condenser to raise the power factor to 0·95 at maximum load. What will the power factor be when the load is halved? (C. and G., 1922.)

Ans.—Assuming a three-phase system with the condensers delta-connected, capacity per phase = 48 mf. Power factor at half-load, 0·83 leading.

(7) Explain, by means of the vector diagram of the motor, why it is that an over-excited synchronous motor cannot supply so much wattless leading kVA when it is doing mechanical work as when it is running idle.

(8) What methods would you recommend for the improvement of the power factor of (*a*) a central station and cable network, (*b*) a large induction motor, (*c*) a group of small motors? Give reasons for the various methods adopted.

(9) A central station is working at a power factor of 0·7. Deduce an expression for the necessary reduction in wattless kVA to improve the power factor to any given value. Hence, plot a curve with values of the reduction as ordinates against the new power factor as abscissae. Draw conclusions from the shape of the curve.

PART THREE
MEASURING INSTRUMENTS

CHAPTER XXV

ELECTRICAL MEASURING INSTRUMENTS

1. **Classification.** Measuring instruments are conveniently classified in terms of the principles upon which they work.

(*a*) Electromagnetic moving-iron instruments.
(*b*) Electromagnetic moving-coil instruments.
(*c*) Electromagnetic dynamometer type instruments.
(*d*) Hot-wire instruments.
(*e*) Electrostatic instruments.

In addition to these there are the shaded pole and rotating field instruments, the latter having been described previously, and which are suitable for alternating-current measurements only. The voltameter, which depends upon the chemical effect of the current, can hardly be regarded as a measuring instrument.

Instruments can be sub-classified according to the nature of the control on the moving system. In deflectional instruments the usual forms of control are—

(*a*) Gravity.
(*b*) Torsion of a spiral spring.

In ampere-hour and watt-hour meters, the moving system rotates at a speed proportional to the current or power respectively, and the control here takes the form of electromagnetic braking produced by the rotation of an aluminium disc between the poles of a permanent magnet.

The best deflectional instruments are very dead-beat, this property being obtained by the addition, if necessary, of a damping device. The methods adopted are—

(*a*) Air friction.
(*b*) Eddy currents.
(*c*) Viscosity of liquids.

2. **Moving-Iron Instruments.** A typical instrument of this type is illustrated in Fig. 417. It consists of a coil W wound on a brass bobbin B, inside which two strips of soft iron M and F are set axially. F is fixed, but M is attached to the spindle S, which also carries the pointer P. Gravity control is used, the instrument being set so that when no current is flowing through W the pointer is at the zero and the iron strips M and F are almost touching. On passing current through W the two strips are magnetized in the same direction and therefore repel each other. The torque set up produces a deflection of the moving system and therefore brings into play a restoring torque due to gravity. The moving

system comes to rest in such a position that the deflecting and restoring torques are equal. With this class of instrument air damping is almost invariably used, being provided by the movement of a piston P_N in a cylindrical air chamber A.

Ammeters of this type have a coil wound with wire or strip of cross section suitable to the current to be carried, while voltmeters are wound with a large number of turns of fine wire. The number of ampere-turns required to produce a full scale deflection is about 300.

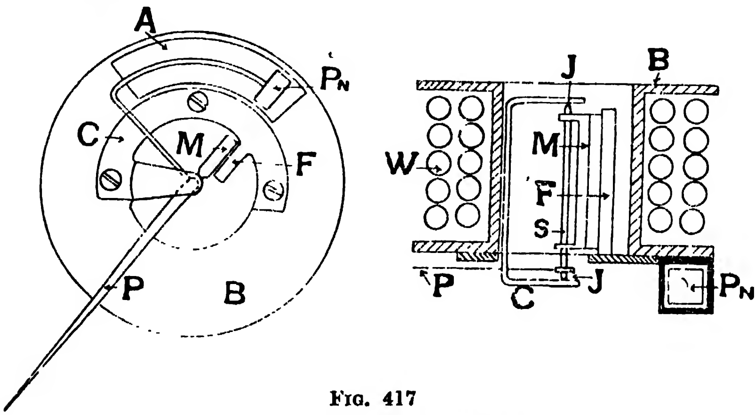


FIG. 417
MOVING SOFT-IRON AMMETER

These instruments can be used for either direct or alternating-current measurements. They are liable to errors due to three different causes.

(*a*) Hysteresis.
(*b*) Stray magnetic fields.
(*c*) Changes in resistance of the coil due to temperature changes.

The effect of hysteresis is to make the readings with falling current higher than with rising current, and also, to produce a small deflection when the current is again zero, because of retained magnetism.

The instrument can be largely screened from the effect of stray fields by the use of a cast-iron case, but this must be arranged so that the minimum flux from the moving iron can pass through the case, otherwise the hysteresis effect will be considerably increased. The error due to the third cause, which only applies to voltmeters, could be eliminated by using wire of negligible temperature coefficient, such as manganin, but such a coil would become very hot because of its high resistance and the small radiating surface. As a compromise, a coil of copper wire in series with a manganin series resistance is commonly used.

Recently a new iron called mumetal has been adopted for the

construction of moving-iron instruments; its hysteresis loss is so very small that this type of instrument can now be manufactured in the form of a precision instrument. It is equally accurate on direct- and alternating-current circuits, and for many purposes is replacing the more expensive moving-coil type.

In the instrument just described, the deflecting torque is proportional to the square of the flux density B, and if B were strictly proportional to the current the instrument would indicate the R.M.S. current (or voltage) on the scale used for direct current working. Actually, B is not proportional to I, and it is necessary to calibrate the scale if the instrument is to be used for both alternating- and direct-current work. Again, in the case of a voltmeter, the impedance of the coil to an alternating current is greater than its ohmic resistance, so that if correct with direct current, the instrument will read low with alternating current This error can be made very small by using a non-inductive series resistance of several times the resistance of the coil.

The scales of these instruments are uneven, being very crowded near the zero. Readings below $\frac{1}{10}$ of the maximum are therefore very unreliable, in fact, according to the specification of the Engineering Standards Committee, the useful range only commences at $\frac{1}{4}$ of the maximum.

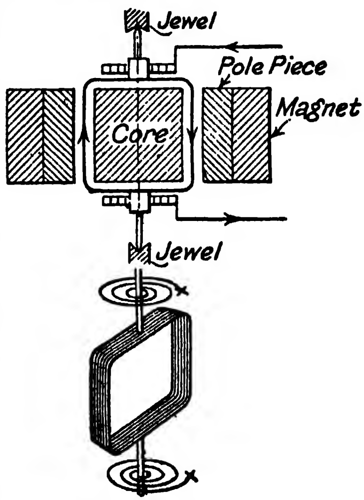


Fig. 418
Moving-Coil Permanent Magnet Instrument

3. Moving-Coil Permanent Magnet Instruments. These consist essentially of a permanent magnet with soft-iron pole pieces, between which a cylindrical iron core is mounted. A rectangular coil of fine wire is suspended so that it can rotate in the two air gaps between the pole pieces and core, and current is led into and out of the coil through two phosphor-bronze hair springs, as shown in Fig. 418. These springs also provide the controlling torque. When the coil carries current a deflecting torque is set up, proportional to the product of the current and the strength of the magnetic field in the air gap. In ordinary instruments the total deflection is only about 60°, in order that the coil sides may be always in a field of uniform strength. From this it follows that since the restoring torque due to the springs is proportional to the deflection, the deflection is strictly proportional to the current in

the coil. Hence, the scale is uniform. Damping is provided by winding the coil on a copper or aluminium frame.

When used as a voltmeter the coil is connected in series with a series resistance, but when used as an ammeter it is connected across a shunt.

In the Record instrument, modifications are made which enable a uniform scale of angular extent 270° to be obtained. It will be seen from Fig. 419 that the air gaps are magnetically in parallel instead of in series as in the ordinary construction, and that the coil is pivoted at one side, the working sides of the coil being horizontal.

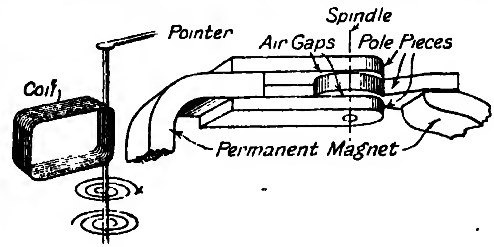


FIG 419
LONG SCALE INSTRUMENT

Moving-coil permanent magnet instruments can be used only for direct current measurements.

4. **Dynamometer Instruments.** The classical examples of this type of instrument are the old Siemens dynamometer and the Kelvin ampere-balance. Except for standard instruments, such as the Kelvin balance, the dynamometer principle is confined mainly to wattmeters. The instrument consists of a fine wire moving coil which can rotate in the magnetic field produced by a second fixed coil. It is usual to divide the fixed coil into two halves as shown in Fig. 420. Since there is no iron, the field strength is proportional to the current in the fixed coil, and therefore, the deflecting torque is proportional to the product of the currents in the fixed and moving coils. When used as a wattmeter the fixed coil is the current coil, and the moving coil the pressure coil, the current in the latter being therefore proportional to the voltage applied. Hence, the deflecting torque is proportional to the product of voltage and current, i.e. the power. Also, the restoring torque, which is provided by spiral springs, is proportional to the deflection, from which it follows that the deflection is proportional to the power. The theory of the instrument for alternating-current working is explained in Chapter XIV.

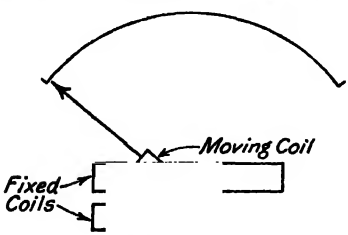


FIG. 420
ARRANGEMENT OF DYNAMOMETER INSTRUMENT

Air damping is often provided, the device consisting of two light

vanes mounted on the spindle and moving in a double sector-shaped box.

When used as a wattmeter the dynamometer instrument has a uniform scale, but if used as an ammeter or a voltmeter the scale is not uniform. This is because the field of the fixed coil is proportional to the current (or voltage) to be measured, the deflecting torque therefore being proportional to the square of current (or voltage). When used as an ammeter there is of necessity a much larger drop of volts in the instrument than in an ammeter of one of other types, and this is a disadvantage. The small ratio of torque to weight of moving part, as compared with instruments using iron, is also a disadvantage.

5. **Hot-Wire Instruments.** These instruments depend upon the increase in length of a wire when traversed by a current due to the heating effect. Consider a wire of initial length l_1 and final length l_2 traversed by a current I. Let its final radius be r. Then rate of generation of heat in calories per second

$$= I^2R \times 0{\cdot}24 = I^2 \times \frac{\rho l_2}{\pi r^2} \times 0{\cdot}24 \quad . \quad . \quad . \quad . \quad (1)$$

If t is the rise of temperature, and k the coefficient of emissivity, then rate of radiation of heat

$$= 2\pi r l_2 kt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Expressions (1) and (2) are equal when the wire has attained its final temperature. Hence

$$2\pi r l_2 kt = I^2 \times \frac{\rho l_2}{\pi r^2} \times 0{\cdot}24$$

$$t = \frac{\rho}{k} \cdot \frac{I^2}{\pi^2 r^3} \times 0{\cdot}12$$

Again, if a is the coefficient of linear expansion the increase in length

$$\delta l = a l_1 t$$

$$\therefore \ \delta l = \frac{\rho a l_1}{k} \cdot \frac{I^2}{\pi^2 r^3} \times 0{\cdot}12 = AI^2$$

where A is a constant. Thus, the increase in length of the wire is proportional to the square of the current.

The mechanism for taking up this increase in length and thus indicating the current is illustrated in Fig. 421. The hot wire W, 16 cm. of platinum-silver, is stretched between a fixed point A and a tension-adjusting screw B, and a second wire W_1 of phosphor-bronze is attached to W and to a fixed point D. Any sag in W causes slack in W_1, and this is taken up by a fibre C attached to a spring S. The fibre passes round a pulley E to which the pointer P is attached, a very small extension of the wire W being thus greatly magnified and conveyed to the pointer. Eddy current

damping is provided by a light aluminium disc L, which moves between the poles of a permanent magnet M.

These instruments can be used for direct- or alternating-current measurements, and they possess the advantage that the deflection depends only upon the R.M.S. value of the current through the wire. It is independent of wave-form and frequency. Their disadvantages are that they are fragile; the zero position often requires adjustment; and the power absorbed is high. Thus, to produce a

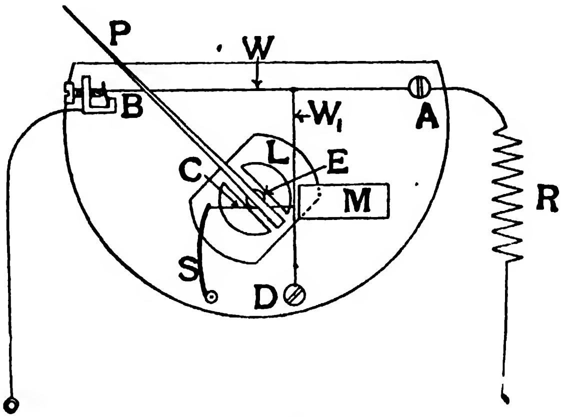


Fig. 421

Hot-Wire Voltmeter

full-scale deflection a current of 0·2 amp. is required. They are also very sluggish in action.

6. **Electrostatic Voltmeter.** The principle of this instrument is the force of attraction between two bodies, between which a potential difference is maintained. In the Kelvin instrument the fixed system consists of a cellular structure composed of two sets of triangular metal plates F (Fig. 422) fixed to brass supports B. The moving system consists of a corresponding number of aluminium vanes M, mounted on a spindle and suspended by a phosphor-bronze wire W. The upper end of this wire is connected to an elliptical spring S which is mounted on a torsion head H, the object of which is to provide a zero adjustment. E is a safety sleeve which comes in contact with a stop G in the event of any sudden jerk which would otherwise tend to break the suspension. The controlling torque is supplied by the torsion in the wire W when the vanes are deflected from the zero position, and the damping is supplied by a vane V immersed in an oil bath, O.

The range of an electrostatic voltmeter can be increased by using a number of condensers in series as a potential slide and connecting the voltmeter across one condenser. Thus, if three

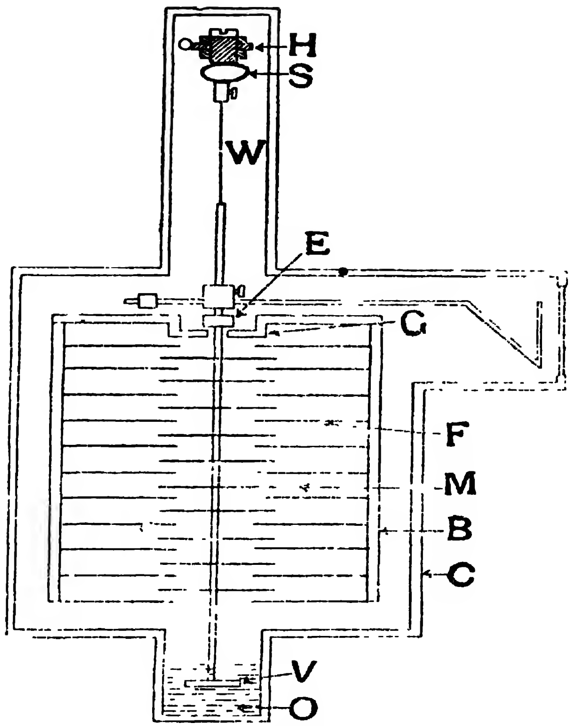


Fig. 422
Multicellular Electrostatic Voltmeter

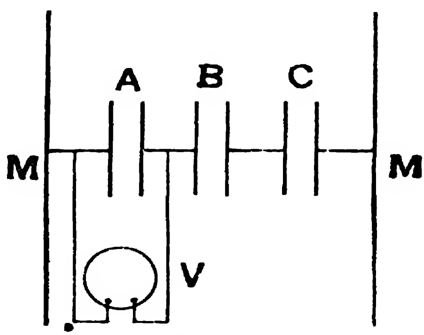


Fig. 423
Multiplier for Electrostatic Voltmeter

equal condensers A, B, and C (Fig. 423) are connected across the points MM whose P.D is required, the voltage across one condenser will be one-third the required P.D. A voltmeter V reading up to one-third the required P.D. can thus be used, since three times its reading will give the P.D. across MM. It is to be noticed that the voltmeter itself is a condenser, and therefore, the capacities of A, B, and C must be large compared with that of V.

7. **Integrating Instruments.** These instruments, usually called "meters," are of two classes: (*a*) Ampere-hour and (*b*) watt-hour meters. The former measure the quantity of electricity which has passed, but when used on constant potential circuits they can be calibrated to read directly in terms of energy. The watt-hour meters are true energy meters.

The specifications of the Engineering Standards Committee for the maximum permissible errors in electricity meters are as follow—

For meters up to 1·25 kW capacity, an error of ± 5 per cent at $\frac{1}{20}$ full load, and ± 2 per cent from full load down to $\frac{1}{10}$ load. For meters above 1·25 kW capacity, an error of between $+2$ per cent and -5 per cent at $\frac{1}{20}$ full load, and ± 2 per cent from full load down to $\frac{1}{10}$ load. Also, the meter must begin to register when 1 per cent of full-load current is flowing, provided that this is not less than 0·05 amp.

8. **The Ferranti Mercury Meter.** This is a very commonly used ampere-hour meter. It is essentially a motor in which the magnetic field is provided by two permanent magnets with mild steel pole pieces NS, NS, which are forced into brass plates A, A (Fig. 424). A fibre ring is placed between these plates, the space between them containing mercury. The rotor, a copper disc, is placed in this mercury bath, the current flowing through it from periphery to centre. The current is led in by a contact C, and out through the supporting screw D. Since the field is vertically upwards and the flow of current through the disc is radial, the disc will rotate as a motor under the influence of the right-hand magnet, and will act as a generator and have eddy currents induced in it by the left-hand magnet. This second magnet therefore provides the necessary controlling torque. With such an arrangement it is easily shown that the speed is proportional to the current.

If H_1 is the field strength of the right-hand magnet, and I the current, then

$$\text{Motoring torque} \propto H_1 I$$

$$\therefore \text{ Power developed by motor} \propto H_1 I N, \text{ where } N = \text{speed.}$$

Let E be the E.M.F. acting round an eddy current path in the disc; then if R is the resistance, the power absorbed in this particular path will be E^2/R. Hence, for all the eddy current paths in the disc the total power absorbed can be written $\Sigma E^2/R$. Now if H_2 is the field strength under the left-hand magnet, the E.M.F.

induced by generator action will be proportional to H_2N, and therefore to N, since H_2 is constant. The power absorbed in any eddy current path is thus proportional to N^2, since R for any path is constant. Hence, the total power absorbed is proportional to N^2. Now constant speed is attained when the motoring power is equal to the power absorbed by generator action. Hence

$$IN \propto N^2,\ H_1 \text{ being constant}$$
$$\text{or } N \propto I$$

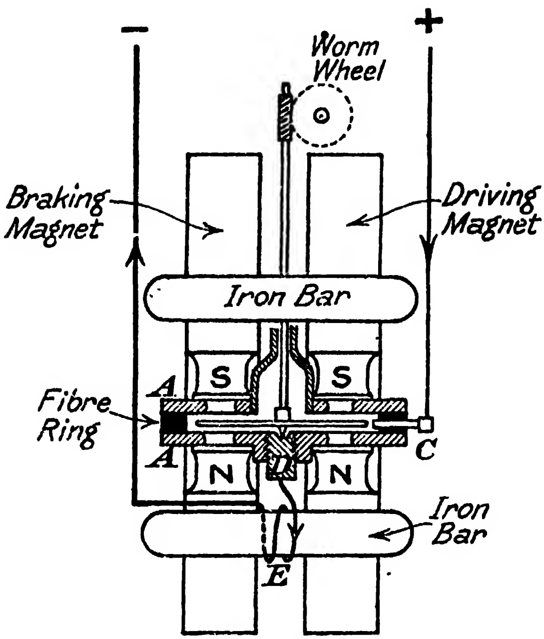


FIG. 424
FERRANTI MERCURY METER

The number of revolutions of the disc in any given time will therefore be proportional to $\int Idt$, that is, to the quantity of electricity.

Owing to the fact that mercury friction is present, the speed would not be proportional to I but for a compensating device. This takes the form of an automatic strengthening of the motor field and a weakening of the generator field with increase of load, and is effected by superposing a local magnetic flux, produced by a compensating coil E, on the main fluxes.

The weight of the moving system is adjusted so that it just sinks in the mercury, the friction on the lower bearing being

therefore very small. This is necessary since no other provision is made for the compensation of bearing friction.

9. **The Elihu Thomson Meter.** This is essentially a small ironless motor having a wound armature with a commutator. The magnetic field is produced by the current coils B and B_1 (Fig. 425) in series with the line, while the armature A with its series resistance R carries a current proportional to the line voltage. Braking is effected by the rotation of an aluminium disc C between the poles of permanent magnets, as shown.

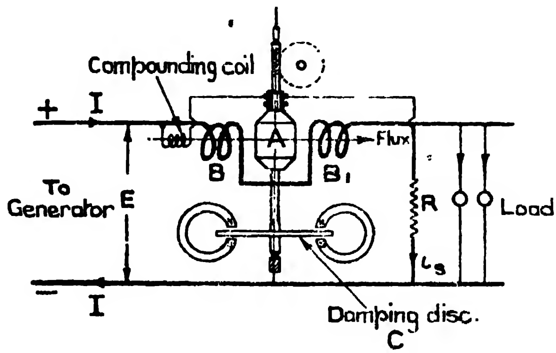


Fig. 425

$$\text{Field strength due to coils } B \text{ and } B_1 \propto I$$
$$\text{Armature current} \propto E$$
$$\therefore \text{ Torque} \propto IE$$
$$\text{and Motor power} \propto IEN$$

The power absorbed in eddy current losses in the disc is proportional to N^2 as before, and therefore, when motion is uniform

$$N^2 \propto IEN$$
$$N \propto IE$$
$$\propto W$$

where W = the power. Hence, the number of revolutions in a given time is proportional to $\int W dt$, that is, to the energy.

The effect of friction is compensated for by means of a small compensating coil placed co-axial with the current coils and connected in series with the armature. The position of this coil is adjusted so that with zero line current the meter just does not rotate.

This type of meter is now mainly used for switchboard use, house service meters being almost invariably of the ampere-hour type.

10. **A.C. Induction Type Instruments.** Suppose that a copper or aluminium disc can rotate freely between the poles of two alternating current electromagnets, as shown in Fig. 426. The current in magnet A will produce an alternating field H_1 which will induce eddy currents in the disc. These currents will link with the lines of force of the field H_2 set up by magnet B, so that under proper conditions a torque will be set up.

(*a*) If the currents in the coils A and B are in phase, then the fields H_1 and H_2 will also be in phase. Now the induced currents I are

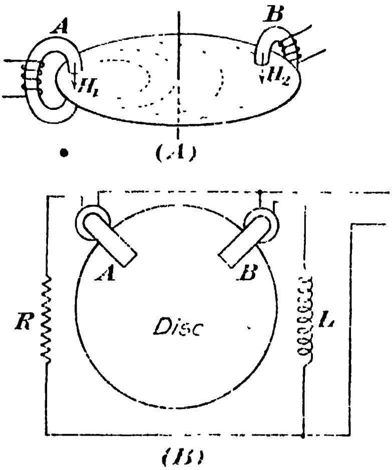


Fig. 426
Principle of Induction Type Instrument

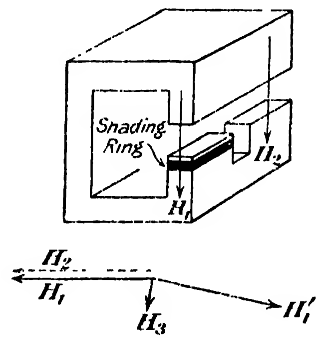


Fig. 427
Shaded Pole

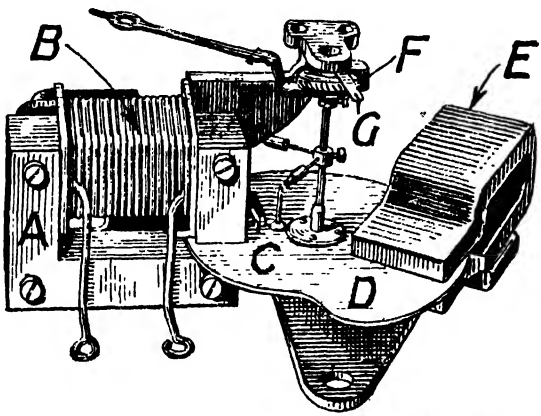
in quadrature with H_1 (*see* the transformer), and therefore, with H_2, and in consequence there will be zero torque. Hence, the first condition is that there shall be a phase difference between the currents in coils A and B.

(*b*) The two magnets must not be at opposite ends of a diameter, otherwise the force acting on the disc will be along a radius, thus producing no torque.

Ammeters and voltmeters are obviously worked on one phase only, and therefore, an artificial method of obtaining the phase difference in the currents in the coils must be adopted. There are two ways. The first is "splitting the phase." The two magnets are connected in parallel, an inductive coil L in series with one, and a resistance R in series with the other, as in Fig. 426 (B). A phase difference of nearly 90° is thus obtained, the torque therefore being a maximum for given values of the fields H_1 and H_2. The second method is to employ a single magnet with "shaded" pole, that is, one of the poles surrounded by a heavy copper ring. (Fig. 427). The field H_1 induces eddy currents in the shading ring

which, in turn, set up another magnetic field H'_1 almost opposite in phase to H_1. The total field, H_3, is the resultant of H_1 and H'_1, and is therefore almost in exact quadrature with H_1. But H_1 and H_2 are in phase, and therefore H_3 and H_2 are nearly in quadrature.

Let a be the phase angle between H_3 and H_2, and let H_3 induce a current I_1 in the disc. Then I_1 is set up by statical action irrespective of the rotation of the disc and is proportional to H_3. Also, I_1 lags 90° behind H_3. But torque is set up by the interaction of I_1 and the field H_2, and because the fields H_3 and H_2 are approximately in quadrature, the phase angle between I_1 and H_2 is $(90 \pm a)$.



(*Everett Edgcumbe and Co., Ltd.*)

FIG. 428

INTERIOR OF INDUCTION TYPE AMMETER

$\therefore$ Couple due to interaction of H_2 and I_1 is

$$\propto H_2 I_1 \cos(90 \pm a)$$

$$\propto H_2 H_3 \cos(90 \pm a)$$

$$\propto H_2 H_3 \sin a.$$

In ammeters or voltmeters based on the above principle the fields H_2 and H_3 are proportional to the current through the magnet coil, and therefore, to the current or the voltage to be measured. Now the torque produced by the interaction of H_2 and the current I_1 induced in the disc by H_3, is proportional to the product $H_2 I_1$, and therefore to the product $H_3 H_2$, as we have just seen. But both fields are proportional to the current in the magnet coil, and therefore the torque is proportional to the square of the current or voltage to be measured. The instruments are provided with spiral control springs, the disc coming to rest when the deflecting torque and the restoring torque of the springs are equal. Since the torque

is proportional to the square of the current, the graduations are not equal, but they can be made nearly so over the major portion of the scale by cutting away some of the disc, as shown in Fig. 428. The figure also shows the permanent magnet E, which is used for damping the motion of the disc D in the event of a sudden change in current. An important point in connection with these instruments is that they must be used only for the frequency for which they were calibrated. They have the advantage of possessing a very long open scale.

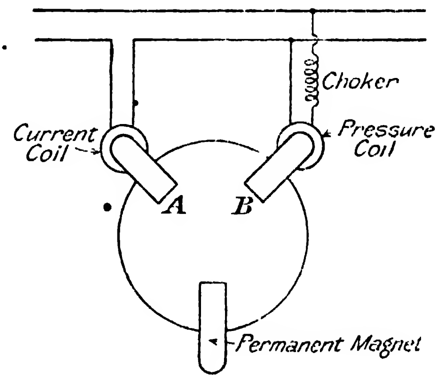
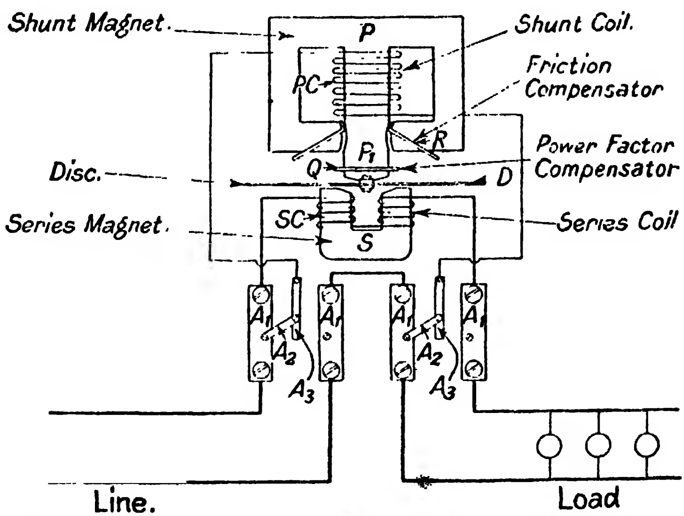


Fig. 429

In alternating-current energy meters of this type



By courtesy of Messrs. Metropolitan Vickers Electrical Co., Ltd.)

Fig. 430

Arrangement of A.C. Energy Meter

the two magnetic fields are produced by the current and pressure coils. The field of the current coil is in phase with the current, while the field of the pressure coil lags one quarter period behind the pressure.

Damping by permanent magnet is provided as in direct current meters, and the uniform motion attained is that for which the deflecting torque is equal to the opposing torque set up by the eddy currents induced by the permanent magnet. The two magnetic fields are produced, one by a current coil and the other by a pressure coil, as in Fig. 429, where separate electromagnets are indicated for clearness. The field H_1 is proportional to the current I, while the field H_2 lags 90° behind the voltage E applied to magnet B, and is proportional to E.

Hence, if φ is the angle of lag or lead of I behind E, the phase angle of the fields H_1 and H_2 is $(90 \pm \varphi)$.

$$\begin{aligned} \therefore \text{ Torque } &\propto H_1 H_2 \sin (90^\circ \pm \varphi) \\ &\propto IE \cos \varphi \\ &\propto \text{power } W. = A \,.\, W, \text{ say.} \end{aligned}$$

Now the power wasted due to eddy currents is proportional to N^2.

$\therefore$ Retarding torque

$$\propto \frac{\text{power}}{\text{speed}}, \propto N, = BN, \text{ say.}$$

Uniform motion is attained when the deflecting and controlling torques are equal.

$$\text{When} \quad BN = AW, \text{ or } N = \frac{A}{B} \times W$$

Thus, the speed of the disc is proportional to the power, and the number of revolutions made by the disc in a given time, proportional to the expression $\int W \,.\, dt$, that is, to the energy.

The actual arrangement of the magnets and the current and pressure coils in a modern meter are shown in Fig. 430. The electromagnets are, of course, built up of thin stampings. Phase compensation, i.e. zero torque at zero power factor, and compensation for friction, are produced by adjustable shading rings, as shown.*

11. Methods of Measuring Power in A.C. Circuit Without the Use of Wattmeters. (*a*) THREE-VOLTMETER METHOD. The circuit, X, in which the power is required is connected in series with a non-inductive resistance R, and the three voltages E_1, E_2, and E_3 measured (Fig. 431). Let $\cos \varphi_1$ be the power factor of the circuit X, then since the drop E_2 along R is in phase with the current, the vector diagram is as shown, and we have

$$E_3^2 = E_1^2 + E_2^2 + 2E_1E_2 \cos \varphi_1$$

$$\text{But} \qquad E_2 = IR$$

$$\begin{aligned} \therefore \qquad E_3^2 &= E_1^2 + E_2^2 + 2(E_1 I \cos \varphi_1)R \\ &= E_1^2 + E_2^2 + 2WR \end{aligned}$$

* Measuring instruments are discussed fully in Golding, *loc. cit.*, and in Drysdale and Jolley, *Electrical Measuring Instruments.*

where W is the power consumption of X

$$\therefore \qquad W = \frac{E_3^2 - E_1^2 - E_2^2}{2R}$$

This method is a very valuable one for the determination of power in coils taking a small current and working at a low power factor;

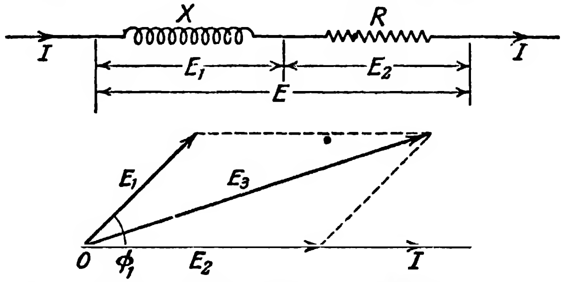


FIG. 431. THREE-VOLTMETER METHOD OF MEASURING POWER

for example, it can be used with success for the determination of the iron loss in an Epstein square, by the alternating-current method. Since the squares of the voltages have to be used it is necessary to

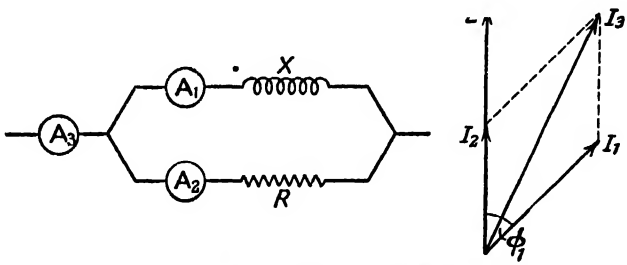


FIG. 432. THREE-AMMETER METHOD OF MEASURING POWER

use accurate instruments, and the possibility of error will be reduced if only one voltmeter is used, a transfer switch connecting it to the points required. It is also desirable that the voltages E_1 and E_2 should be as nearly equal as possible, for which reason the supply voltage must be in the neighbourhood of 100 per cent higher than the voltage to be applied to X.

(*b*) THREE-AMMETER METHOD. In this method the circuit X in which the power is required is connected in parallel with a non-inductive resistance, and the total current I_3, and the two branch currents I_1 and I_2 are measured by means of three ammeters A_1, A_2, and A_3 (Fig. 432). The current I_2 is in phase with the applied voltage E, so that the vector diagram is as shown. We have

$$I_3^2 = I_1^2 + I_2^2 + 2I_1I_2 \cos \varphi_1$$

But $$I_2 = E/R$$

$$\therefore \quad I_3^2 = I_1^2 + I_2^2 + 2EI_1 \cos \varphi_1 \div R$$
$$= I_1^2 + I_2^2 + 2W/R$$
$$\therefore \quad W = \frac{R}{2}(I_3^2 - I_1^2 - I_2^2).$$

This method can be used when the current I_2 is too large for the three-voltmeter method to be applied. The greatest accuracy is obtained when the currents I_1 and I_2 are of the same order.

(c) ELECTROSTATIC VOLTMETER, OR ELECTROMETER METHOD. This method has been used fairly extensively for the measurement of small amounts of power at high voltages; for example, the power absorbed by samples of dielectric. Fig. 433 shows one method of

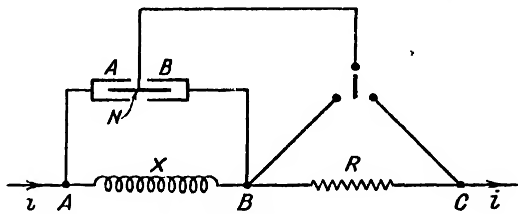


FIG. 433. ELECTROMETER METHOD OF MEASURING POWER

determining the power in a circuit X, by means of the quadrant electrometer. One pair of quadrants is connected to the end A, and the other pair to the end B of the circuit X. A non-inductive resistance R is placed in series with X, and readings are taken, first with the needle connected to B, and secondly with the needle connected to the far end C of R, as shown.

Let V_A, V_B, and V_N be the potentials of the pairs of quadrants A and B, and the needle N respectively. Then, the deflection of the needle is given by

$$\theta = K(V_A - V_B)\left(V_N - \frac{V_A + V_B}{2}\right)$$

where K is a constant depending on the instrument. Let v_a, v_b, v_n, and v_c be the instantaneous potentials of the points A, B, N, and C respectively, and let the deflections with the needle connected first to B, and then to C, be θ_1 and θ_2 respectively. Then, since the steady deflection with alternating potentials will be the average value of the above expression, we have

$$\theta_1 = K \times \text{av. of } (v_a - v_b)\left(v_b - \frac{v_a + v_b}{2}\right)$$

$$\theta_2 = K \times \text{av. of } (v_a - v_b)\left(v_c - \frac{v_a + v_b}{2}\right)$$

$\therefore\ (\theta_1 - \theta_2) = K \times \text{av. of } (v_a - v_b)\,(v_b - v_c)$

Now

$(v_b - v_c) = Ri$, where i is the instantaneous current

$$\therefore\ \theta_1 - \theta_2 = K \times \text{av. of } \{(v_a - v_b)i\} \times R$$

$$\propto \text{av. of } (v_a - v_b)i$$

$$\propto \text{power in } X.$$

PART FOUR
ILLUMINATION

CHAPTER XXVI

ILLUMINATION

1. **Radiant Efficiency.** When the temperature of a body is gradually increased, the body begins to radiate energy into space, the nature of the radiation depending on its temperature. Thus, when the temperature is low, heat energy will be radiated only, but when a certain temperature is reached, light will also be emitted and the body will now become luminous. At first the light radiated will consist of red waves only, but with increasing temperature more and more of the shorter waves will be given off, until eventually the body is white hot. The heat waves still given off are identical in nature to the light waves, but as they produce no impression on the retina, it is obvious that from the point of view of light-giving, they represent so much wasted energy. The energy radiated as light, expressed as a percentage of the total energy radiated, is called the radiant efficiency of the body. As the temperature is increased above that at which light waves are first given off, the radiant efficiency will increase, because the energy of the light waves will increase in greater proportion than the total radiated energy. When the light radiations include all the visible wave lengths from extreme red to extreme violet, then a further increase in temperature produces further radiations which are invisible, this time of wave lengths smaller than the extreme violet. The radiant efficiency now decreases. The temperature for maximum efficiency is about 6,000°C.; it is far above the temperature usually attained by sources of light.

2. **Definitions.** The *Flux* of light emitted from a luminous body is the energy radiated per second in the form of light waves.

The *luminous intensity* in any given direction is equal to the flux per unit *solid angle.*

It is obvious that some standard must be used in order that the luminous intensity can be measured, or compared, experimentally. This unit is derived from the candle, the unit of flux being that flux contained within unit solid angle from a source of 1 candle-power. This unit is called the *Lumen.* From this definition we see that the candle-power of a source of light in any direction is the number of lumens contained within unit solid angle in that direction.

The candle-power of a source of light is always different in different directions, these differences being taken into account in the following definitions. The *Mean Horizontal Candle-power* (M.H.C.P.) is the mean of the candle-powers in all directions in the horizontal plane containing the source of light.

The *Mean Spherical Candle-power* (M.S.C.P.) is the mean of the candle-powers in all directions in all planes. Hence, if a second source having equal intensity in all directions produced the same total flux as actual source of light, then the candle-power of this second source would be equal to the M.S.C.P. of the actual source. The *Mean Hemispherical Candle-power* (M.H.C.P.) is the mean of the candle-powers in all directions below the horizontal.

The *Reduction Factor* is the ratio

$$\frac{\text{M.S.C.P.}}{\text{M.H.C.P.}}$$

The *degree of illumination* of an illuminated surface is the luminous flux per unit area received by it. The British unit is the *candle-foot*, being the illumination of the inner surface of a sphere of 1 ft. radius at the centre of which there is a source of 1 candle-power.

It is preferable to define the efficiency of a lamp as the ratio of the lumens per watt, instead of the ratio of watts per candle-power, which is really a measure of the inefficiency. The ratio lumens per watt is also called the specific output. The following relationships are important—

$$\text{Lumens per watt} = \frac{4\pi}{\text{Watts per M.S.C.P.}}$$

$$= \frac{4\pi f}{\text{Watts per M.H.C.P.}}$$

when f is the reduction factor

$$\text{Watts per M.S.C.P.} = \frac{4\pi}{\text{Lumens per watt}}$$

$$= \frac{\text{Watts per M.H.C.P.}}{f}$$

$$\text{Watts per M.H.C.P.} = \frac{4\pi f}{\text{Lumens per watt}}$$

$$= f \times \text{watts per M.S.C.P.}$$

For a 100 watt, 200 volt, gas-filled lamp, representative figures for the performance are—

Lumens—about 1,100.
M.S.C.P.—about 90.
Lumens per watt—about 11.
Watts per M.S.C.P.—about 1·1*

* See Meares and Neale, *Loc. cit.*

The illuminations required for indoor and outdoor lighting under different conditions naturally vary very considerably. For important streets, representative values are ·1 ft.-candle for the average, and ·05 ft.-candle as a minimum. Half these values will do for unimportant streets. For ordinary interiors, e.g. living rooms, it may vary between 1 and 6 or 7 ft.-candles, and for offices, from 10 to 20.

The degree of illumination is the flux *received* by each unit area, the brightness is the flux *emitted* per unit area in a direction at right angles to the surface. This definition of brightness obviously applies to luminous sources themselves and also to bodies which are made visible through receiving light from a primary source, but not actually emitting any light of their own.

The *brightness* in a given direction of a surface emitting light is the quotient of the intensity measured in that direction by the area of this surface projected on a plane perpendicular to the direction considered. The unit of brightness is the candle per unit area of surface.

Consider an element dS of surface of a luminous position, let I be the intensity, and θ the angle between the normal to the surface at the element considered and the line of sight. Then

$$\text{Brightness} = \frac{dI}{ds \cos \theta}$$

It follows from this that a luminous sphere has the appearance of a luminous disc.*

3. **Laws of Illumination.** The illumination of a surface receiving its flux from a source whose distance is sufficiently great for its being regarded as a point source, is inversely proportional to the square of the distance between the surface and the source.

The illumination of a surface at any point is proportional to the cosine of the angle between the normal at that point and the direction of the luminous flux. This is Lambert's Cosine Law. Fig. 434 represents a rectangular tube of flux F.

$$\text{Intensity of surface } ABCD = \frac{F}{ABCD}$$

$$\text{Intensity of surface } ABC'D' = \frac{F}{ABC'D'} = \frac{F \cos \theta}{ABCD}$$

$$\therefore \quad \frac{\text{Intensity of } ABCD}{\text{Intensity of } ABC'D'} = \frac{1}{\cos \theta}$$

* See also *Photometry*, by Walsh, and *Theory and Design of Engineering Illuminating Equipment*, by Jolley, Waldram, and Wilson.

Example. Two arc lamps of 1,000 and 500 candle-power respectively (assumed the same in all directions) are suspended 40 ft. above the ground

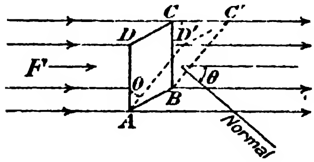


FIG. 434

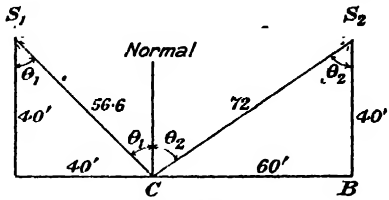


FIG. 435

and are 100 ft. apart. Find the intensity of illumination at a point on the ground in line with the two lamps and 40 ft. from the base of the more powerful lamp.

From Fig. 435, we have

$$\text{Illumination at } C \text{ due to } S_1 = \frac{1{,}000}{(S_1C)^2} \times \cos\theta_1$$

$$= \frac{1{,}000}{(56{\cdot}6)^2} \times {\cdot}707$$

$$= 0{\cdot}22 \text{ candle-foot}$$

$$\text{Illumination at } C \text{ due to } S_2 = \frac{500}{(S_2C)^2} \times \cos\theta_2$$

$$= \frac{500}{(72)^2} \times \frac{40}{72}$$

$$= 0{\cdot}05 \text{ candle-foot}$$

$$\therefore \text{ Total illumination at } C = 0{\cdot}22 + 0{\cdot}05$$

$$= 0{\cdot}27 \text{ candle-foot}$$

4. **Measurement of Candle Power.** In order to determine the candle-power of a source of light it is necessary to compare it with a source of known candle-power. A standard lamp is therefore required. At one time the candle-power of a candle of pure spermaceti wax, weighing $\frac{1}{6}$ lb. and burning 120 grains per hr., was taken as the unit. More modern standards are the Harcourt pentane lamp, which burns a mixture of pentane vapour and air, and has a candle-power of about 10 ; the Heffner lamp, which burns amyl-acetate at a wick and has a candle-power of 1 ; and the Carcel lamp, which burns colza oil and has a candle-power of about 10. In practice, it is not convenient to use such standards, a more satisfactory way being to determine the candle-power of an aged electric lamp by comparison with one of the above standards, and then to use this lamp as a secondary standard. It is usual to make periodic tests on a lamp which is to be used as such a standard, and

when it is consistent, to remove the filament from the old bulb and place it in a new cylindrical bulb.

The most convenient form of photometer for general purposes is the Lummer-Brodhun Photometer. This consists essentially of a plaster of Paris screen S, (Fig. 436), the two sides of which are illuminated by the standard lamp and the lamp under test, respectively The light is scattered at the two surfaces, and the rays which leave at an angle of 45° reach the right-angled prisms A and B, where they are totally reflected to two more prisms, C and D. In one form of photometer a portion of the base of C is ground away, so that the contact area between C and D is a circle. Rays from prism A entering this circle pass through to the telescope, whereas those from B which enter the circle do not reach the telescope.

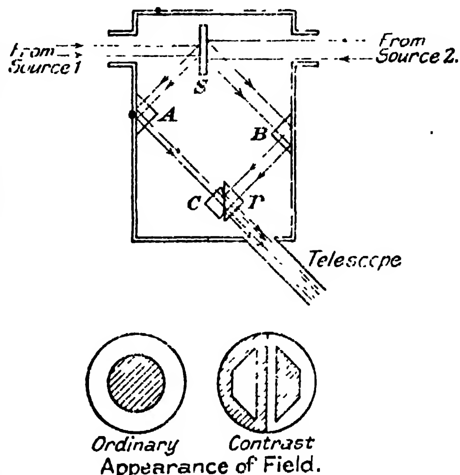


FIG. 436

On the other hand, of the rays which reach the face of D outside this circle, only those from B reach the telescope. Hence, the field consists of an inner circle and an outer annular ring illuminated by the standard and the test lamp respectively. The position of the photometer is adjusted until the field is of uniform brightness and its distance from the two lamps read off. If CP_1 and CP_2 are the candle-powers of the standard and test lamp, and d_1 and d_2 their respective distances from the photometer, then, from the inverse square law

$$\frac{CP_2}{CP_1} = \frac{d_2^2}{d_1^2} \text{ or } . CP_2 = CP_1 \times \left(\frac{d_2}{d_1}\right)^2$$

In the more modern form of Lummer-Brodhun Photometer, the arc of contact between the two prisms C and D is so modified that the field of view consists of trapezoidal patches in semicircles. This pattern is more accurate and is called the contrast pattern.

When the candle-power of very powerful lamps, such as large half-watt lamps, is being determined, the distance between the test and standard lamps must be very large, otherwise the photometer will have to come so close to the standard that accurate

observations are impossible. A photometer room is not large as a rule, so that, in order to obtain the required distance, the test lamp can be slung from the ceiling and the light from it reflected by a mirror on to the photometer.

5. **The Illumination Photometer.** To enable the intensity of illumination of surfaces to be measured, special portable photometers have been evolved. One of these, the Trotter Photometer, is illustrated in Fig. 437. The photometer is placed on the surface whose illumination is required, and a small screen S_1 is viewed through slits S_2 in the case. The screen S_1 is illuminated by means of a small standard lamp L supplied from a portable battery B. S_1 is then rotated until its intensity of illumination is judged to be equal to that of the top of the case, and its position on the scale read off by means of the pointer P. The scale is graduated directly in candle-feet. The instrument is very easily calibrated by producing known intensities of illumination on the top of the case, and finding the position of the pointer which gives equality of illuminations of S_1 and S_2. It is thus not necessary to know the candle-power of L, but the lamp must be aged before it can be used in the photometer.

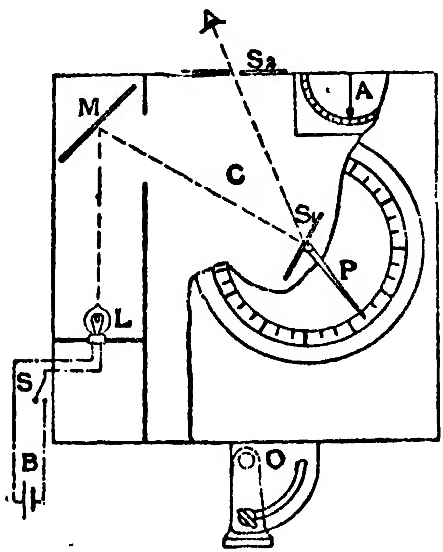


Fig. 437

To measure the illumination of sloping surfaces, the case is pivoted at O and is capable of rotation through an angle of 90°, the angle being indicated by a plumb-bob A, moving over a graduated scale.

The above type of illumination photometer is now little used, since more convenient instruments incorporating photoelectric cell have been developed. The type of cell generally used is the barrier-layer type. The light falls on a very thin layer of gold (1), Fig. 438, which is sputtered on to a thin layer of selenium (2), the whole being supported on an iron back plate (3). The gold layer is so thin that it is transparent, and light can therefore reach the gold-selenium boundary where it liberates electrons. These electrons pass from the gold

to the selenium with the result that if a micro-ammeter is connected to the cell as shown in the figure there will be an electron flow from the iron plate through the instrument to the gold layer. The electron flow is (within limits) proportional to the light flux falling on the surface of the cell, and for a given external resistance is a function of the light flux. Thus the micro-ammeter can be calibrated directly in illuminations by means of a standard lamp. The commercial cell of 2 in. diameter gives about 450 micro-amps. per lumen. By the use of a special glass filter in front of the cell the spectral sensitivity of the device approximates closely to that of the eye. It

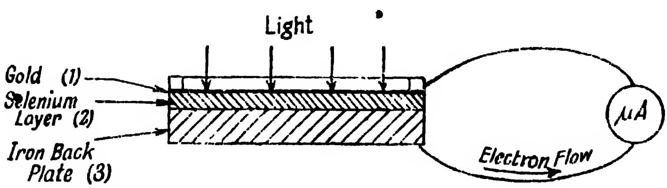


FIG. 438. PRINCIPLE OF PHOTRONIC CELL

should be noted that the barrier-layer type of photoelectric cell is most suited for dealing with a steady light flux; other types are more useful for dealing with fluctuating light as in television and talking film work.

6. Determination of M.H.C.P. and M.S.C.P. In order to determine the M.H.C.P., the candle-power is measured in a horizontal direction, the lamp being rotated through, say, 10° about a vertical axis for each observation. The resulting candle-powers plotted on a polar diagram give the horizontal curve of illumination. Such a curve is, roughly, circular for most lamps. The M.H.C.P. is the mean of the observed values. If these values differ considerably, so that the diagram departs from the circular form, it is preferable to plot candle-power against angle with rectangular co-ordinate axes. The base is then divided into an even number of steps of equal width, and the ordinates measured. There will be an odd number. Applying Simpson's Rule, we have

$$\text{Area under curve} = \frac{h}{3}\left\{\begin{matrix} y_1 + y_n + 2(y_3 + y_5 + y_7 + \ldots) \\ + 4(y_2 + y_4 + y_6 + \ldots) \end{matrix}\right\}$$

where y_1, y_2, y_3, etc., are the various ordinates and h is the horizontal distance between consecutive ordinates. By dividing the area by the base the mean ordinate is found, and this is the required M.H.C.P. Usually, 10 steps, i.e. 11 ordinates, will be sufficient.

If the lamp is mounted so that it can be rotated in a vertical plane the resulting polar curve will exhibit two lobes, the cone of

shadow about the position 0° being due to the holder. The M.S.C.P. can be determined from this polar diagram by Rousseau's construction. A circle is drawn on the diagram, as shown in Fig. 439, and an axis XY drawn parallel to the vertical diameter AB. Points on the semicircle with uniform angular displacement are projected on to XY, and the lines produced past it by an amount equal to the candle-power corresponding to the angular position. Thus, GH is made equal to OF, and so on. The mean ordinate of the curve so obtained (CD being regarded as its base) gives the

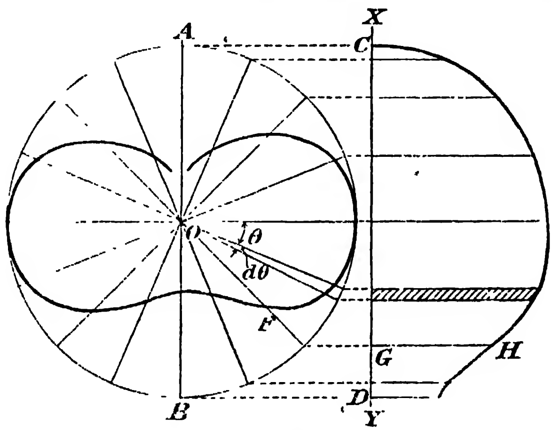


FIG. 439

M.S.C.P. For, suppose the intensity at any angular position θ is I, then, if an element of angular width $d\theta$ is taken and this element rotated about AB as axis, it will describe a zone of a sphere of area

$$2\pi r \cos\theta \times r d\theta = 2\pi r^2 \cos\theta \,.\, d\theta$$

The illumination of the inner surface of this zone is therefore

$$2\pi I r^2 \cos\theta \,.\, d\theta$$

Hence, for the total spherical illumination, we have

$$2\pi \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} I r^2 \cos\theta d\theta$$

I, of course, being a function of θ. Now, $r \cos\theta \,.\, d\theta$ is the width of the shaded strip in the figure, and therefore, $Ir \cos\theta \,.\, d\theta$ is the area of the shaded strip. The total area under the derived curve is therefore

$$\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} Ir \cos \theta d\theta$$

$$= \frac{\text{Total spherical illumination}}{2\pi r}.$$

Now the base of this curve is of length $2r$. Hence, the length of the mean ordinate

$$= \frac{\text{Total spherical illumination}}{2\pi r \times 2r}$$

$$= \frac{\text{Total spherical illumination}}{4\pi r^2}$$

$$= \frac{\text{Total spherical illumination}}{4\pi}$$

if the sphere has unit radius. But 4π is the total solid angle subtended by the surface of a sphere, from which we see from the definition of M.S.C.P., that the mean ordinate of the derived curve is the required value. The M.H.S.C.P. can obviously be determined by the same construction.

Instead of determining the M.S.C.P. by the above somewhat laborious method, the measurement can be made directly by means of an integrating photometer. The Ulbricht photometer of this type consists of a large, hollow sphere with a smooth surface which is coated evenly with white paint. The diameter of the sphere should be large in comparison with the lamp to be tested, certainly not less than six times the diameter of the bulb of the lamp. When a lamp is placed inside the sphere the light is diffused in such a manner that the resulting illumination is uniform over the whole of the surface, the result of this being that, if a small window is arranged in the sphere, the illumination of this window is proportional to the M.S.C.P. of the lamp.

Consider the lamp placed at any position S, in the sphere (Fig. 440). The flux from S reaching any small area $\triangle S$ produces a certain illumination. Light is reflected and diffused at the surface $\triangle S$, which in consequence becomes a secondary source of light. Let $\triangle F$ be the flux reaching $\triangle S$ directly from the source, then the luminous intensity along the normal to $\triangle S$ is equal to $K \cdot \triangle F$, where K is a constant whose value is fixed by the nature of the interior surface of the sphere. Now consider any point P on the sphere. The illumination of a surface at P due to the reflected rays from $\triangle S$ is for normal incidence

$$\frac{K \cdot \triangle F}{x^2} \cdot \cos \theta$$

But as is shown in the figure, the normal to the surface at P makes an angle θ with the incident rays, so that for the actual illumination at P, due to the light reflected from ΔS, we have

$$\frac{K \cdot \Delta F}{x^2} \cdot \cos\theta \cdot \cos\theta$$

$$= \frac{K \cdot \Delta F}{x^2} \times \left(\frac{x}{2r}\right)^2$$

$$= \frac{K \cdot \Delta F}{4r^2}$$

This is independent of x, proving that all small areas such as ΔS which receive an equal flux from the source S produce identical illuminations at the point P. Hence, since P is any point, we see that the illumination at any point due to light reflected from the surface of the sphere is uniform. But by taking all the elementary

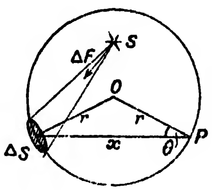


Fig. 440

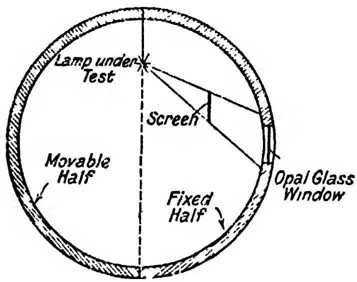


Fig. 441

surfaces ΔS we include the total flux of light from the source S, and therefore the illumination at any point of the sphere *due to reflected light* is proportional to the M.S.C.P.

The window where the observation is made must, therefore, be screened from direct rays as indicated in Fig. 441. The sphere is made in two halves, one of which is fixed and the other movable on rails so that different lamps can be easily inserted and the surface inspected periodically.

The sphere photometer is naturally very expensive; especially when very large, and therefore when very accurate results are not required a cubical box, as first suggested by Dr. Sumpner, can be used instead. The box has to be perfectly white on the inside and the surface uniform, as in the case of the sphere photometer, and it is used in exactly the same way. It is found that the cube gives very good results when the lamps under test have similar light distributions. If it is necessary to compare lamps with distributions which are not similar, it is necessary to obtain a correction factor for each distribution, and when this is done the measurements are of a high order of accuracy.

7. **Incandescent Lamps.** The bulbs of incandescent lamps are evacuated for two reasons, first to prevent the filament burning away, second to prevent the lowering of temperature of the filament through convection.

There is a definite relationship between the diameter of a given filament and the current. Consider a filament working at a fixed temperature; then the power intake of the filament will be equal to the rate of dissipation of heat, chiefly by radiation.

$$\text{Power intake} = I^2R = I^2 \times \frac{4\rho l}{\pi d^2}$$

where l and d are the length and diameter respectively.

Now, the heat radiated per second

$$= \text{Surface area} \times K \times F(t)$$

where K is the emissivity of the filament. $F(t)$ is a constant only so long as the temperature t is constant.

Equating the intake to the rate of radiation of heat, we have

$$I^2 \times \frac{4\rho l}{\pi d^2} = \pi dl \times K \times F(t)$$

$$\therefore\ I^2 \times \frac{l}{d^2} = A \times dl$$

where A is a constant.

$$\therefore\ d = B \times \sqrt[3]{I^2} \text{ or } I = C \times d^{1.5}$$

where B and C are constant. Note that similar expressions hold for the fusing current of a wire of given material under stated conditions.

Again, for filaments at the same temperature, the flux per unit area is the same. Hence

$$\begin{aligned}\text{Candle-power} &= D \times \text{surface area}\\ &= D \times ld\end{aligned}$$

where D is another constant.

The materials used for the filaments of incandescent lamps are carbon, tantalum, and tungsten, the latter now being the most important. Carbon vaporizes at 3,500° C., but the working temperature is much lower than this, otherwise particles would be driven off at a very high rate; the bulb would blacken and the life of the lamp would be short. Because of the comparatively low temperature of the filament, a large electrical intake is required to produce a given candle-power, or in other words, the commercial efficiency is low. This efficiency is usually expressed in watts per candle-power, that is, more strictly speaking, the inefficiency. For carbon filament lamps the value is about 3½. Carbon filament lamps have

a negative temperature coefficient, that is, their resistance decreases with increase of temperature. The resistance at working temperature is about $\frac{2}{3}$ times the cold resistance.

Tantalum has a melting point of 2,800° C. Its specific resistance cold is 16·5 microhms per cm. cube, and at the working temperature, about 38 microhms per cm. cube. Its temperature coefficient is therefore positive, and has a value of 0·00234. Tantalum lamps have an average efficiency of 1·6 to 1·7 watts per candle-power; they are now largely superseded by tungsten lamps.

Tungsten has a melting point of 3,400° C., a specific resistance of 6 cold and 70 at the working temperature. Its temperature coefficient is 0·0051. These are average figures, as there are several processes of preparing tungsten filaments, and the electrical properties depend on the nature of the process. Tungsten is extremely hard and difficult to work, and the early filaments were very fragile and variable in performance. The most recent method of preparing the filament is as follows. Chemically pure tungsten oxide is reduced at red heat in an atmosphere of hydrogen to tungsten metal. The tungsten is obtained in the form of a grey powder, which is next pressed in steel moulds under hydraulic pressure into small bars. These bars, which are very fragile, are then heated to a temperature of 1,100° C., again in an atmosphere of hydrogen, and the particles "sinter" together, thereby imparting a certain amount of mechanical strength to the bars. They are next raised almost to the melting point by passing an electric current through them, this process producing a mechanically strong bar. The metal is still of insufficient ductility to be drawn into filaments, but this quality is imparted to it by hammering or rolling at red heat. The efficiency of a tungsten filament lamp is about 1·2, the superior operation being due to the possibility of a higher working temperature.

8. **Gas-filled Lamps.** It was stated that the filaments of incandescent lamps are worked in a vacuum. Although this holds for lamps of ordinary construction, there is now in common use a "gas-filled" lamp. A metal filament worked in an evacuated bulb must not exceed a temperature of about 2,000° C., or the metal will volatilize quickly and blacken the bulb. For a high efficiency it is necessary to use a working temperature much greater than 2,000° C., and to make such a temperature possible, it is necessary to keep down evaporation. This is done by filling the bulb with an inert gas, argon, with a small percentage of nitrogen. This latter gas is added to reduce the possibility of arcing. In some high wattage projector lamps in which the tungsten coils are very close together the gas is all nitrogen. The presence of gas molecules causes convection currents, but it is found that these can be reduced to a minimum by winding the filament into a very close spiral and suspending it horizontally. The bulb itself hangs vertically, and

in the large sizes is provided with a neck into which convection currents carry any particles from the filaments. Hence, blackening only takes place in the neck, and the candle-power in directions below the horizontal is unimpaired. The efficiency of these lamps is much higher than that of ordinary lamps, because of their higher working temperature. An average value is 0·6, hence the name "half-watt" lamps.*

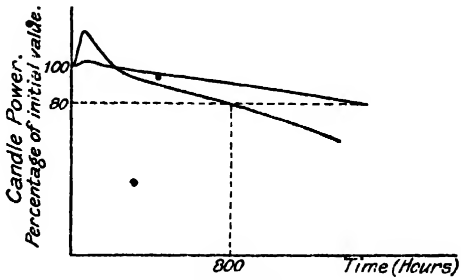


FIG. 442

9. Life of Incandescent Lamps. The life of a carbon filament lamp is ended when its candle-power is reduced to 80 per cent of the original value, because it is then cheaper to replace the lamp than continue working with the old one. The average life of the carbon lamp is 800 hours. Lamps of all kinds experience an increase in candle-power during the first few hours of working, as shown by the curves in Fig. 442. This initial increase is due to consolidation of the filament, and after it has taken place there is a gradual deterioration in candle-power, due partly to slow vaporization of the filament, and partly to the absorption of light by the black deposit formed on the inside of the bulb. The falling off in candle-power is slower for metal filament than for carbon filament lamps, and since the metal filament is more fragile, the life of these lamps is usually ended by breakage.

The term "smashing-point" is used to denote the percentage of initial candle-power to which the lamp should be allowed to decline before it is replaced, the falling off in candle-power beyond this point causing the efficiency to be so poor that it is cheaper to take the lamp from service and buy a new one. The smashing-point is generally taken as 80 per cent, although when tests have been made it has been found that lamps generally fail before this point is reached.

10. Effect of Variations in Voltage on the Operation. The candle-power can be expressed as a function of the voltage E in the form

$$\text{C.P.} \propto E^{n_1}$$

where n_1 is about 6 or 7 for a carbon filament and 4 or 5 for a metal filament. The difference in behaviour in the two types of lamp is due to the fact that the temperature coefficient of carbon is negative, so that if E increases the resistance decreases, and the percentage increase in current is therefore greater than the

* In the "coiled-coil" gas-filled lamp a single helix is coiled again on itself. This arrangement gives a slight increase in the candle-power per watt.

percentage increase in E. The temperature coefficient of metal filaments being positive, the percentage increase in current is less than the percentage increase in E.

We also have the relationships

$$\text{C.P.} \propto I^{n_2}$$

where n_2 is about 5 for both carbon and metal filament lamps.

And power $\quad W \propto I^{n_3}$ where n_3 is about 2·5.

Hence, dividing, we have

$$\text{Watts per C.P.} \quad \frac{W}{\text{C.P.}} \propto I^{\cdot 2.5}$$

Hence, an increase in the current produces a very large decrease in the watts per candle-power, and therefore, an increase in efficiency. The life of the lamp is considerably reduced by doing this because of the increased vaporization of the filament, and it is therefore not commercially economical.

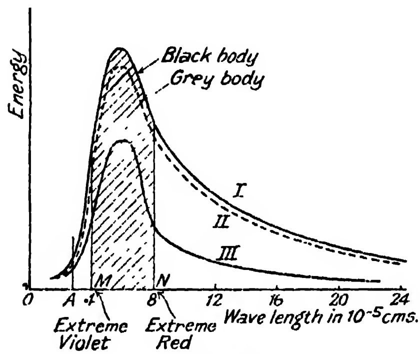


FIG. 443

11. Selective Emission. We have seen that incandescent bodies emit radiations of very different wavelengths and frequencies. From the point of view of light giving, the only useful radiations are those whose frequencies are comprised between the narrow limits of 4×10^{14} per sec. (red) and 8×10^{14} (violet). All other radiations are outside the visible spectrum and therefore represent so much wasted energy. The theoretically ideal radiator, known as a "black body," is one which absorbs all the radiation which falls upon it. If such a body is radiating energy, then, for a given temperature, the energy radiated per unit area of surface is a function of the wavelength of the radiation as shown by Fig. 443. It is clear that if the body is emitting radiations up to the wavelength OA, the total energy of all the radiations emitted will be represented by the area under the curve bounded by the limits ∞ and A. If OM and ON are the extreme visible wavelengths, the whole of the visible spectrum being included in the band MN, then the total energy radiated as light will be represented by the shaded area. The radiant efficiency of a black body emitting rays of

all lengths up to OA will thus be equal to the ratio : Shaded area, divided by total area between the limits ∞ and A. With most bodies, the shape of the curve is identical to that for the black body, the lengths of ordinates to the two curves at different wavelengths being in the same proportion. The curve for an ordinary body, or "grey" body as it is called, is therefore as shown by the dotted curve, and the radiant efficiency will be the same as for a black body. There are certain bodies whose radiation characteristics differ considerably from that of a black body, the percentage of energy radiated at some particular wavelengths being greater than the percentage of the energy radiated at the same wavelength by an ordinary body. Such a body is said to have "selective emission." If the selection occurs within the visible spectrum the characteristic will be similar to that represented by curve III. The radiant efficiency of such a body is obviously greater than that for an ordinary body or a black body, thus showing that such a characteristic is desirable in substances used as light emitters. Tungsten possesses this property to a small extent, but it is exhibited mainly by incandescent gases.

In Fig. 444 are drawn the characteristics for a black body at different temperatures, the visible spectrum being again represented by the shaded band. It is obvious from these curves that up to about 6,000° C. the radiant efficiency increases with the temperature.

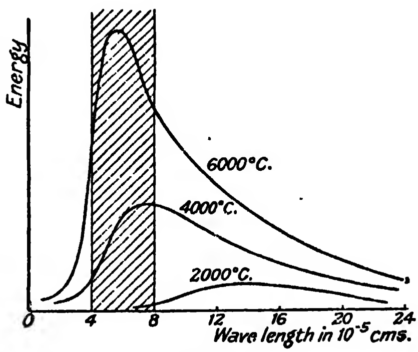


FIG. 444

12. **Requirements for Filament Materials.** The most important factor in determining the material of which to make the filament of a lamp, is the temperature at which the filament can be worked. Tungsten has the highest melting point of any of the materials so far used, and for this reason it is now used almost exclusively, especially as a satisfactory manufacturing process has at last been evolved. The working temperature is also determined by the amount of vaporization which takes place. Carbon actually has a higher melting point than tungsten, but its working temperature is limited to about 1,800° C. because of the rapid vaporization at temperatures greater than this. It thus appears that vaporization is even more important than melting point. The gas-filled lamp has been introduced in order to keep down vaporization and to enable filaments to be worked at much higher temperatures than

are possible in vacuo. It can be stated in general that the ideal filament material is one having a high melting point, a low vapour pressure, high specific resistance, ductility, and mechanical strength.

13. Discharge Lamps. When dealing with filament lamps we saw that the efficiency of the lamp could be increased by raising the temperature of the filament. The obvious limit to the temperature is the melting-point of the material; but the practical limit is appreciably below this, because a high temperature is associated with a short life. Again, even at high filament temperatures only a small portion of the energy emitted appears as visible light: a very little is given out in the ultra-violet region, but by far most of the energy is given out in the infra-red region, as is shown in Fig. 443. Now, an ordinary incandescent lamp gives a continuous spectrum, i.e. there is present light of every colour, and there are no gaps in which there is no radiation. This is due to the fact that with solids the atoms are so closely packed together that they are unable to radiate the frequency characteristic of their free state, as for example when the atoms are widely separated, as in a rarefied gas, and can therefore radiate without interference from one another. As a result of the continuous nature of the spectrum the ratio of the energy emitted as visible light to the total energy emitted is given by the ratio of the shaded portion to the total area under the curve. With a modern coiled-coil lamp this ratio is only about 10 per cent, showing that in order to obtain any appreciable gain in efficiency it is necessary to break away from the traditional heated-filament lamp and adopt some other principle.

This principle is available in the discharge lamp, in which the light is obtained from a luminous column of rarefied gas or vapour. Furthermore, this luminous column is not heated to a high temperature, but is excited electrically. In order to appreciate the meaning of this consider the nature of the electrical discharge through a rarefied gas or vapour. These phenomena are similar to those already studied in the case of the mercury-arc rectifier. Electrons travel from the cathode to the anode, and under the influence of the potential gradient they acquire a velocity, and therefore a kinetic energy, dependent upon the length of the mean free path. While in motion they will collide with gas or vapour atoms, and the result of the collision will depend on the kinetic energy of the electron just prior to the collision. One of three things can happen. Firstly, if the kinetic energy is small, the electron will bounce off the atom and there will be little result beyond a possible change of direction. Secondly, if the kinetic energy is sufficiently high, the impact will be sufficiently great to displace one of the electrons of the atom from its normal orbit to another orbit round the positive nucleus. The atom is then said to be in an excited state and light of a certain wavelength will be emitted, the wavelength depending on the

change in the orbits of the electrons round the nucleus. The lowest kinetic energy to produce excitation raises the atom to what is called the first excited state, and if the colliding electron has this amount of energy it will leave the atom with zero velocity. If the energy before collision is greater than this, then the electron will leave the atom with a velocity corresponding to the excess energy, or, if the excess energy is sufficiently great, the atom will be raised to what is called its second excited state and light of another colour will be emitted. Hence the emission of light corresponds to the absorption by the atom of definite amounts of energy from the

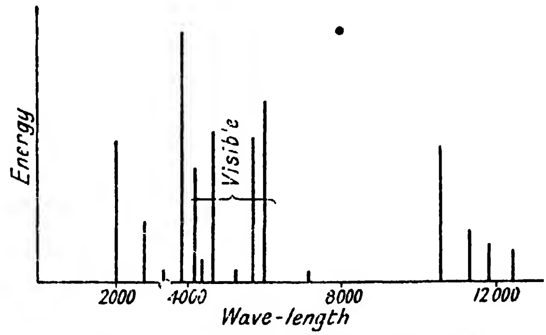


FIG. 445. MERCURY-VAPOUR SPECTRUM

colliding electron; therefore the spectrum, instead of being continuous, as with an incandescent solid, will be a line spectrum, each line corresponding to one of the excited states of the atom.

Thirdly, if the velocity of the colliding electron is sufficiently high, it will remove an electron from the atom of gas or vapour so that an additional free electron is produced and also the atom becomes preponderatingly positive: in other words it becomes an ion. This process is called ionization. Since, with each collision the number of electrons is increased, the process is cumulative and as a result it is characterized by the passage of heavy currents, unless there is some limiting device such as a resistance or a choke. In the case of sodium the atomic energy necessary for ionization is just over 5 electron-volts.

Let an electron having an initial velocity of zero travel between two points having a P.D. of 1 volt. Then it will acquire a definite kinetic energy which is defined as the electron-volt. In the case of sodium vapour an energy of 2·1 electron-volts in the colliding electron will be just sufficient to raise the atom to its first excited state, and light of wavelength 5,890 Ångström units* is emitted. This gives the characteristic yellow light of sodium vapour. If the electron has a kinetic energy of 3·18 electron-volts, then the atom is raised to its second excited state, and there is an additional

* The Ångström unit is one ten-millionth of a millimetre.

emission corresponding to a wavelength of 11,404 Ångström units. This is not within the visible spectrum and is therefore of no value from the point of view of light-production.

In the case of mercury vapour, light is emitted at several frequencies in the visible range, the general colour being bluish-green or bluish-white, according to the value of the vapour pressure. The spectrum of mercury vapour is given in Fig. 445, the lengths of the lines representing the energy at each wavelength. The point to notice is that the energy emitted in the visible region is a greater proportion of the whole than in the case of a filament

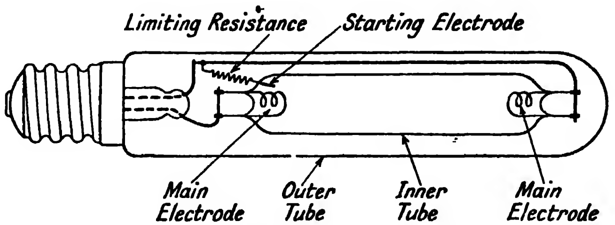


FIG. 446. TYPICAL MERCURY-DISCHARGE LAMP

lamp, and in consequence a mercury-discharge lamp is more efficient than an incandescent lamp. Thus a mercury-discharge lamp of 400-watt size will have an initial efficiency of 45 lumens per watt, while a gas-filled lamp of the same lumen output will have an efficiency of about 18 lumens per watt. In the case of the sodium lamp the efficiency is even better, the 150-watt size having an initial efficiency of 64 lumens per watt.

The relative intensities of the radiation at the characteristic wavelengths depend on the pressure. In the case of sodium there is a very strong yellow line at low pressure, but if the pressure is increased the radiations in the invisible regions increase while the visible yellow decreases. Thus the sodium-discharge lamp is, of necessity, a low-pressure lamp. With mercury, on the other hand, a rise in pressure causes a greater proportion of the total radiation to take place in the visible spectrum, so that increased pressure is accompanied by increased efficiency. Thus, for industrial and street-lighting purposes the mercury-discharge lamp operates at a pressure of about one atmosphere.

An examination of the mercury-vapour spectrum shows that there is a very strong line at a wavelength of 3,650, and therefore just within the ultra-violet region. The wavelength is thus too short for visibility. There are some materials which have the property of fluorescence, that is, they can absorb radiation of a certain wavelength and re-emit energy at a longer, and visible, wavelength. By coating the lamp container with such a material

invisible radiation is converted into visible radiation with corresponding gain in the efficiency. Also, by a suitable choice of fluorescent material, based on the colour of the light emitted by this material, it is possible to correct the colour of the mercury-discharge lamp in such a way as to reduce the predominance of the blue.

One form of mercury-discharge lamp consists of a hard-glass tube containing argon at low pressure and a little mercury. There is an electrode at each end and also an auxiliary starting electrode at one end. This tube is supported centrally in an outer glass tube, the space between the two tubes being exhausted so as to prevent inadvertent condensation of mercury due to the formation of cool

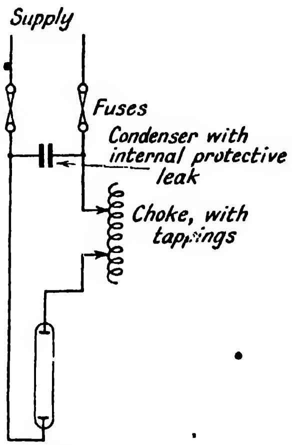


Fig. 447. Circuit of Mercury-discharge Lamp

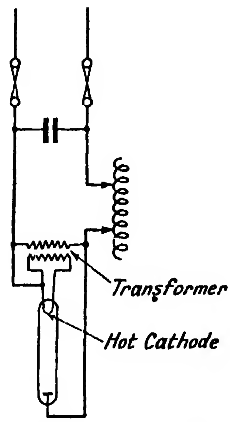


Fig. 448. Circuit of Sodium-discharge Lamp

places on the inner tube. This construction is shown in Fig. 446. When the lamp switch is closed, the full voltage is applied between the auxiliary electrode and the adjacent main electrode. This is sufficient to break down the small gap, and an argon discharge takes place. This discharge sets up sufficient ionization for the discharge to spread between the two main electrodes, after which there is sufficient heat production to vaporize the mercury and so permit the required mercury-vapour discharge. In order to limit the current at the starting electrode a high resistance of about 50,000 ohms is placed in series with it. Since the discharge is in the nature of an arc, the lamp has a negative temperature coefficient and it is therefore necessary to use a series resistance or reactance, the latter being usual because of the energy loss in a resistance. In the G.E.C. "Osira" lamp there is no starting electrode, the full supply voltage, which is applied to the electrodes at the moment of starting, being sufficient to cause the discharge in the rare-gas filling. There is an initial switching surge which lasts for a few cycles only, after which

the current in the cold lamp will be of the order of twice the normal current, this current being attained after a few minutes. The connection diagram is shown in Fig. 447. The condenser is included in the circuit for the purpose of improving the power factor.

The mercury lamp is a cold-cathode lamp, since the cathode is maintained in an incandescent state by ionic bombardment, as in the case of the mercury-arc rectifier. In the case of the sodium lamp this is not possible, and therefore the lamp must be of the hot-cathode type. The circuit therefore includes a step-down transformer for cathode heating, in addition to the stabilizing choke. In order that the lamp may be self-starting, the sodium lamp also has a filling of neon or argon, and, after switching on, the discharge takes a few minutes completely to change over to the sodium discharge. The essential connections are shown in Fig. 448.

The starting characteristics of a discharge lamp are, in some ways, analogous to those of a D.C. motor. At the moment of switching on, the back voltage of the arc is not established and the current is limited by the series choke. As ionization proceeds, the arc voltage rises and the current falls. Thus in the case of a 230 volt, 400 watt mercury-vapour lamp the starting current is 5·5 amperes and the arc voltage 15. After about seven minutes the lamp has settled down

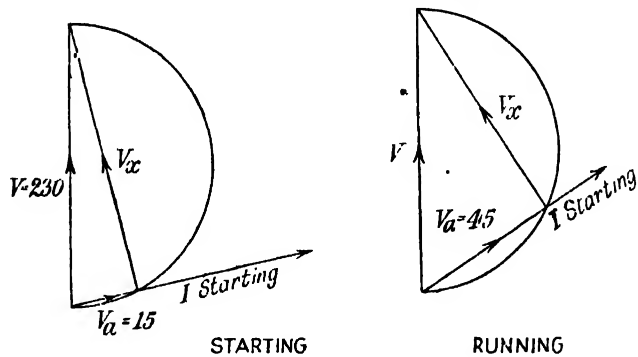


FIG. 449

to steady state conditions, and the current has fallen to 3 amperes while the arc voltage has risen to 145.

As with all A.C. arcs an approximately sinusoidal current gives rise to a flat-topped arc voltage, but so far as the expression can be applied to two quantities of different wave forms, the arc voltage is in phase with the current. The drop across the choke is in quadrature with the current, and therefore the voltage triangle whose sides represent the applied voltage V, arc voltage V_a, and choker drop V_x, will be right-angled, having V as the hypotenuse. The vector diagram for starting and running conditions for the above 400 watt lamp will therefore be as shown in Fig. 449.

EXAMPLES ON CHAPTER XXV.

(1) A lamp gives 30 c.p. in every direction. Draw a curve showing the illumination produced on the floor of a room if the lamp is 8 ft. above the floor.

(2) What considerations determine the best height of suspension when a number of similar arc lamps are to be used for street lighting and open spaces? A single arc lamp is such that, when suspended with the light source 22 ft. above the ground, it produces an illumination I on the ground, which varies with the distance d, reckoned from the foot of the lamp as follows—

If d is 0, 10, 20, and 30 ft., the corresponding values of I are 1·8, 1·7, 1·5, and 1·2 foot-candles, while for values of d from 40 to 100, the values of I may be assumed given by the equation $I = 1{\cdot}4 - 0{\cdot}01d$.

Draw a curve for the above illumination, and another curve for it when the lamp is raised 8 ft. (London Univ., 1913.)

(3) Define (*a*) illumination, (*b*) luminous flux, and (*c*) brightness of a luminous surface. State the units in terms of which these quantities are expressed. In what respect does the reflecting action of a diffusing surface differ from that of a mirror?

A large area is illuminated by a number of lamps each placed 12 ft. above the ground on posts erected all over the area, so as to stand at the corners of squares of 50 ft. side. The candle-power of each lamp is 300 and may be assumed uniform. Calculate the illumination of the ground (*a*) at the base of each lamp, (*b*) at the centre of each square, and (*c*) as an average for the whole area. (London Univ., 1914.)

Ans.—(*a*) 2·26; (*b*) 0·35; (*c*) ·75 candle-feet.

(4) A metal-filament lamp has the following light distribution in a vertical plane, angles being reckoned from the vertical—

Angle - -	0	15	30	45	60	75	90	105	120	135	150	165
C.-p. - -	8	10	26	35	44	52	49	46	40	30	14	0

Deduce its mean spherical candle-power and mean hemispherical candle-power for directions below the horizontal. If the candle-power for all directions in the horizontal plane through the lamp is constant at 49, find also its reduction factor.

(5) Define carefully the terms: mean horizontal candle-power, mean hemispherical candle-power, lumen, and luminous flux. Describe, with the help of a sketch, the method you would employ to determine the polar curve of illumination due to a large source of light, such as a "half-watt" lamp, with its reflector, the position of which is to remain fixed during your test. How would you find the mean spherical candle-power of the light from the readings taken? (London Univ., 1915.)

(6) If the filament of a 32 c.-p., 100 volt lamp has a length l and diameter d, calculate the length and diameter of the filament of a 16 c.p., 200 volt lamp, assuming that the two lamps are run at the same intrinsic brilliancy. (London Univ., 1916.) *Ans.*—1·25 l; 0·4 d.

(7) Describe recent improvements in the manufacture of metallic filament lamps. Find an expression connecting current and diameter of filament for a given maximum temperature. (London Univ., 1912.)

(8) Describe the half-watt gas-filled metal filament lamp. What conditions affect the efficiency of such a lamp, and how may the efficiency in candle-power per watt be determined? (London Univ., 1922.)

(9) Compare the effects of variable pressure of supply on glow lamps with carbon filaments, and on those with filaments of metal such as tungsten, as regards (*a*) fluctuations in light; (*b*) probable influence on the life; (*c*) fluctuations in the current flowing through. (London Univ., 1908.)

(10) Find the height at which a light having uniform spherical distribution should be placed over a floor in order that the intensity of horizontal illumination at a given distance from its vertical centre line may be the greatest. (London Univ., 1911.)

Ans.—$h = {\cdot}707\ l$, where l = distance from vertical centre line.

SUMMARY OF FORMULAE

PAGE

1. The Magnetic Circuit—

$$H = \frac{4\pi}{10} \times \frac{IN}{l}, \text{ or } \frac{1{\cdot}26 \times AT}{l} \quad . \quad . \quad . \quad . \quad 6$$

$$B = \mu H, = \frac{1{\cdot}26 \times AT}{l/\mu} \quad . \quad . \quad . \quad . \quad . \quad 6$$

$$\Phi = Ba, = \frac{1{\cdot}26 \times AT}{l/a\mu} \quad . \quad . \quad . \quad . \quad . \quad 9$$

$$\text{M.M.F.} = 1{\cdot}26 \times AT \quad . \quad . \quad . \quad . \quad . \quad . \quad 7$$

$$at = {\cdot}8H \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 7$$

$$\text{ctor} = \frac{\text{Total Flux}}{\text{Useful Flux}} \quad . \quad . \quad . \quad . \quad . \quad . \quad 8$$

$$P = \frac{B^2 a}{8\pi} \text{ dynes} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 11$$

$$P = \frac{B^2 a}{11{,}200{,}000} \text{ lb. wt.} \quad . \quad . \quad . \quad . \quad . \quad . \quad 11$$

Energy per c.c. in a magnetic field

$$= \frac{B^2}{8\pi} \text{ ergs} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 16$$

Hysteresis loss $= \eta B_{max}^{1{\cdot}6}$ ergs per c.c. per cycle 17

$= \eta v f B_{max}^{1{\cdot}6} \times 10^{-7}$ watts 17

Demagnetizing force for round bars

$$H_d = \frac{K}{4\pi \, . \, B} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 23$$

2. Production of an E.M.F.: Self and Mutual Induction—

$$E_e = Hlv \times 10^{-8} \text{ volts} \quad . \quad . \quad . \quad . \quad . \quad . \quad 26$$

$$\begin{pmatrix}\text{Self induced}\\ \text{E.M.F.}\end{pmatrix} = -\left\{\begin{pmatrix}\text{Flux set up}\\ \text{by 1 amp.}\end{pmatrix} \times N \times 10^{-8}\right\} \times \begin{pmatrix}\text{Rate of change}\\ \text{of current}\end{pmatrix} \quad 29$$

$$= -L\frac{dI}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 29$$

For two neighbouring coils A and B of N_1 and N_2 turns respectively,

$$\begin{pmatrix}\text{Mutually induced}\\ \text{E.M.F. in } B\end{pmatrix} = -\left\{\begin{pmatrix}\text{Flux through } B \text{ due}\\ \text{to 1 amp. in } A\end{pmatrix} \times N_2 \times 10^{-8}\right\} \times \begin{pmatrix}\text{Rate of change}\\ \text{of current in } A\end{pmatrix} \quad 29$$

Rise of current, $i = I\left(1 - e^{-\frac{R}{L}.t}\right)$ 30

Decay of current, $i = I \, . \, e^{-\frac{R}{L}.t}$ 30

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Inductance of a pair of parallel conductors

$$L = 14{\cdot}8 \times 10^{-4} \log_{10} \frac{d}{r} \text{ henrys per mile} \quad \ldots \quad 37$$

3. Capacitance—

Capacitance of isolated sphere $= \kappa a$ 55

" concentric spheres $= \kappa \dfrac{ab}{b-a}$ 55

parallel plates $= \dfrac{\kappa A}{4\pi d}$ 56

" coaxial cylinders $= \dfrac{\kappa}{2 \log_e R/r}$ 57

" parallel wires in air $= \dfrac{1}{4 \log_e \dfrac{d-r}{r}}$ 59

Electrical Oscillations –

E.M.F. equation of an oscillatory circuit having an E.M.F. impressed on it,

$$Ri + L\frac{di}{dt} + \frac{1}{c}\int i\,dt = e \quad \ldots \quad 69$$

If $e = E_{max} \sin \omega t$

then $$R\frac{di}{dt} + L\frac{d^2 i}{dt^2} + \frac{i}{c} = \omega E_{max} \cos \omega t,$$

the solution of which is

$$i = A \sin(\omega t - \alpha) + B\varepsilon^{-\frac{R}{2L}t} (\sin \omega' t - \beta)$$

where

$$A = \frac{E_{max}}{\sqrt{R^2 + \left(L\omega - \dfrac{1}{C\omega}\right)^2}}$$

$$\omega^1 = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

and B, α, and β are constants.

For the discharge of a condenser,

$$R\frac{di}{dt} + L\frac{d^2 i}{dt^2} + \frac{i}{c} = 0 \quad \ldots \quad 70$$

the solution of which is

(a) if $\dfrac{1}{LC} > \dfrac{R^2}{4L^2}$

$$i = a\varepsilon^{-\frac{R}{2L}t} (\cos \omega t + \varphi) \quad \ldots \quad 71$$

(b) if $\dfrac{1}{LC} < \dfrac{R^2}{4L^2}$

$$i = A\varepsilon^{-\left(\frac{R}{2L} - m\right)t} + B\varepsilon^{\left(\frac{R}{2L} + m\right)t} \quad \ldots \quad 72$$

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(c) if $\frac{1}{LC} = \frac{R^2}{4L^2}$

$$i = (At + B)\varepsilon^{-\frac{R}{2L}t} \quad . \quad . \quad . \quad . \quad . \quad 72$$

Rise of voltage at a condenser charged through a resistance

$$v = E\left(1 - \varepsilon^{-\frac{t}{CR}}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad 68$$

Decay of voltage on discharge through a resistance

$$v = E\,\varepsilon^{-\frac{t}{CR}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 68$$

4. E.M.F. Induced in a D.C. Armature—

$$E = \frac{\Phi ZN}{60} \times \frac{P}{A} \times 10^{-8} \text{ volts} \quad . \quad . \quad . \quad . \quad 95$$

5. Armature Reaction and Commutation—

$$AT_d = \frac{\theta}{360} \times ZI$$

$$AT_c = ZI\left(\frac{1}{4p} - \frac{\theta}{360}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad 99$$

Reactance voltage $= L \times \frac{2I}{t}$ 99

Formulae for self-induction of an armature coil under different conditions,

$$L = n^2(4l_1 + \cdot 8l_2) \times 10^{-8} \text{ henrys} \quad . \quad . \quad . \quad . \quad 101$$

$$L = n^2(8l_1 + \cdot 8l_2) \times 10^{-8} \text{ henrys} \quad . \quad . \quad . \quad . \quad 102$$

$$L = mn^2(8l_1 + \cdot 8l_2) \times 10^{-8} \text{ henrys} \quad . \quad . \quad . \quad 102$$

$$t = \frac{w_b - w_m}{v} \text{ seconds} \quad . \quad . \quad . \quad . \quad . \quad 102$$

6. Efficiency of a D.C. Machine—

$$\eta = \frac{E_tI}{E_tI + R_aI^2 + W_c} \quad . \quad . \quad . \quad . \quad . \quad . \quad 116$$

7. Direct Current Motor—

$$E_b = \frac{\Phi ZN}{60} \times \frac{P}{A} \times 10^{-8} \text{ volts} \quad . \quad . \quad . \quad . \quad 121$$

$$N = \frac{E - R_aI_a}{\Phi Z} \times \frac{60A}{P} \times 10^8 \quad . \quad . \quad . \quad . \quad 121$$

$$T = \frac{\Phi I_a Z}{20\pi} \times \frac{P}{A} \text{ dyne cm.} \quad . \quad . \quad . \quad . \quad . \quad 123$$

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Speed of shunt motor for any armature current I,

$$N = N_o \times \frac{E - R_a I}{E - R_a I_o} \quad \ldots \quad 128$$

Rules for calculating the steps of motor starters.
For a shunt motor,

$$\frac{R_n}{R_{n-1}} = \frac{R_{n-1}}{R_{n-2}} = \ldots \frac{R_2}{R_1} \quad \ldots \quad 133$$

Time required to attain full speed,

$$= \left(\frac{20\pi}{\Phi Z} \times \frac{A}{P}\right) \times \frac{K\omega}{I} \quad \ldots \quad 136$$

8. Direct Current Distribution—

Efficiency of transmission $= \dfrac{E_2}{E_1}$ 159

Losses in feeder $= 2I^2R$ 159

Differences in voltages on two sides of a three wire system with balancer, $E_2 - E_1 = R_a(I_1 - I_2)$ 163

Total annual financial loss

$$= £\left(Pa + \frac{Q}{a}\right) \quad \ldots \quad 168$$

Drop in distributor fed from one end

$= \sum(iR)$ 172

Drop in uniformly loaded distributor fed from one end

$= \frac{1}{2}IR$ 177

Drop in uniformly loaded distributor fed at both ends

$= \frac{1}{8}IR$ 177

9. Testing of D.C. Machines—

$$\eta = 17{\cdot}3\,\frac{N(W_1 - W_2)r}{EI}\,\% \quad \ldots \quad 179$$

$$W = I\omega \times \frac{d\omega}{dt} \times 10^{-7} \text{ watts} \quad \ldots \quad 186$$

$$= 0{\cdot}0109\, IN \frac{dN}{dt} \text{ watts} \quad \ldots \quad 186$$

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Efficiencies of an accumulator.

Quantity effic. $= \dfrac{\text{Discharge current} \times \text{time}}{\text{Charging current} \times \text{time}}$ 202

Energy effic. $= \dfrac{\text{Discharge current} \times \text{volt-hours of discharge}}{\text{Charge current} \times \text{volt-hours of charge}}$. 202

11. Alternating Quantities: Laws of Alternating Current Circuits—

$T = \frac{1}{f}$ and $f = \frac{1}{T}$ 209

$e = E_{max} \sin \omega t$
$= E_{max} \sin 2\pi f t$ 209

Form Factor $= E_{eff}/E_{av}$ 212

Inductance Reactance $= L\omega$ 216

(Impedance)^2 = (Resistance)^2 + (Reactance)^2 218

Power, $W = EI \cos \varphi$ 218

Power factor $= \cos \varphi$ 218

Capacity reactance $= 1/C\omega$ 221

In a circuit with resistance, inductance, and capacity in series,

$X = L\omega - \frac{1}{C\omega}$ 223

$Z = \sqrt{R^2 + \left(L\omega - \frac{1}{C\omega}\right)^2}$ 223

$\varphi = \tan^{-1}\frac{L\omega - 1/C\omega}{R}$ 224

Power factor $= \frac{R}{\sqrt{R^2 + \left(L\omega - \frac{1}{C\omega}\right)^2}}$ 224

Frequency for resonance,

$f = \frac{1}{2\pi}\sqrt{\frac{1}{LC}}$ 224

For a circuit consisting of parallel branches,

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or $Y = A_1 + A_2 + A_3 + \ldots$ (vector sum) . . 226

Conductance, $g = \frac{R}{Z^2}$ mhos. 226

Susceptance, $b = \frac{X}{Z^2}$ mhos. 226

$g = g_1 + g_2 + g_3 + \ldots$ 227

$b = b_1 + b_2 + b_3 + \ldots$ 227

$Y = \sqrt{g^2 + b^2}$ 227

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